Accelerator Physics*

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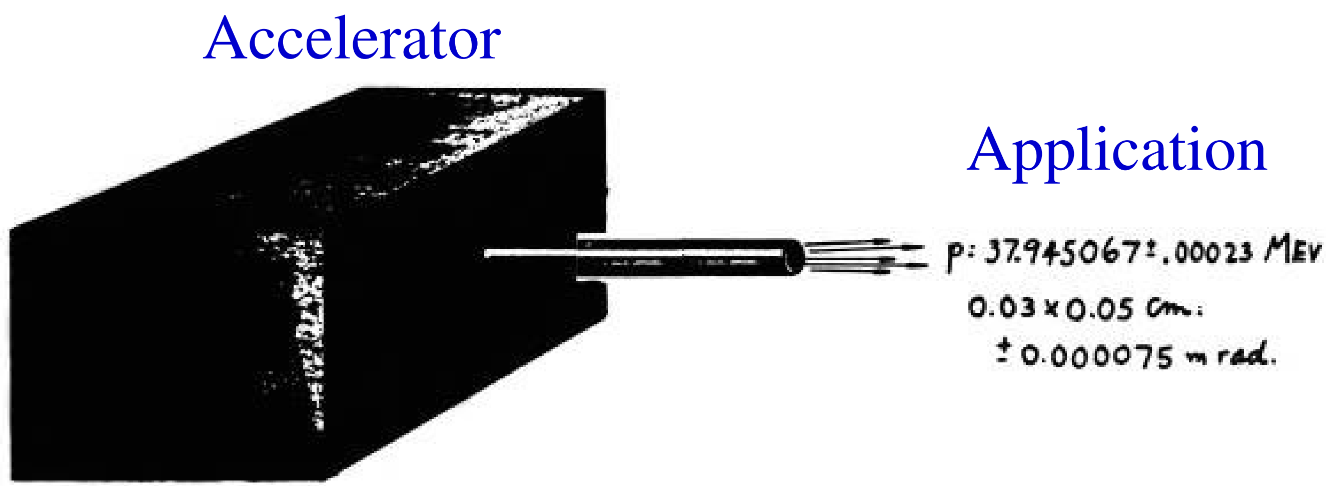
Exotic Beam Summer School
Michigan State University
National Super Conducting Cyclotron Laboratory (NSCL)
18-19 July, 2016
(Version 20160728)

* Research supported by:
FRIB/MSU, 2014 onward via: U.S. Department of Energy Office of Science Cooperative Agreement DE-SC0000661and National Science Foundation Grant No. PHY-1102511

Accelerator Physics Overview: Outline

1. Overview

Accelerators tend to be viewed by specialists in other fields as a “black box” producing particles with some parameters



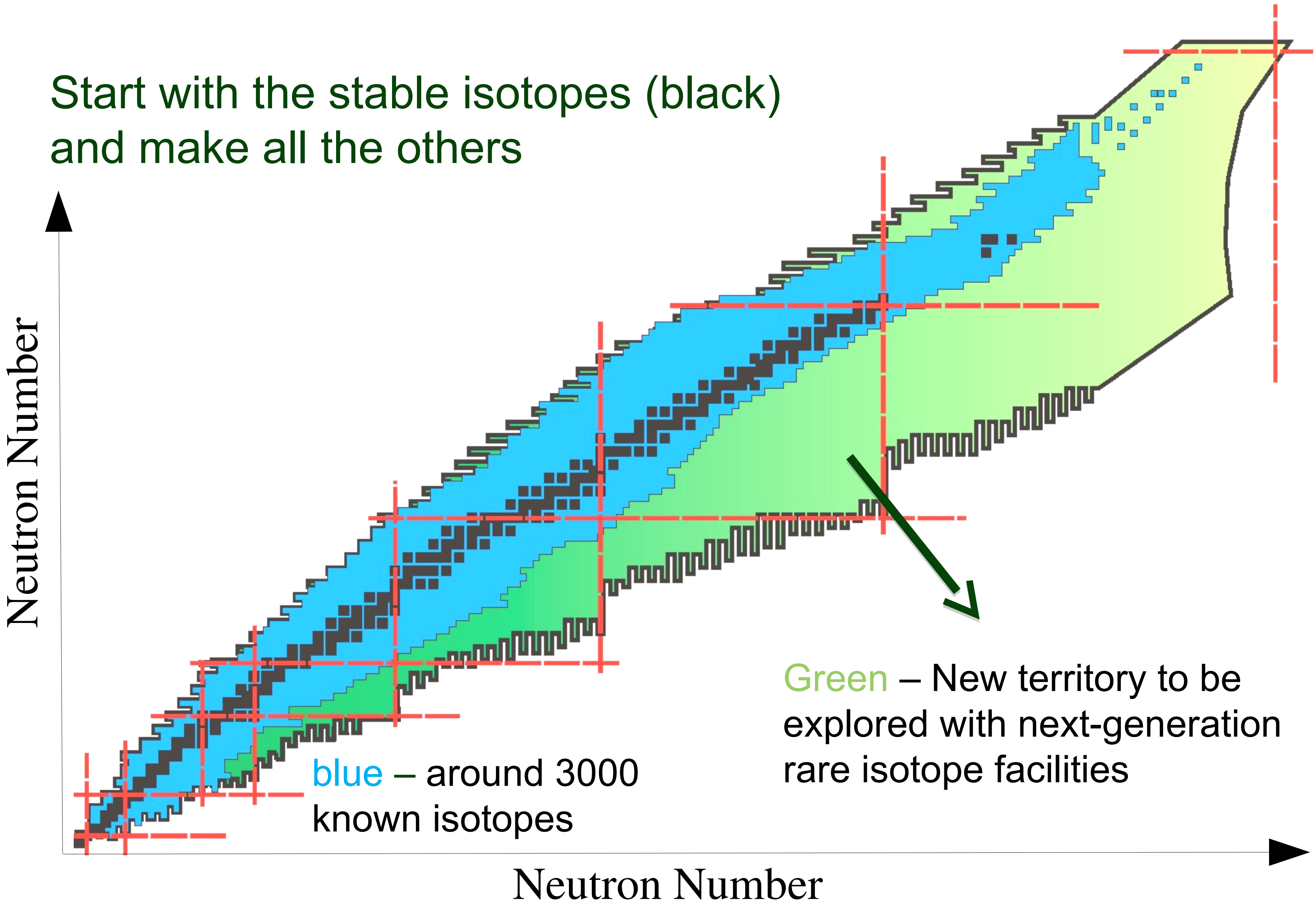
But accelerator science and technology is a highly developed field enabling a broad range of discovery science and industry

- Discovery Science:
  High Energy (Colliders) and Nuclear Physics (Cyclotrons, Rings, Linacs)
  Materials Science (Light Sources)
- Industrial: Semiconductor Processing, Material Processing, Welding
- Medical: X-Rays, Tumor Therapy, Sterilization

Modern, large-scale accelerator facilities are a monument to modern technology and take a large number of specialists working effectively together to develop and maintain

- Only briefly survey a small part of linear optics in this lecture …. much much more !

In nuclear physics, accelerators are used to produce beams of important rare isotopes



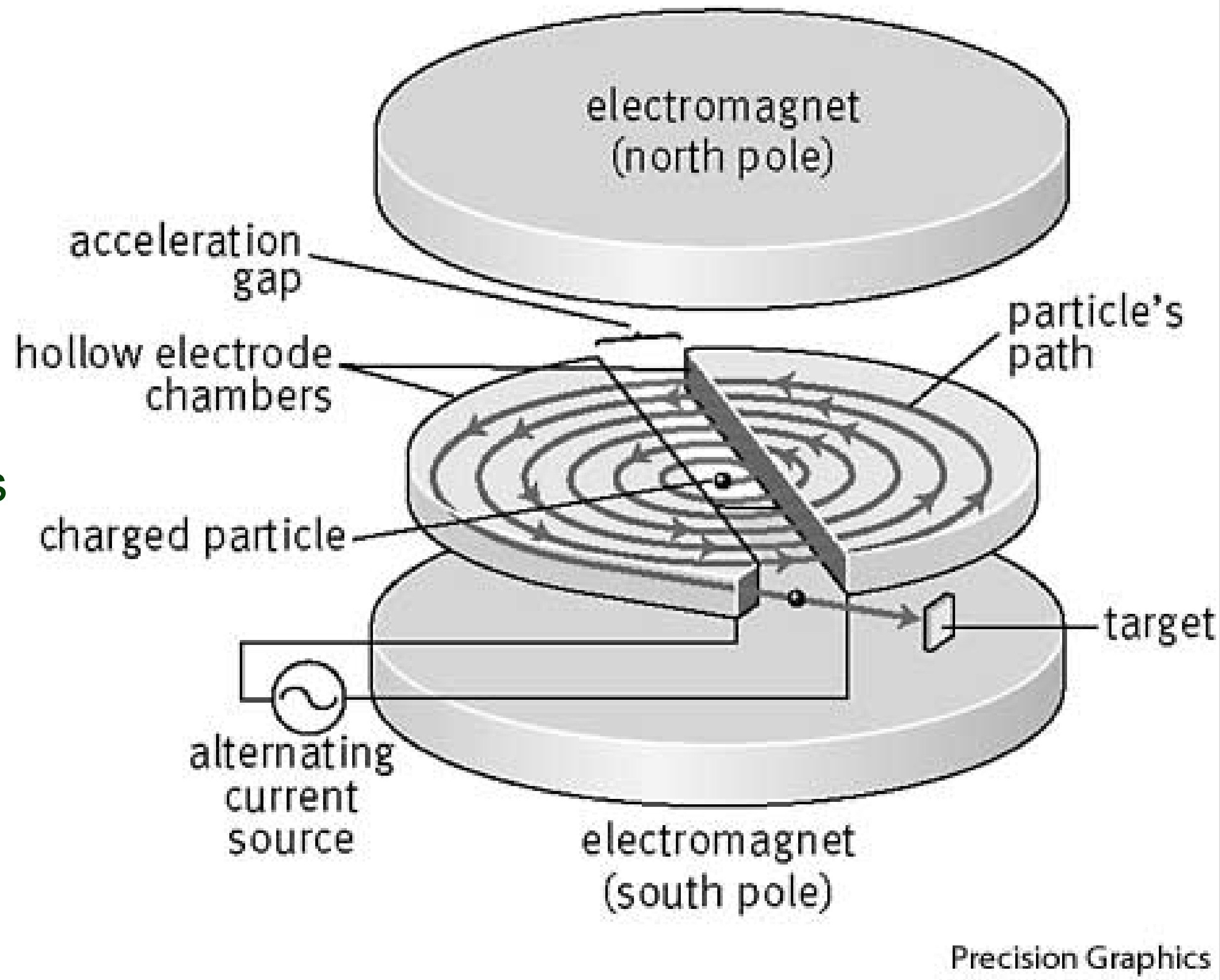
Accelerators for rare isotope production

- The particle accelerator used for production is called the “driver”
- Types
  - Cyclotron — NSCL (USA; MSU), GANIL (France), TRIUMF (proton; Canada), HRIBF (proton; USA ORNL), RIKEN RIBF (Japan)
  - Synchroton — GSI, FAIR-GSI (Germany); IMP (China)
  - LINAC — (LINear ACcelerator) FRIB (USA; MSU), ATLAS (USA; ANL), RAON (Korea)
  - Others — like FFAGs (Fixed-Field Alternating Gradient) not currently used but considered
- Main Parameters
  - Max Kinetic Energy (e.g. FRIB will have 200 MeV/u uranium ions)
  - Particle Range (TRIUMF cyclotron accelerates hydrogen, used for spallation)
  - Intensity or Beam Power (e.g. 400 kW = $8 \times 6 \times 10^{12}$/s x 50GeV
  - Power = pμA x Beam Energy (GeV) (1pμA = 6×10^{12} /s)

Cyclotrons

Continuous train of particle bunches injected from center and spiral outward on RF acceleration over many laps. Exits machine on last lap to impinge on target.

- Relatively easy to operate and tune (few parts)
- Used for isotope production and applications where reliable and reproducible operation are important (medical)
- Intensity low (but continuous train of bunches) due to limited transverse focusing, acceleration efficiency is high, cost low
- Relativity limits energy gain, so energy is limited to a few hundred MeV/u.
- State of art for heavy ions: RIKEN (Japan) Superconducting Cyclotron 350 MeV/u

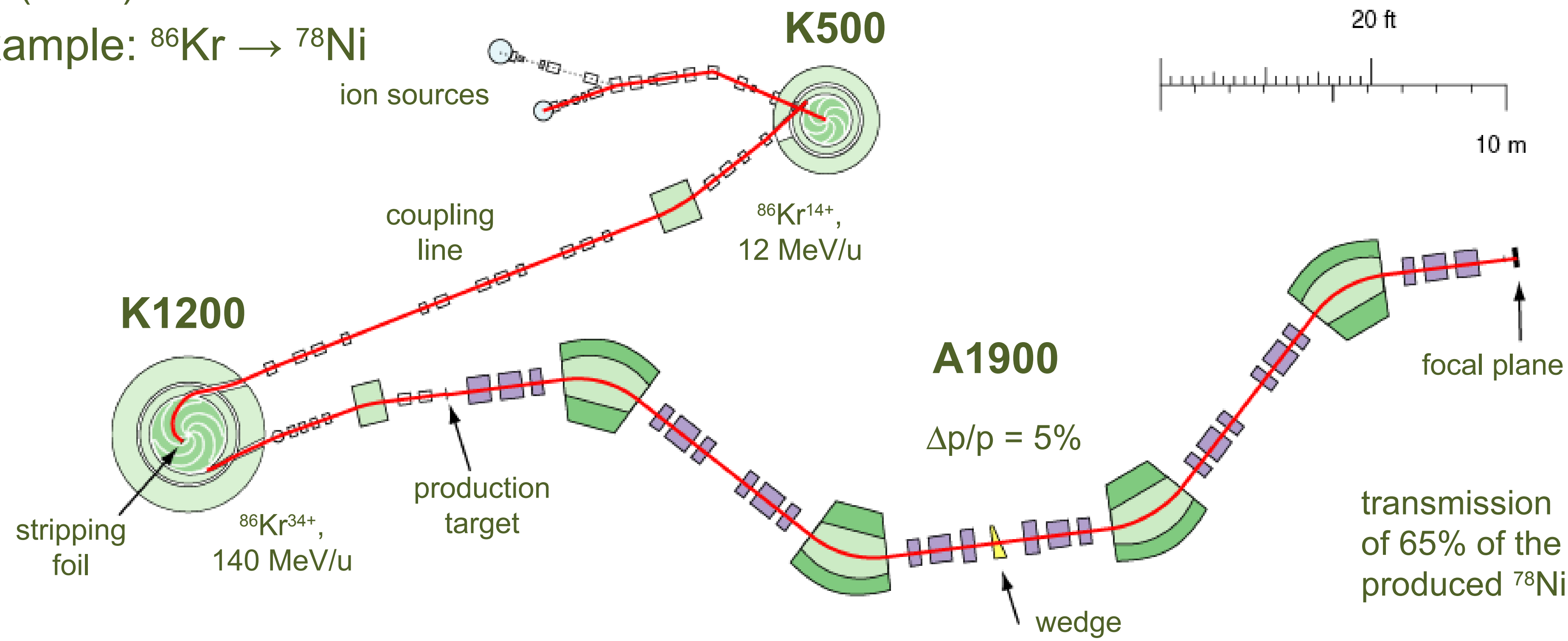


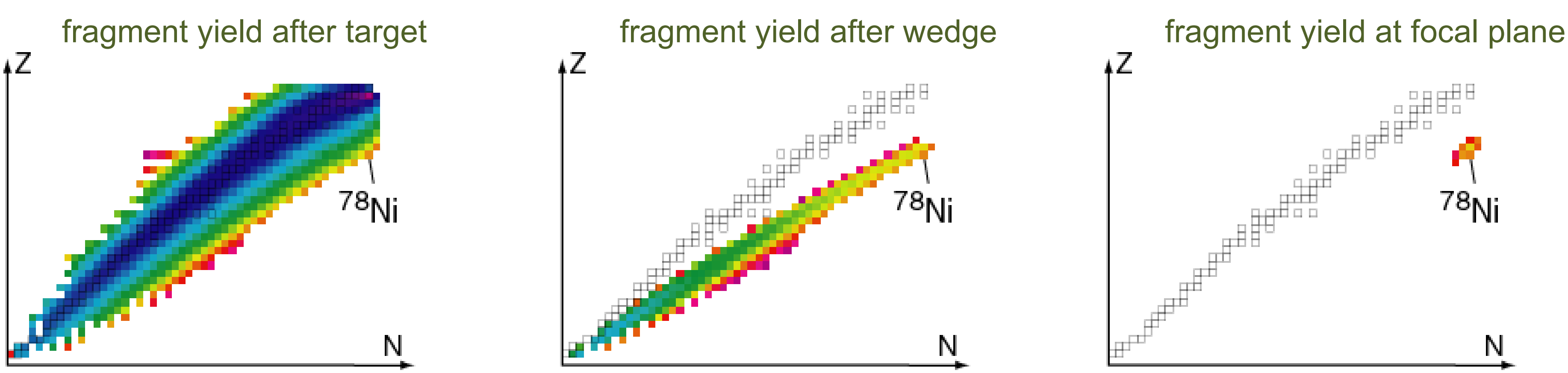
http://images.yourdictionary.com/cyclotron

Cyclotron example: NSCL's coupled cyclotron facility

D.J. Morrissey, B.M. Sherrill, Philos. Trans. R. Soc. Lond. Ser. A. Math. Phys. Eng. Sci. 356 (1998) 1985.

Example: $^{86}Kr \rightarrow {}^{78}Ni$



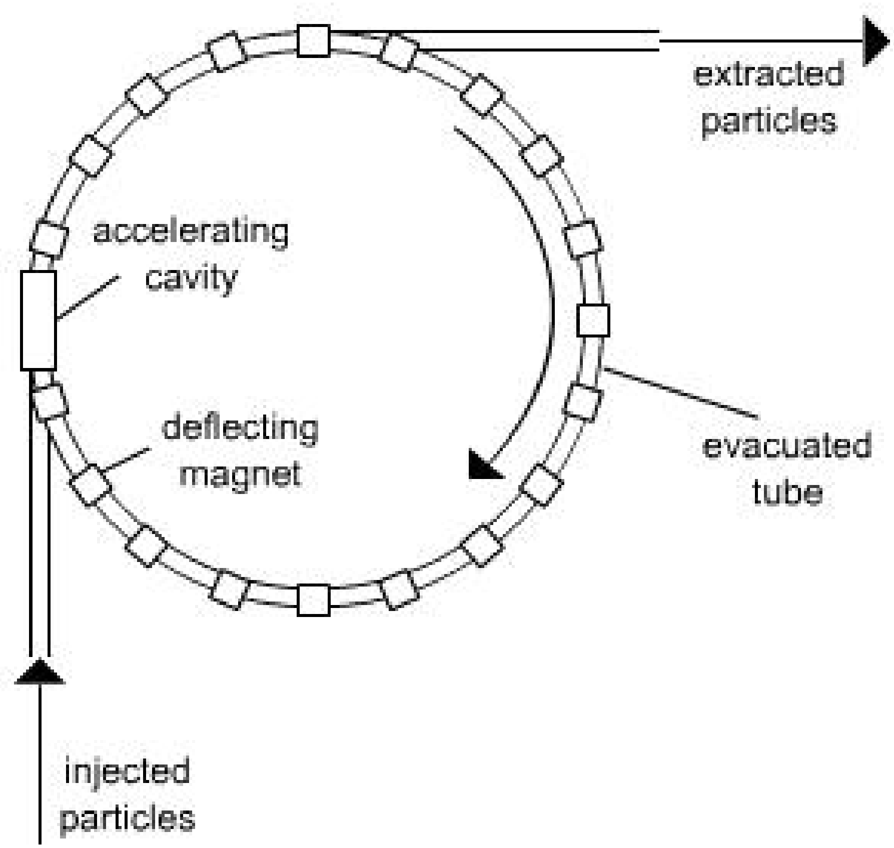


Synchrotrons

A "train" of bunches injected to fill ring and then bunches in ring accelerated while bending and focusing rise synchronously. At max energy bunch train is kicked out of ring and impinges on target. Then next cycle is loaded.

- Can achieve high energy at modest cost – tend to be used to deliver the highest energies
- Intensity is limited by the Coulomb force of particles within bunches (Space Charge)
- The magnets (bend and focus) must rapidly ramp and this can be difficult to do for superconducting magnets
- Machine must be refilled for next operating cycle giving up average intensity due to overall duty factor
- State of the art for heavy ions now under construction: FAIR (Germany) and IMP/Lanzhou

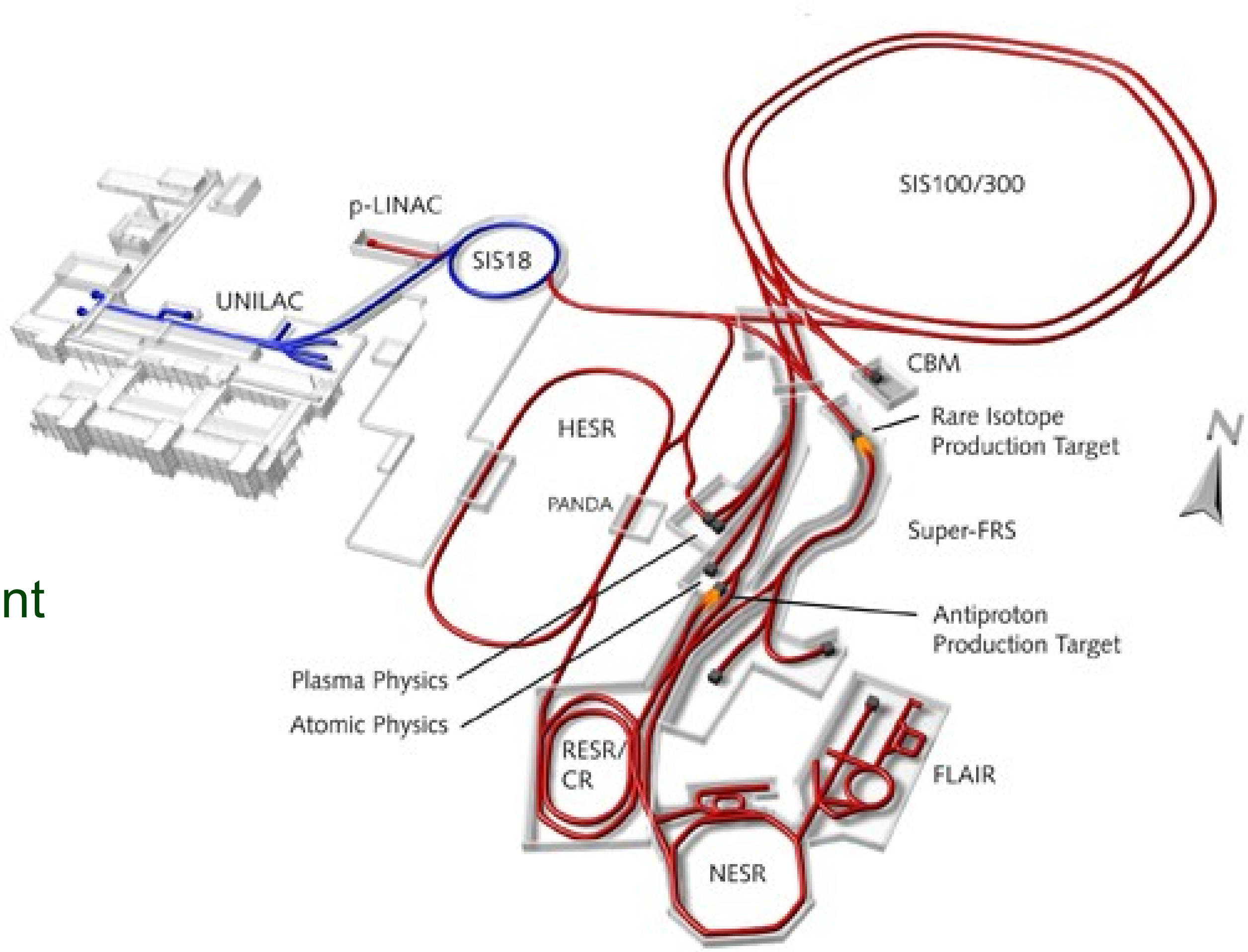
  + CERN LHC for p-p (Higgs)



http://universe-review.ca/R15-20-accelerators.htm

Example synchrotron: Facility for Antiproton and Ion Research (FAIR), GSI Germany

- Beams at 1.5 GeV/u
- 10^{12}/s Uranium
- Research
  - Rare isotopes
  - Antiproton
  - Atomic physics
  - Compressed matter
  - Plasma physics
- Under construction with significant delays but front end working/upgraded



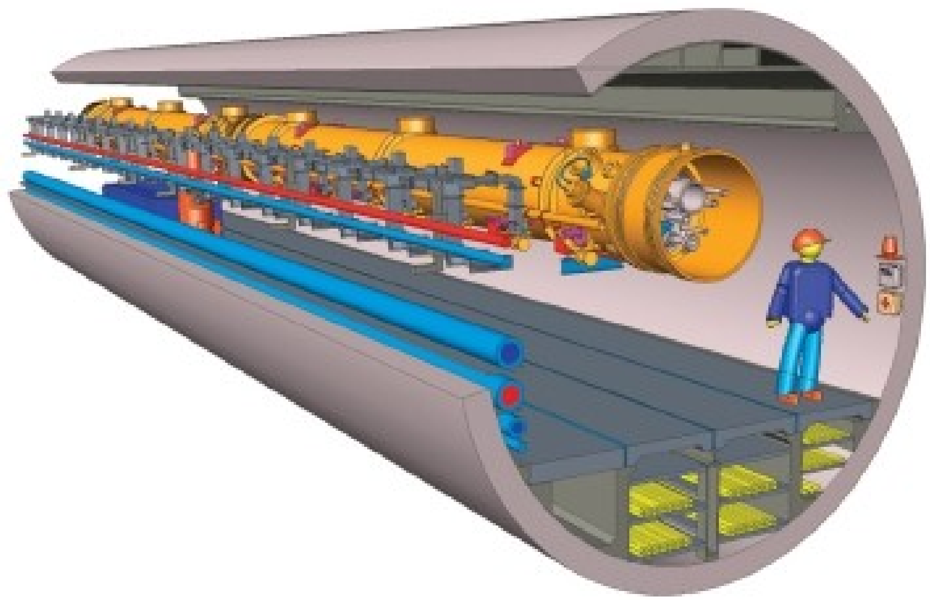
http://www.fair-center.de/index.php?id=1

LINAC (LINear ACcelerator)

A continuous "train" of bunches injected into a straight lattice for acceleration with strong transverse focusing and RF acceleration

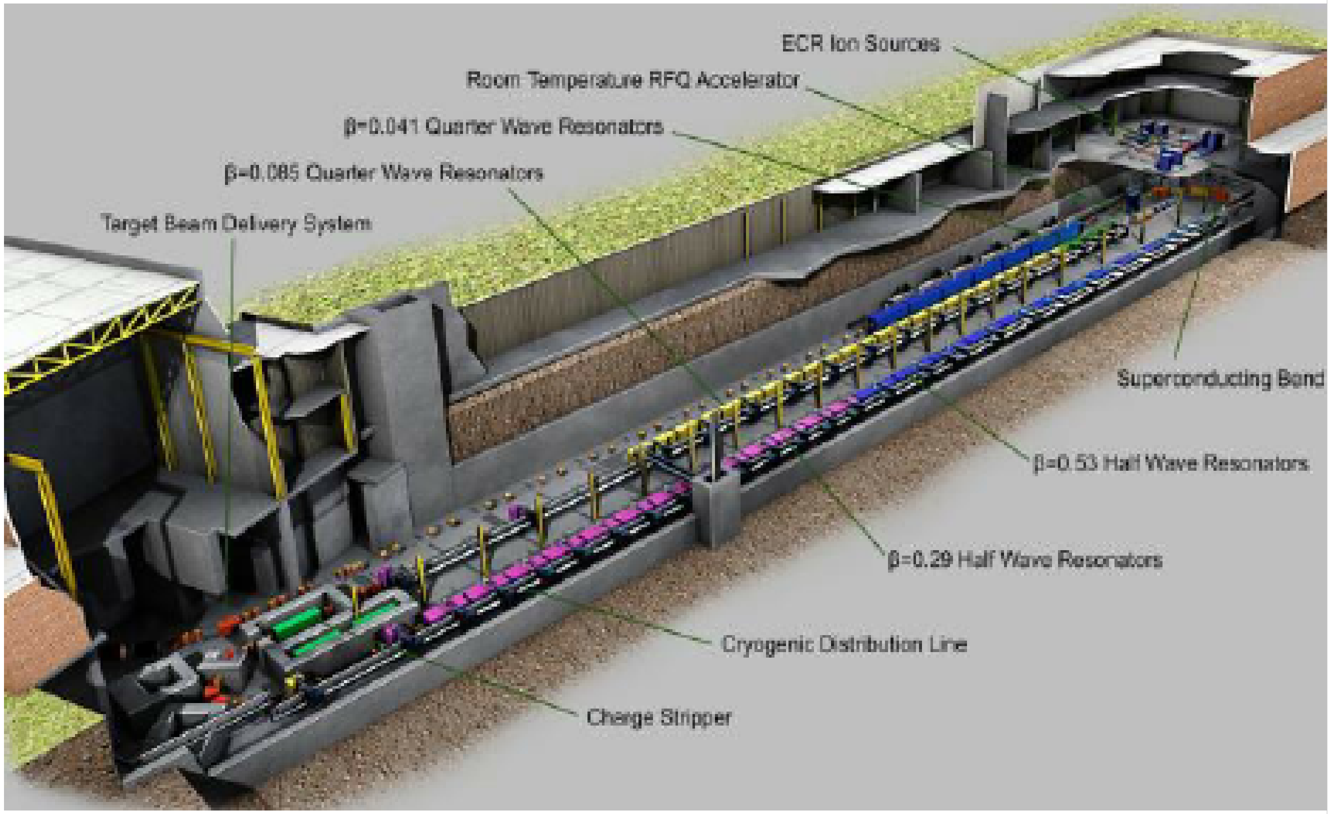
- Many types for ions, protons, and electrons and has simpler physics than rings. Applications from industrial and small medical to discovery science.
- Intensity can be very high since compatible with strong transverse focusing and a continuous train of bunches
- Retuning for ions can be complex
- Can use superconducting RF cavities for high efficiency but at high cost/complication
- Cost can be high (RF cavities one pass: need high gradient)
- Used to provide the highest intensities
- State of art for heavy ions: FRIB folded linac (MSU) under construction

DESY LINAC segment (XFEL driver)



Example LINAC: Facility for Rare Isotope Beams (FRIB)

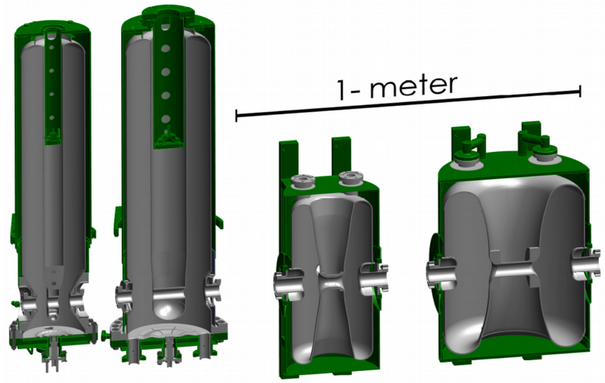
- 200 MeV/u, 400 KW continuous power on-target with heavy ions
- Superconducting RF cavities and solenoid focusing
- Novel liquid Li stripper to boost charge state and simultaneously accelerates several species to increase power on target
- Under construction: front end undergoing early commissioning in 2016



Superconducting RF cavities

4 types ≈ 344 total

E_{peak} ≈ 30 MV/m



β = 0.04 β = 0.08 β = 0.2 β = 0.5

2. Quadrupole and Dipole Fields and the Lorentz Force Equation

Consider a long static magnet where we can approximate the fields as 2D transverse within the vacuum aperture:

$$\boxed{\mathbf{B} = B_x(x,y)\hat{\mathbf{x}} + B_y(x,y)\hat{\mathbf{y}}}$$

Taylor expand for small x,y about origin and retain only linear terms of “right” symmetry:

$$\mathbf{B} \simeq \left[\cancelto{0}{B_x(0)} + \cancelto{0}{\left.\frac{\partial B_x}{\partial x}\right|_0 x} + \left.\frac{\partial B_x}{\partial y}\right|_0 y + \cancelto{0}{\cdots}\right]\hat{\mathbf{x}} + \left[B_y(0) + \cancelto{0}{\left.\frac{\partial B_y}{\partial y}\right|_0 y} + \left.\frac{\partial B_y}{\partial x}\right|_0 x + \cancelto{0}{\cdots}\right]\hat{\mathbf{y}}$$

Via: symmetry choices and design

Maxwell equation for a static magnetic field in a vacuum aperture:

$$\nabla\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ B_x & B_y & 0 \end{vmatrix} = \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\hat{\mathbf{z}} = 0 \qquad \Longrightarrow \left.\frac{\partial B_x}{\partial y}\right|_0 = \left.\frac{\partial B_y}{\partial x}\right|_0 \equiv G$$

$$\nabla\cdot\mathbf{B} = 0 \quad \text{satisfied}$$

$G =$ Quadrupole Field Gradient

Giving:

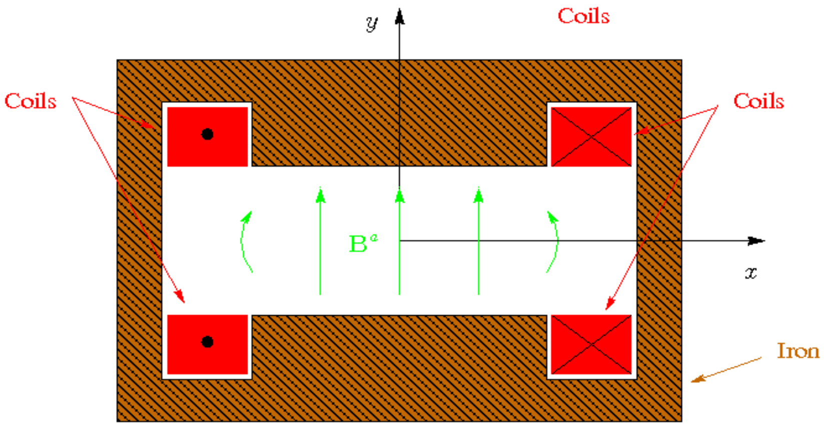
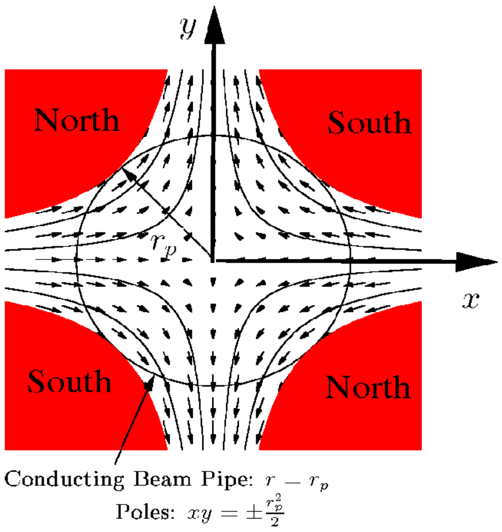
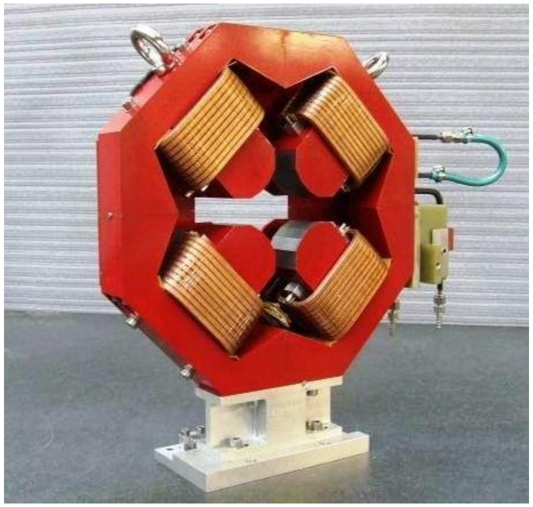
$$\boxed{\mathbf{B} \simeq Gy\hat{\mathbf{x}} + \left[B_y(0) + Gx\right]\hat{\mathbf{y}}}$$

$$B_y(0) \quad [\text{Tesla}] \qquad \Longleftrightarrow \qquad \text{Dipole x-plane Bend}$$

$$G \quad [\text{Tesla/meter}] \qquad \Longleftrightarrow \qquad \text{Quadrupole Magnetic Focus}$$

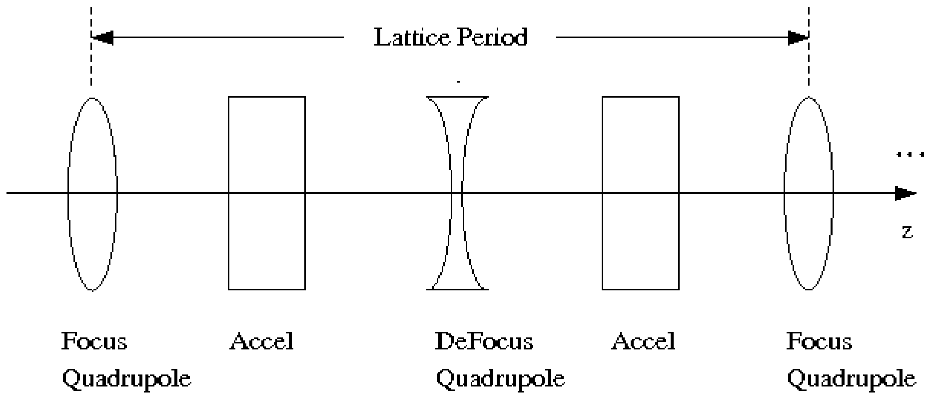
Magnets

Magnet design is a complicated topic … but some examples of elements to produce these static magnetic fields:

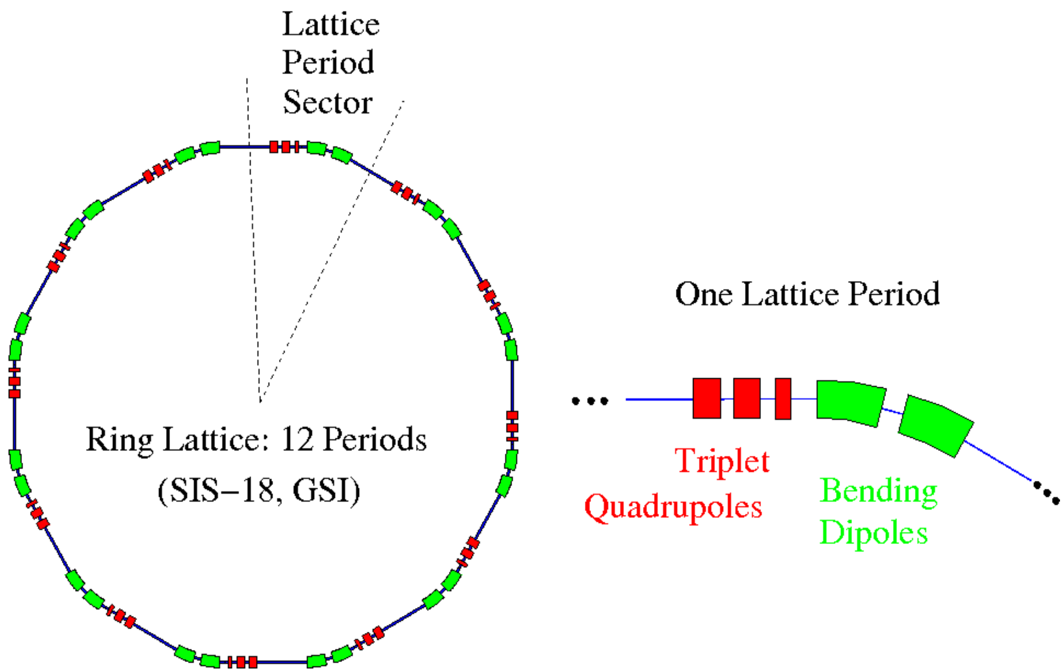
	Idealized Structure	Laboratory Magnet
Dipole		 India: Dept Atomic Eng
Quadrupole		 Canada: TRIUMF

Elements of accelerator are typically separated by function into a sequence of elements making up a “lattice”

Example – Linear FODO lattice (symmetric quadrupole doublet) for a LINAC



Example – Synchrotron lattice with quadrupole triplet focusing



Lorentz force equation for particle of charge q and mass m evolving in magnetic field:

- Magnetic field only bends particle without change in energy:

$$\frac{d}{dt}\mathbf{p} = q\mathbf{v} \times \mathbf{B} \qquad \mathbf{p} = m\gamma\mathbf{v} = m\gamma\dot{\mathbf{x}} \qquad \dot{} = \frac{d}{dt}$$

$$\gamma = \text{const} \qquad \longrightarrow \qquad \frac{d}{dt}\mathbf{p} = m\gamma\ddot{\mathbf{x}}$$

Simplified Lorentz force equation giving the particle evolution in the field:

$$\boxed{m\gamma\ddot{\mathbf{x}} = q\dot{\mathbf{x}} \times \mathbf{B}}$$

Putting in the expanded field of our specific form of interest:

$$m\gamma\ddot{\mathbf{x}} = q\dot{\mathbf{x}} \times \{Gy\hat{\mathbf{x}} + [B_y(0) + Gx]\,\hat{\mathbf{y}}\}$$

Beam is directed, so assume particle primarily moving longitudinally with:

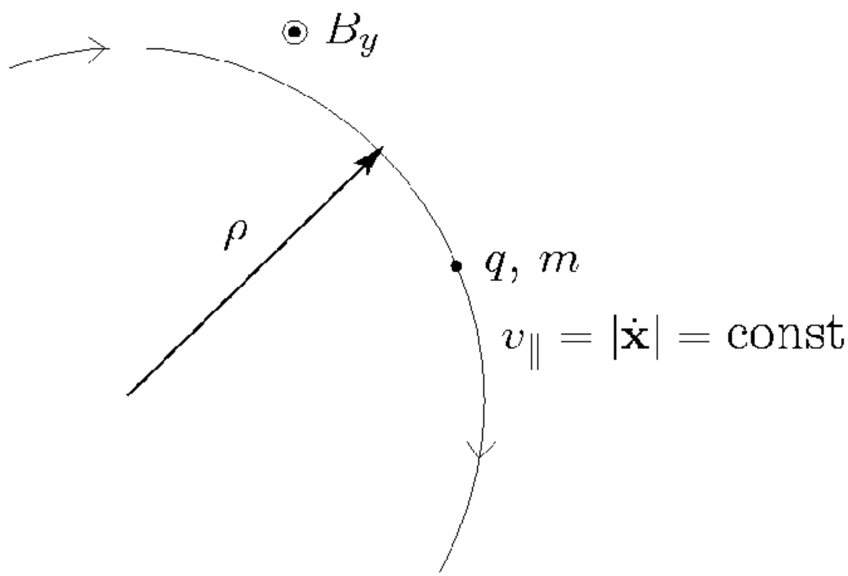
$$[\dot{\mathbf{x}} \times \mathbf{B}_\perp]_\perp = -v_{||}B_y\hat{\mathbf{x}} + v_{||}B_x\hat{\mathbf{y}}$$

$$\Longrightarrow \quad \boxed{\begin{aligned} \gamma m\ddot{x} &= -qv_{||}[B_y(0) + Gx] \\ \gamma m\ddot{y} &= qv_{||}Gy \end{aligned}}$$

3. Dipole Bending and Particle Rigidity

Illustrative Case: Particle bent in a uniform magnetic field

$$B_y(0) \neq 0, \qquad G = 0$$



Particle is bent on a circular arc so Lorentz force equation gives:

$$m\gamma\ddot{\mathbf{x}} = q\dot{\mathbf{x}} \times B_y(0)\hat{\mathbf{y}}$$

$$\Longrightarrow \quad -\gamma m \frac{v_{\parallel}^2}{\rho} = -q v_{\parallel} B_y(0)$$

$$\boxed{\begin{aligned} \frac{1}{\rho} &= \frac{B_y(0)}{[B\rho]} \\ [\mathrm{B}\rho] \equiv \frac{\gamma m v_{\parallel}}{q} = \frac{p}{q} &= \frac{\mathrm{Momentum}}{\mathrm{Charge}} \\ &= \mathrm{Rigidity} \end{aligned}}$$

Dipole bends are used to manipulate "reference" path

- Rings
- Transfer Lines

and also manipulate focusing properties since bend radius ρ depends on energy

- Fragment Separators for nuclear physics

Rigidity measures the particle coupling strength to magnetic field

$$[B\rho] = \frac{p}{q} = \frac{\text{Momentum}}{\text{Charge}} \equiv \text{Rigidity}$$
$$= \frac{\gamma m v}{q} = \frac{mc}{q}\gamma\beta$$

Set in terms of:

- Particle Species: $q, \quad m$
- Particle Kinetic Energy: $\mathcal{E} = (\gamma - 1)mc^2 \quad \Longleftrightarrow \gamma, \quad \beta$
- Units are Tesla-meters and $[B\rho]$ is read as one symbol "B-rho"

Heavy ions much more "rigid" than electrons and require higher fields to move:

Electron: $mc^2 = 511 \text{ KeV} \quad q = -e \quad \rightarrow \quad \dfrac{mc}{q} = -1.705^{-3} \text{ Tesla-m}$

$m = Am_u \quad q = Qe$

Ion: $m_u c^2 = 931 \text{ MeV} \quad \rightarrow \quad \dfrac{mc}{q} = 3.11 \dfrac{A}{Q} \text{ Tesla-m}$

Particle kinetic energy sets $\gamma, \quad \beta$:

$$\gamma = 1 + \frac{\mathcal{E}}{mc^2} \qquad \beta = \sqrt{1 - 1/\gamma^2}$$

$$\mathcal{E} \equiv (\gamma - 1)mc^2 = \text{Kinetic Energy}$$

$$\beta\gamma = \sqrt{\gamma^2 - 1} = \sqrt{\left(\frac{\mathcal{E}}{mc^2}\right)^2 + 2\left(\frac{\mathcal{E}}{mc^2}\right)}$$

For ions take:

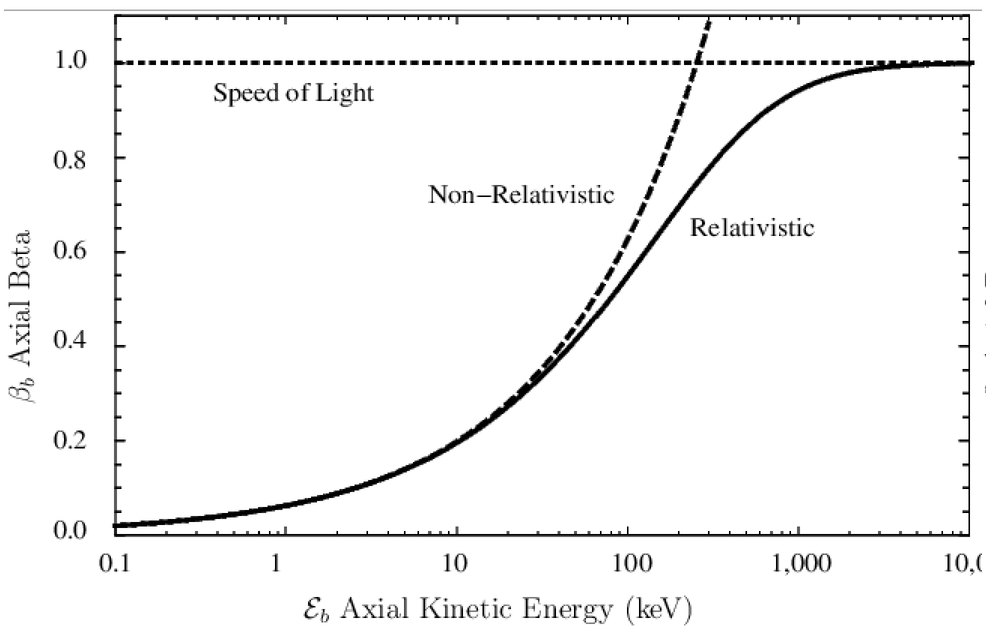
$$\boxed{m \simeq A m_u \qquad A = \text{Mass Number (Number of Nucleons)}}$$

and kinetic energy per nucleon $\mathcal{E}/A$ [MeV/u] fixes $\gamma,\ \beta$

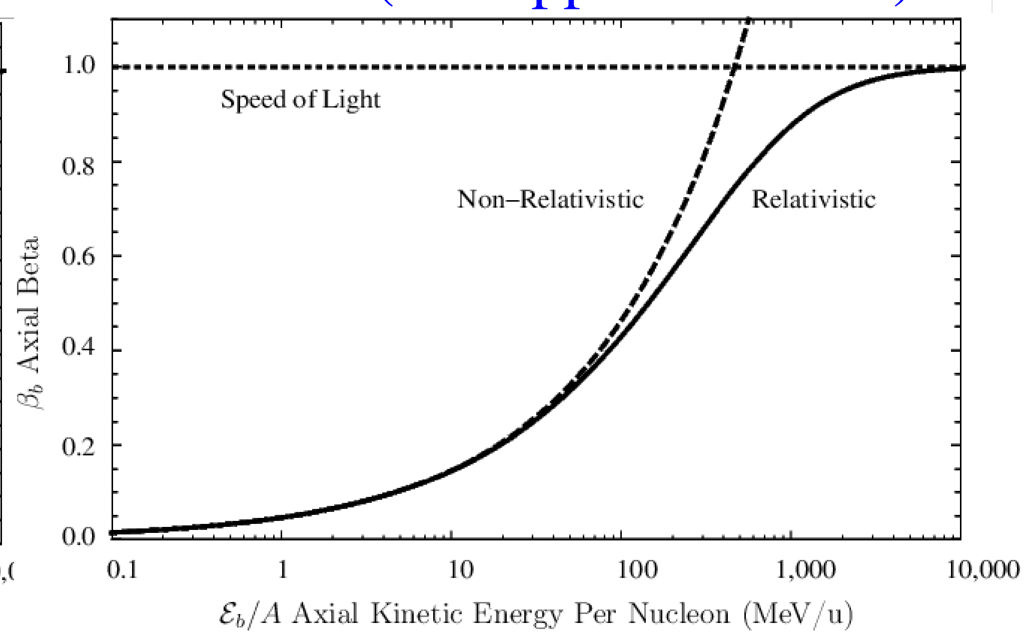
$$\boxed{\mathcal{E}/A = (\gamma - 1) m_u c^2}$$

Common measure of energy since β determines synchronism with RF fields for acceleration and bunching

Electrons



Ions (and approx Protons)



Electrons are much less "rigid" than ions and are deflected with lower field strength

4. Quadrupole Focusing

Illustrative Case: Focused within a quadrupole magnetic field

$$B_y(0) = 0 \qquad G \neq 0$$

$$\begin{aligned} \gamma m \ddot{x} &= -q v_{\|}[B_y(0) + Gx] \\ \gamma m \ddot{y} &= q v_{\|} G y \end{aligned} \quad \Longrightarrow \quad \boxed{\begin{aligned} \gamma m \ddot{x} + q v_{\|} G x &= 0 \\ \gamma m \ddot{y} - q v_{\|} G y &= 0 \end{aligned}}$$

Let s be the axial coordinate (will later bend on curved path in dipole) and assume beam motion is primarily longitudinally (s) directed

$$\dot{} = \frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} = v_{\|}\frac{d}{ds} \simeq v\frac{d}{ds} \qquad v_{\|} \simeq v \simeq \text{const}$$

(no acceleration)

$$\Longrightarrow \quad \ddot{x} \simeq v^2 \frac{d^2}{ds^2} x$$

Giving the particle trajectory equations in a quadrupole magnet:

$$\boxed{\begin{aligned} &\frac{d^2}{ds^2}x + \frac{G}{[B\rho]}x = 0 \qquad [B\rho] = \frac{p}{q} = \frac{mc}{q}\gamma\beta = \text{Rigidity} = \text{const} \\ &\frac{d^2}{ds^2}y - \frac{G}{[B\rho]}y = 0 \qquad \frac{G}{[B\rho]} \equiv \kappa(s) = \text{Lattice Focus Function} \\ &\text{Allow } G \text{ to vary in } s \qquad [[\kappa]] \sim 1/(\text{length})^2 \end{aligned}}$$

Transfer Matrix Solutions

Integrate equation from initial condition:

$$x(s_i) = x_i = \text{ Initial coordinate} \qquad y(s_i) = y_i = \text{ Initial coordinate}$$
$$x'(s_i) = x'_i = \text{ Initial angle} \qquad y'(s_i) = y'_i = \text{ Initial angle}$$

Free drift: $\kappa = 0 \qquad x'' = 0 \qquad \cdots' \equiv \frac{d}{ds}\cdots$

$$x = x_i + x'_i(s - s_i)$$
$$x' = x'_i$$
$$\mathbf{M}_x(s|s_i) = \begin{bmatrix} 1 & s - s_i \\ 0 & 1 \end{bmatrix}$$

+ analogous for *y*-plane

Write linear phase-space solutions in 2x2 "Transfer Matrix" form:

$$\begin{bmatrix} x \\ x' \end{bmatrix}_s = \mathbf{M}_x(s|s_i) \cdot \begin{bmatrix} x \\ x' \end{bmatrix}_{s_i}$$
$$\begin{bmatrix} y \\ y' \end{bmatrix}_s = \mathbf{M}_y(s|s_i) \cdot \begin{bmatrix} y \\ y' \end{bmatrix}_{s_i}$$

x-Focusing Plane: $\kappa = \hat{\kappa} = \text{const} > 0 \qquad x'' + \hat{\kappa}x = 0$ simple harmonic oscillator

$$x = x_i \cos[\sqrt{\hat{\kappa}}(s - s_i)] + (x'_i/\sqrt{\hat{\kappa}}) \sin[\sqrt{\hat{\kappa}}(s - s_i)]$$
$$x' = -\sqrt{\hat{\kappa}}x_i \sin[\sqrt{\hat{\kappa}}(s - s_i)] + x'_i \cos[\sqrt{\hat{\kappa}}(s - s_i)]$$
$$\mathbf{M}_x(s|s_i) = \begin{bmatrix} \cos[\sqrt{\hat{\kappa}}(s - s_i)] & \frac{1}{\sqrt{\hat{\kappa}}} \sin[\sqrt{\hat{\kappa}}(s - s_i)] \\ -\sqrt{\hat{\kappa}} \sin[\sqrt{\hat{\kappa}}(s - s_i)] & \cos[\sqrt{\hat{\kappa}}(s - s_i)] \end{bmatrix}$$

y-DeFocusing Plane: $-\kappa = \hat{\kappa} = \text{const} > 0 \qquad y'' - \hat{\kappa} x = 0$

Exponential growth and decay

$$y = y_i \cosh[\sqrt{\hat{\kappa}}(s - s_i)] + (y_i'/\sqrt{\hat{\kappa}}) \sinh[\sqrt{\hat{\kappa}}(s - s_i)]$$

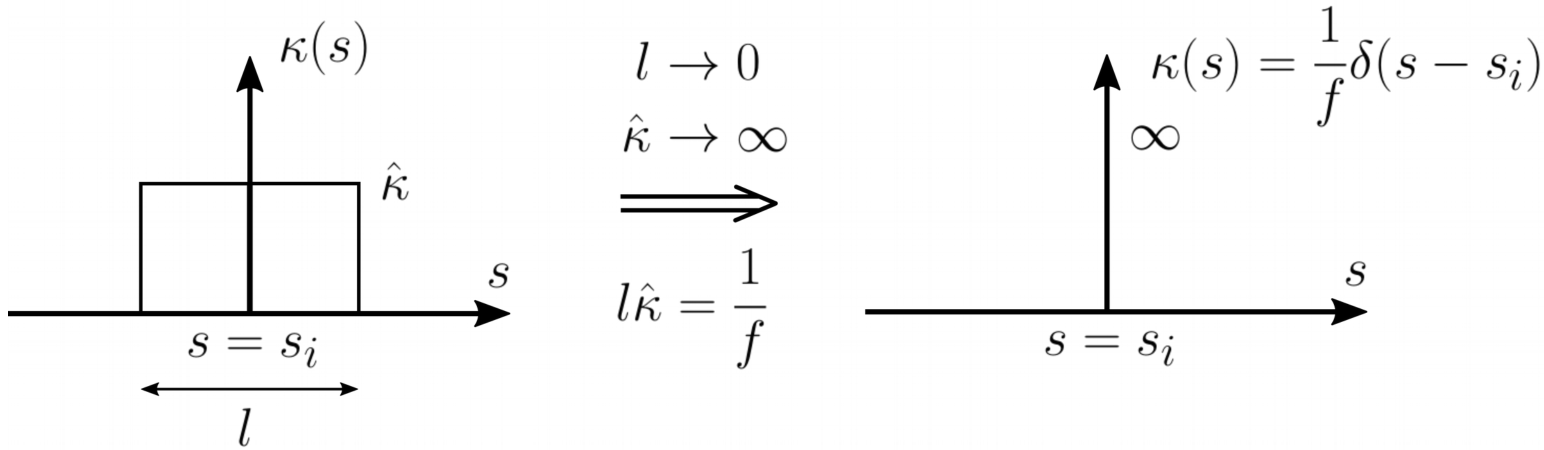
$$y' = \sqrt{\hat{\kappa}} y_i \sinh[\sqrt{\kappa}(s - s_i)] + y_i' \cosh[\sqrt{\hat{\kappa}}(s - s_i)]$$

$$\boxed{\mathbf{M}_y(s|s_i) = \begin{bmatrix} \cosh[\sqrt{\hat{\kappa}}(s - s_i)] & \frac{1}{\sqrt{\hat{\kappa}}} \sinh[\sqrt{\hat{\kappa}}(s - s_i)] \\ \sqrt{\hat{\kappa}} \sinh[\sqrt{\hat{\kappa}}(s - s_i)] & \cosh[\sqrt{\hat{\kappa}}(s - s_i)] \end{bmatrix}}$$

Exchange x and y when sign of focusing function reverses

Thin lens limit: thick quadrupole lens can be replaced by a thin lens kick + drift for equivalent focusing

Replace finite length quadrupoles by a short impulse with same integrated gradient



Results in kick approximation transfer matrices for transport through the element

<u>Focusing Plane</u>

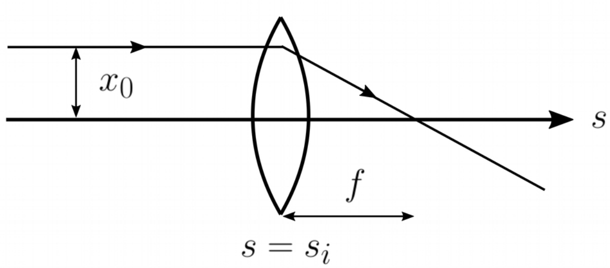
$$\mathbf{M}_x = \begin{bmatrix} \cos[\sqrt{\hat{\kappa}}\ell] & \frac{1}{\sqrt{\hat{\kappa}}}\sin[\sqrt{\hat{\kappa}}\ell] \\ -\sqrt{\hat{\kappa}}\sin[\sqrt{\hat{\kappa}}\ell] & \cos[\sqrt{\hat{\kappa}}\ell] \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$

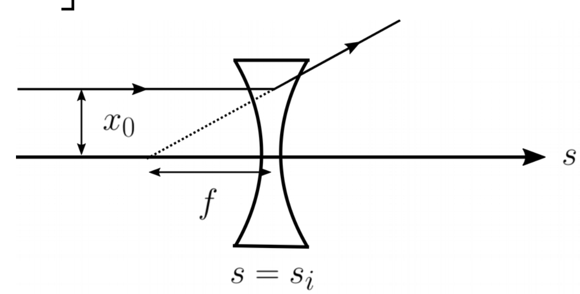
<u>DeFocusing Plane</u>

$$\mathbf{M}_y = \begin{bmatrix} \cosh[\sqrt{\hat{\kappa}}\ell] & \frac{1}{\sqrt{\hat{\kappa}}}\sinh[\sqrt{\hat{\kappa}}\ell] \\ \sqrt{\hat{\kappa}}\sinh[\sqrt{\hat{\kappa}}\ell] & \cosh[\sqrt{\hat{\kappa}}\ell] \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 \\ 1/f & 0 \end{bmatrix}$$

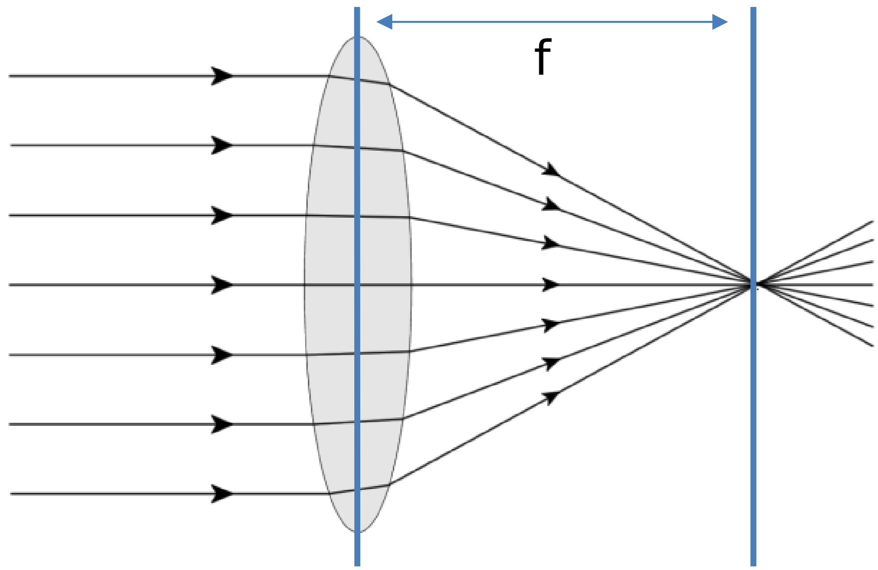


$$\frac{1}{f} = \int \kappa(s)ds$$
$$\propto \int (\text{Gradient } G')ds$$



Reminder: What is a focal point?

M. Couder, Notre Dame, 2015

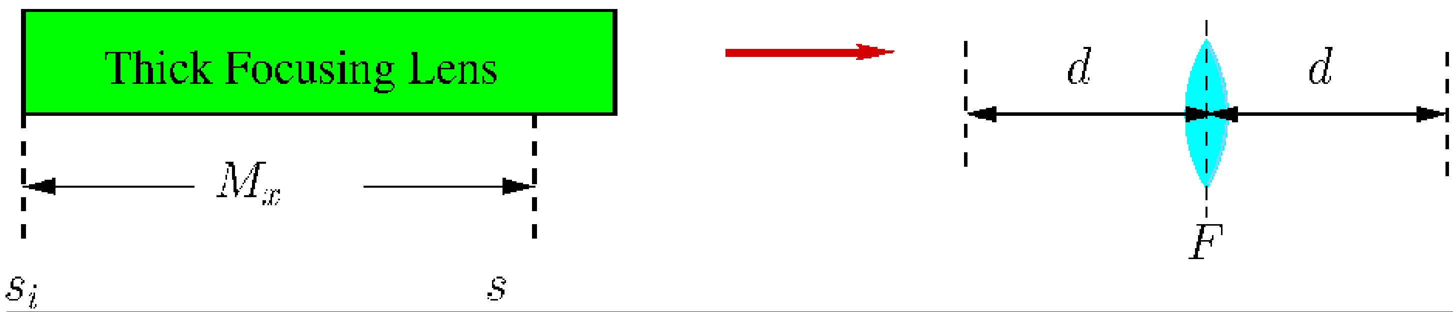


Rays that enter the system parallel to the optical axis are focused such that they pass through the "rear focal" point.

Any ray that passes through it will emerge from the system parallel to the optical axis.

This kick approximation may seem extreme, but works well

- Can show the net focus effect any continuously varying $\kappa(s)$ can be exactly replaced by a kick + drift
- Replacement breaks down in detail of orbit within quadrupole but can work decently there for a high energy particle

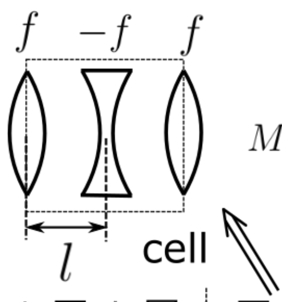


generally,
$2d > s - s_i$

$$
\begin{aligned}
\mathbf{M}_\mathbf{x}(s|s_i) &= \begin{bmatrix} C(s) & S(s)/\sqrt{\kappa} \\ -\sqrt{\kappa}S(s) & C(s) \end{bmatrix} \\
&= \begin{bmatrix} 1 & d(s) \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ -1/f(s) & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & d(s) \\ 0 & 1 \end{bmatrix} \\
&\equiv \mathbf{M}_{\mathrm{drift}} \cdot \mathbf{M}_{\mathrm{kick}} \cdot \mathbf{M}_{\mathrm{drift}}
\end{aligned}
$$

where
$$
\begin{aligned}
C(s) &= \cos[\sqrt{\kappa}(s - s_i)] & d(s) &= \tan[\sqrt{\kappa}(s - s_i)/2]/\sqrt{\kappa} \\
S(s) &= \sin[\sqrt{\kappa}(s - s_i)] & 1/f(s) &= \sqrt{\kappa}S(s)
\end{aligned}
$$

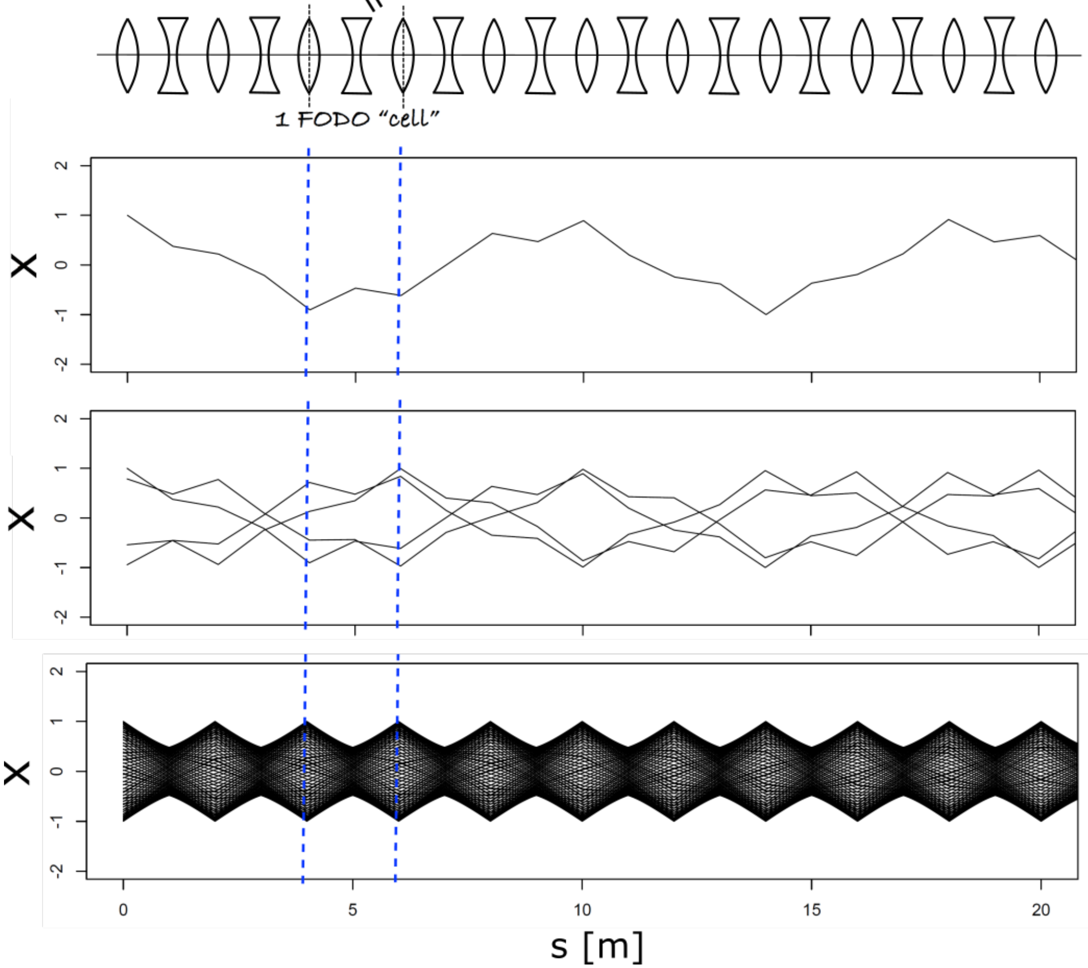
Alternating gradient quadrupole focusing: use sequence of focus and defocus optics in a regular lattice to obtain net focusing in both directions

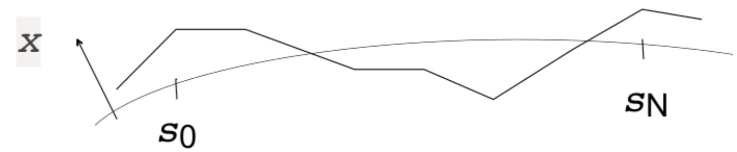


$$M_{cell} = \begin{pmatrix} 1 & 0 \\ \frac{-1}{2f} & 1 \end{pmatrix} \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-1}{2f} & 1 \end{pmatrix}$$

Multiply sequence of transfer matrices to where you are in lattice to obtain evolution

$$\begin{pmatrix} x_N \\ x'_N \end{pmatrix} = M_N M_{N-1} \cdots M_2 M_1 \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$





Plots: Syphers USPAS

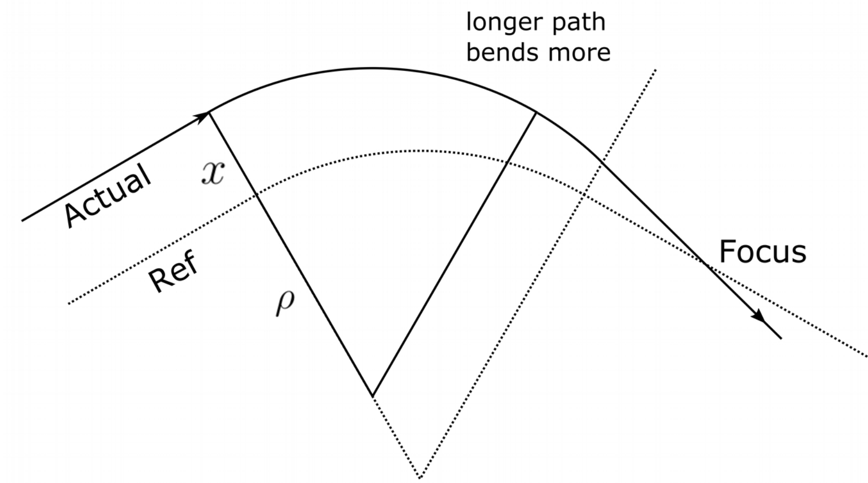
Envelope of many particle orbits repeats regularly: suggestive of a simpler way to analyze of particle via phase-amplitude methods

5. Combined Focusing and Bending

Focused due to combined quadrupole and dipole fields

$$B_y(0) \neq 0 \qquad B' \neq 0$$

More complicated, and no time to go into details, but expect a focusing effect from dipoles if particles enter off reference trajectory since they will bend with same radius about a different center:



When expressed with respect to the reference (design) particle of the lattice, leads to a corrected equation of motion:

Flat System $(\rho \rightarrow \infty)$

$$\boxed{\begin{aligned} \frac{d^2}{ds^2}x + \frac{B'}{[B\rho]}x &= 0 \\ \frac{d^2}{ds^2}y - \frac{B'}{[B\rho]}y &= 0 \end{aligned}}$$

Lattice Function

$$\kappa(s) \equiv \frac{B'(s)}{[B\rho]}$$

Bend + Focusing $(\rho$ finite)

$$\boxed{\begin{aligned} \frac{d^2}{ds^2}x + \left[\frac{1}{\rho^2} + \frac{B'}{[B\rho]}\right]x &= 0 \\ \frac{d^2}{ds^2}y - \frac{B'}{[B\rho]}y &= 0 \end{aligned}}$$

Essentially redefines the lattice function in a bend. Both eqns have Hill's Equation form:

$$\boxed{x''(s) + \kappa_{\text{new}}(s)x(s) = 0}$$

- Previous results and thin-lens limits can be applied

6. Stability of Particle Orbits in a Periodic Focusing Lattice

The transfer matrix must be the same in any period of the lattice:

$$\mathbf{M}(s + L_p | s_i + L_p) = \mathbf{M}(s|s_i)$$

For a propagation distance $s - s_i$ satisfying

$$N L_p \leq s - s_i \leq (N+1) L_p \qquad N = 0, 1, 2, \cdots$$

the transfer matrix can be resolved as

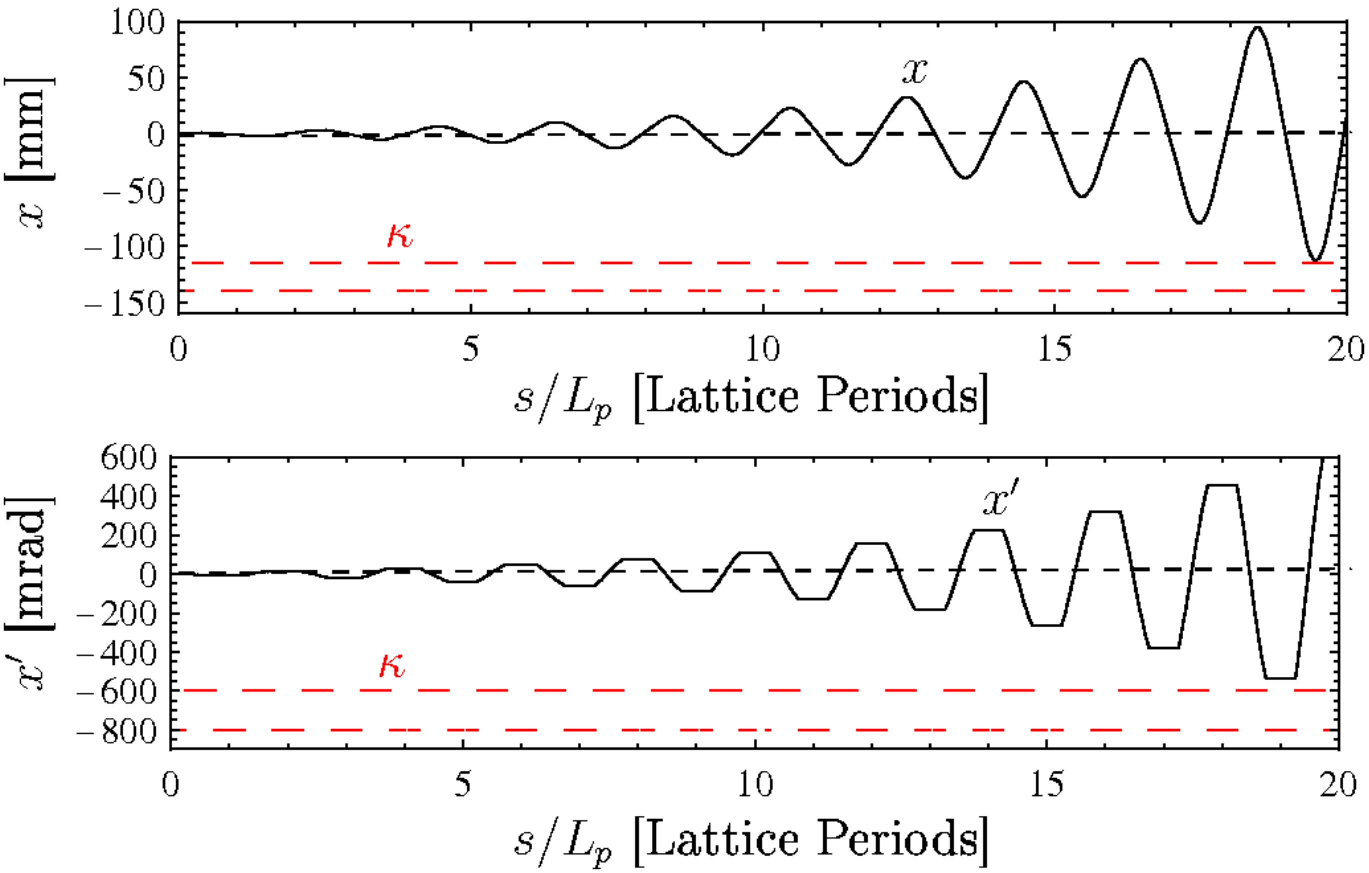
$$\boxed{\begin{aligned} \mathbf{M}(s|s_i) &= \mathbf{M}(s - N L_p | s_i) \cdot \mathbf{M}(s_i + N L_p | s_i) \\ &= \underbrace{\mathbf{M}(s - N L_p | s_i)}_{\text{Residual}} \cdot \underbrace{[\mathbf{M}(s_i + L_p | s_i)]^N}_{N \text{ Full Periods}} \end{aligned}}$$

For a lattice to have stable orbits, both $x(s)$ and $x'(s)$ should remain bounded on propagation through an arbitrary number N of lattice periods. This is equivalent to requiring that the elements of **M** remain bounded on propagation through any number of lattice periods:

$$\mathbf{M}^N \equiv [\mathbf{M}^N{}_{ij}]$$

$$\boxed{\lim_{N \to \infty} \left| \mathbf{M}^N{}_{ij} \right| < \infty \quad \Longrightarrow \text{Stable Motion}}$$

Clarification of stability notion: Unstable Orbit



$$L_p = 0.5 \text{ m}$$

$$\eta = 0.5$$

$$\kappa = \begin{cases} 48/\text{m}^2 & \text{where } \kappa \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$x(0) = 1 \text{ mm}$$

$$x'(0) = 0$$

For energetic particle:

$$H = \frac{1}{2}x'^2 + \frac{1}{2}\kappa x^2 \quad \sim \text{Large, but} \quad \neq \text{const}$$

$$\text{where } |x'| \text{ small, } |x| \text{ large}$$

$$\text{where } |x| \text{ small, } |x'| \text{ large}$$

The matrix criterion corresponds to our intuitive notion of stability: as the particle advances there are no large oscillation excursions in position and angle.

To analyze the stability condition, examine the eigenvectors/eigenvalues of $\mathbf{M}$ for transport through one lattice period:

$$\boxed{\begin{aligned} &\mathbf{M}(s_i + L_p|s_i) \cdot \mathbf{E} \equiv \lambda \mathbf{E} \\ &\mathbf{E} = \text{Eigenvector} \\ &\lambda = \text{Eigenvalue} \end{aligned}}$$

- Eigenvectors and Eigenvalues are generally complex
- Eigenvectors and Eigenvalues generally vary with s_i
- Two independent Eigenvalues and Eigenvectors

Derive the two independent eigenvectors/eigenvalues through analysis of the characteristic equation:

Abbreviate Notation

$$\mathbf{M}(s_i + L_p|s_i) = \begin{bmatrix} C(s_i + L_p|s_i) & S(s_i + L_p|s_i) \\ C'(s_i + L_p|s_i) & S'(s_i + L_p|s_i) \end{bmatrix} \equiv \begin{bmatrix} C & S \\ C' & S' \end{bmatrix}$$

Nontrivial solutions exist when:

$$\det \begin{bmatrix} C - \lambda & S \\ C' & S' - \lambda \end{bmatrix} = \lambda^2 - (C + S')\lambda + (CS' - SC') = 0$$

But we can apply the Wronskian condition:

$$\det \mathbf{M} = CS' - SC' = 1$$

and we make the notational definition

$$C + S' = \mathrm{Tr}\,\mathbf{M} \equiv 2\cos\sigma_0$$

The characteristic equation then reduces to:

$$\boxed{\lambda^2 - 2\lambda\cos\sigma_0 + 1 = 0 \qquad \cos\sigma_0 \equiv \frac{1}{2}\mathrm{Tr}\,\mathbf{M}(s_i + L_p|s_i)}$$

- The use of $2\cos\sigma_0$ to denote Tr **M** is in anticipation of later results where σ_0 is identified as the phase-advance of a stable orbit

There are two solutions to the characteristic equation that we denote $\lambda_\pm$

$$\boxed{\begin{aligned} \lambda_\pm &= \cos\sigma_0 \pm \sqrt{\cos^2\sigma_0 - 1} = \cos\sigma_0 \pm i\sin\sigma_0 = e^{\pm i\sigma_0} \\ \mathbf{E}_\pm &= \text{Corresponding Eigenvectors} \qquad i \equiv \sqrt{-1} \end{aligned}}$$

Note that: $\lambda_+\lambda_- = 1$

$$\lambda_+ = 1/\lambda_-$$

Consider a vector of initial conditions:

$$\begin{bmatrix} x(s_i) \\ x'(s_i) \end{bmatrix} = \begin{bmatrix} x_i \\ x'_i \end{bmatrix}$$

The eigenvectors $\mathbf{E}_\pm$ span two-dimensional space. So any initial condition vector can be expanded as:

$$\boxed{\begin{aligned} \begin{bmatrix} x_i \\ x'_i \end{bmatrix} &= \alpha_+ \mathbf{E}_+ \;+\; \alpha_- \mathbf{E}_- \\ \alpha_\pm &= \text{Complex Constants} \end{aligned}}$$

Then using $\mathbf{M}\mathbf{E}_\pm = \lambda_\pm \mathbf{E}_\pm$

$$\boxed{\mathbf{M}^N(s_i + L_p | s_i) \begin{bmatrix} x_i \\ x'_i \end{bmatrix} = \alpha_+ \lambda_+^N \mathbf{E}_+ \;+\; \alpha_- \lambda_-^N \mathbf{E}_-}$$

Therefore, if $\lim_{N\to\infty} \lambda^N$ is bounded, then the motion is stable. This will always be the case if $|\lambda_\pm| = |e^{\pm i\sigma_0}| \leq 1$, corresponding to σ_0 real with $|\cos\sigma_0| \leq 1$

This implies for stability or the orbit that we must have:

$$\boxed{\begin{aligned}\frac{1}{2}\,|\mathrm{Trace}\,\mathbf{M}(s_i + L_p|s_i)| &= \frac{1}{2}|C(s_i + L_p|s_i) + S'(s_i + L_p|s_i)| \\ &= |\cos\sigma_0| \leq 1\end{aligned}}$$

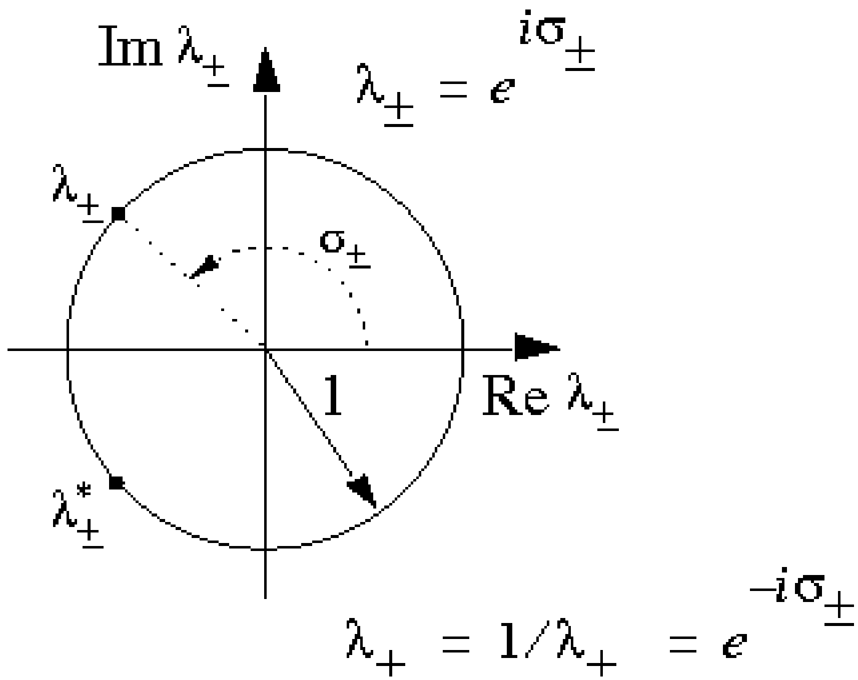
In a periodic focusing lattice, this important stability condition places restrictions on the lattice structure (focusing strength) that are generally interpreted in terms of phase advance limits

- Accelerator lattices almost always tuned for single particle stability to maintain beam control

Extra: Eigenvalue interpretation

See: Dragt, *Lectures on Nonlinear Orbit Dynamics*, AIP Conf Proc 87 (1982) show that symplectic 2x2 transfer matrices associated with Hill's Equation have only two possible classes of eigenvalue symmetries:

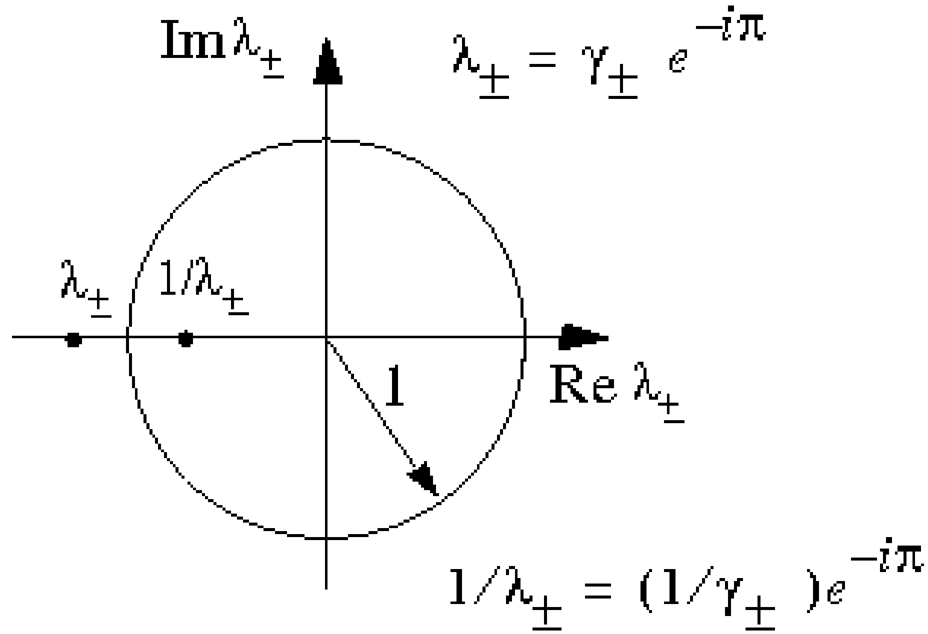
1) Stable



Occurs for:

$$0 \leq \sigma_0 \leq 180^\circ/\text{period}$$

2) Unstable, Lattice Resonance



Occurs in bands when focusing strength is increased beyond

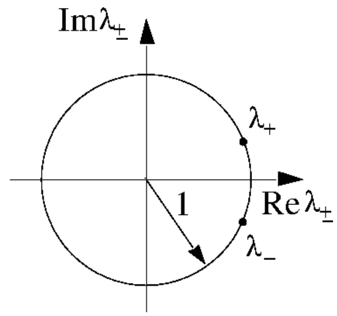
$$\sigma_0 = 180^\circ/\text{period}$$

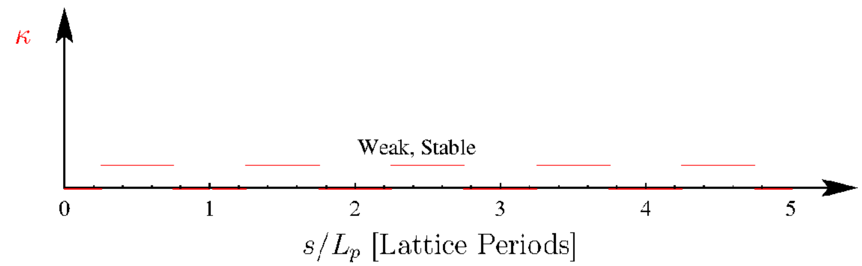
- Limited class of possibilities simplifies analysis of focusing lattices

Eigenvalue structure as focusing strength is increased

Weak Focusing:

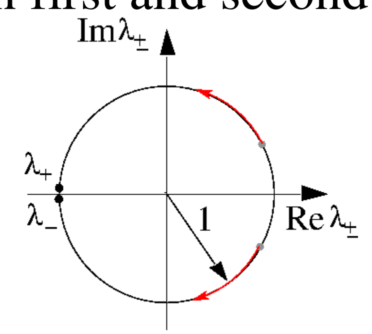
- Make κ as small as needed (low phase advance σ_0)
- Always first eigenvalue case: $|\lambda_\pm| = 1, \quad \lambda_+ = 1/\lambda_- = \lambda_-^*$

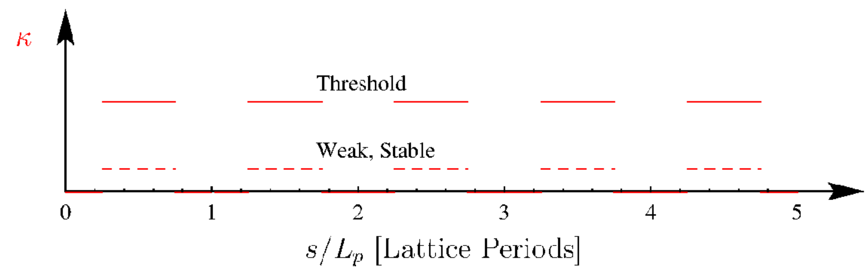




Stability Threshold:

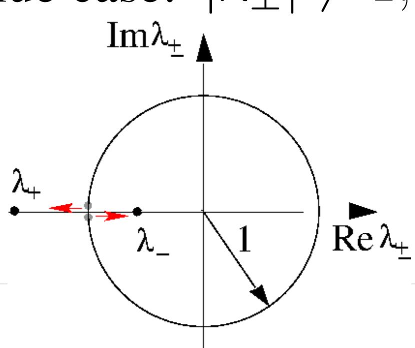
- Increase κ o stability limit (phase advance $\sigma_0 = 180^\circ/\text{Period}$)
- Transition between first and second eigenvalue case: $\lambda_\pm = -1$

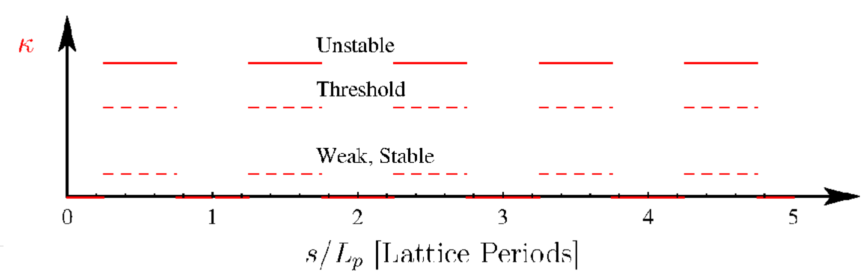




Instability:

- Increase κ beyond threshold (phase advance $\sigma_0 = 180^\circ/\text{Period}$)
- Second eigenvalue case: $|\lambda_\pm| \neq 1, \quad \lambda_+ = 1/\lambda_-$ $\lambda_\pm$ both real and negative





7. Phase-Amplitude Form Particle Orbit

As a consequence of Floquet's Theorem, any (stable or unstable) nondegenerate solution to Hill's Equation can be expressed in phase-amplitude form as:

- Same form for y-equation and κ includes all focusing terms (quad and bend)

$$\boxed{x''(s) + \kappa(s)x(s) = 0}$$

can be expressed in phase-amplitude form as:

$$x(s) = A(s)\cos\psi(s) \qquad A(s) = \text{Real-Valued Amplitude Function}$$
$$A(s + L_p) = A(s) \qquad \psi(s) = \text{Real-Valued Phase Function}$$

Derive equations of motion for A, ψ by taking derivatives of the phase-amplitude form for $x(s)$:

$$x = A\cos\psi$$

$$x' = A'\cos\psi - A\psi'\sin\psi$$

$$x'' = A''\cos\psi - 2A'\psi'\sin\psi - A\psi''\sin\psi - A\psi'^2\cos\psi$$

then substitute in Hill's Equation:

$$\boxed{x'' + \kappa x = \left[A'' + \kappa A - A\psi'^2\right]\cos\psi - \left[2A'\psi' + A\psi''\right]\sin\psi = 0}$$

$$\boxed{x'' + \kappa x = \left[A'' + \kappa A - A\psi'^2\right]\cos\psi - [2A'\psi' + A\psi'']\sin\psi = 0}$$

We are free to introduce an additional constraint between A and ψ:

- Two functions A, ψ to represent one function x allows a constraint

Choose:

Eq. (1) $$\boxed{2A'\psi' + A\psi'' = 0} \quad \Longrightarrow \quad \text{Coefficient of } \sin\psi \text{ zero}$$

Then to satisfy Hill's Equation for all ψ, the coefficient of $\cos\psi$ must also vanish giving:

Eq. (2) $$\boxed{A'' + \kappa A - A\psi'^2 = 0} \quad \Longrightarrow \quad \text{Coefficient of } \cos\psi \text{ zero}$$

Eq. (1) Analysis (coefficient of $\sin\psi$): $2A'\psi' + A\psi'' = 0$

Simplify:

$$2A'\psi' + A\psi'' = \frac{(A^2\psi')'}{A} = 0$$

$$\Longrightarrow \quad (A^2\psi')' = 0$$

Assume for moment:

$A \neq 0$

Will show later that this assumption met for all *s*

Integrate once:

$$\boxed{A^2\psi' = \text{const}}$$

One commonly rescales the amplitude $A(s)$ in terms of an auxiliary amplitude function $w(s)$:

$$\boxed{A(s) = A_i w(s) \qquad A_i = \text{const} = \text{Initial Amplitude}}$$

such that

$$\boxed{w^2\psi' \equiv 1}$$

This equation can then be integrated to obtain the phase-function of the particle:

$$\boxed{\psi(s) = \psi_i + \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})} \qquad \psi_i = \text{const} = \text{Initial Phase}}$$

Eq. (2) Analysis (coefficient of $\cos\psi$): $A'' + \kappa A - A\psi'^2 = 0$

With the choice of amplitude rescaling, $A = A_i w$ and $w^2\psi' = 1$, Eq. (2) becomes:

$$\boxed{w'' + \kappa w - \frac{1}{w^3} = 0}$$

Floquet's theorem tells us that we are free to restrict *w* to be a periodic solution:

$$\boxed{w(s + L_p) = w(s)}$$

Reduced Expressions for *x* and *x*':

Using $A = A_i w$ and $w^2\psi' = 1$:

$$x = A\cos\psi$$

$$x' = A'\cos\psi - A\psi'\sin\psi$$

$$\Longrightarrow \boxed{\begin{aligned} x &= A_i w \cos\psi \\ x' &= A_i w' \cos\psi - \frac{A_i}{w}\sin\psi \end{aligned}}$$

Summary: Phase-Amplitude Form of Solution to Hill's Eqn

$$x(s) = A_i w(s) \cos\psi(s)$$

$$x'(s) = A_i w'(s) \cos\psi(s) - \frac{A_i}{w(s)} \sin\psi(s)$$

$A_i = \text{const} = \text{Initial Amplitude}$

$\psi_i = \text{const} = \text{Initial Phase}$

where $w(s)$ and $\psi(s)$ are amplitude- and phase-functions satisfying:

Amplitude Equations

$$w''(s) + \kappa(s)w(s) - \frac{1}{w^3(s)} = 0$$

$$w(s + L_p) = w(s)$$

$$w(s) > 0$$

Phase Equations

$$\psi'(s) = \frac{1}{w^2(s)}$$

$$\psi(s) = \psi_i + \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})}$$

$$\psi(s) = \psi_i + \Delta\psi(s)$$

Initial ($s = s_i$) amplitudes are constrained by the particle initial conditions as:

$$x(s = s_i) = A_i w_i \cos\psi_i$$

$$x'(s = s_i) = A_i w_i' \cos\psi_i - \frac{A_i}{w_i} \sin\psi_i$$

or

$$A_i \cos\psi_i = x(s = s_i)/w_i$$

$$A_i \sin\psi_i = x(s = s_i)w_i' - x'(s = s_i)w_i$$

$$w_i \equiv w(s = s_i)$$

$$w_i' \equiv w'(s = s_i)$$

Undepressed Particle Phase Advance

Some analysis shows that the quantity σ_0 occurring in the stability criterion for a periodic focusing lattice

$$\boxed{\cos\sigma_0 \equiv \frac{1}{2}\text{Tr}\,\mathbf{M}(s_i + L_p|s_i)}$$

$\mathbf{M}(s_i + L_p|s_i) =$ Tranfer Matrix (through one period)

is related to the phase advance of particle oscillations in one period of the lattice:

$$\boxed{\sigma_0 = \Delta\psi(s_i + L_p) = \int_{s_i}^{s_i+L_p} \frac{ds}{w^2(s)}}$$

Consequence:

Any periodic lattice with undepressed phase advance satisfying

$$\sigma_0 < \pi/\text{period} = 180^\circ/\text{period}$$

will have stable single particle orbits.

- The phase advance σ_0 is useful to better understand the bundle or particle oscillations in the focusing lattice

Illustration: Particle orbits in a periodic FODO quadrupole lattice

Parameters:

L_p = Lattice Period

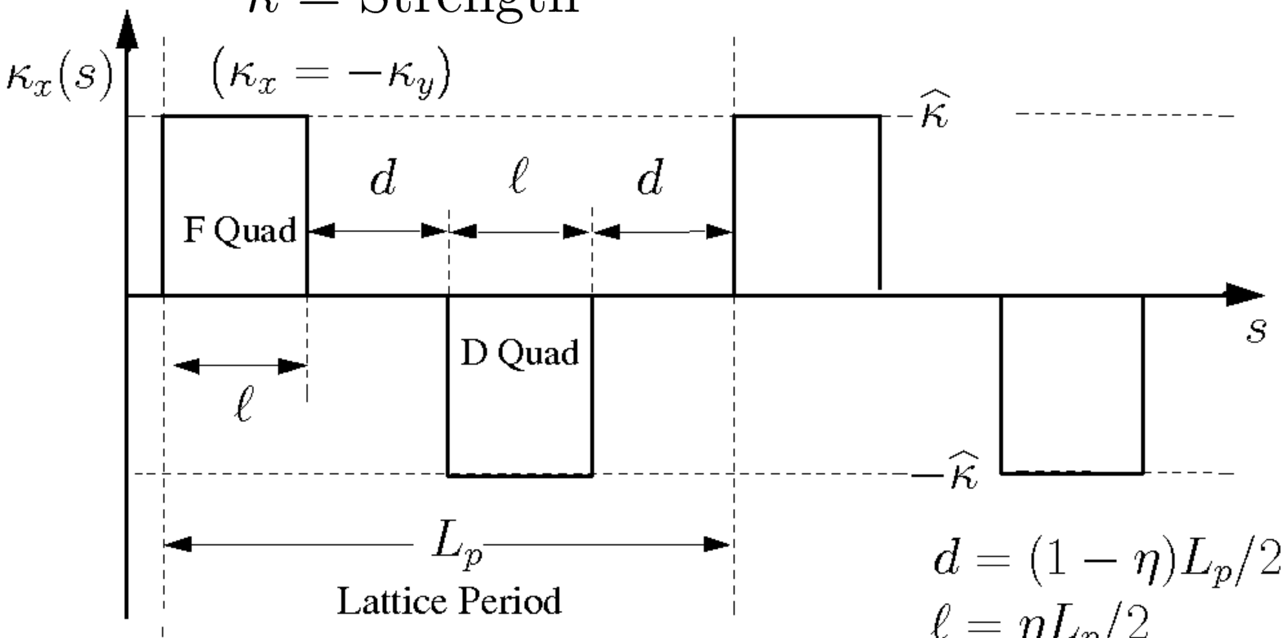
$\eta \in (0,1]$ = Occupancy

$\hat{\kappa}$ = Strength

Characteristics:

$\eta L_p/2 = \ell$ = F/D Len

$(1-\eta)L_p/2 = d$ = Drift Len



Phase advance formula derived to set lattice focus strength $\hat{\kappa}$ for target value of σ_0

$$\cos\sigma_0 = \cos\Theta\cosh\Theta + \frac{1-\eta}{\eta}\Theta(\cos\Theta\sinh\Theta - \sin\Theta\cosh\Theta) - \frac{(1-\eta)^2}{2\eta^2}\Theta^2\sin\Theta\sinh\Theta$$

$$\Theta \equiv \frac{\eta}{2}\sqrt{|\hat{\kappa}|}L_p$$

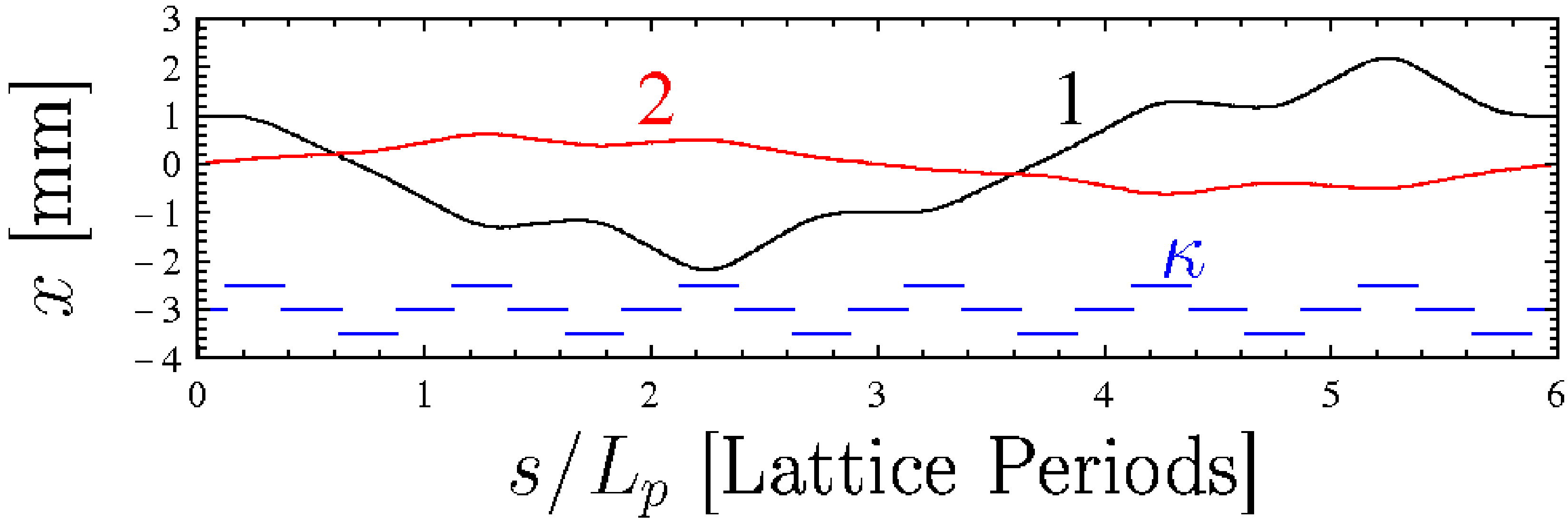
Rescaled Principal Orbit Evolution FODO Quadrupole:

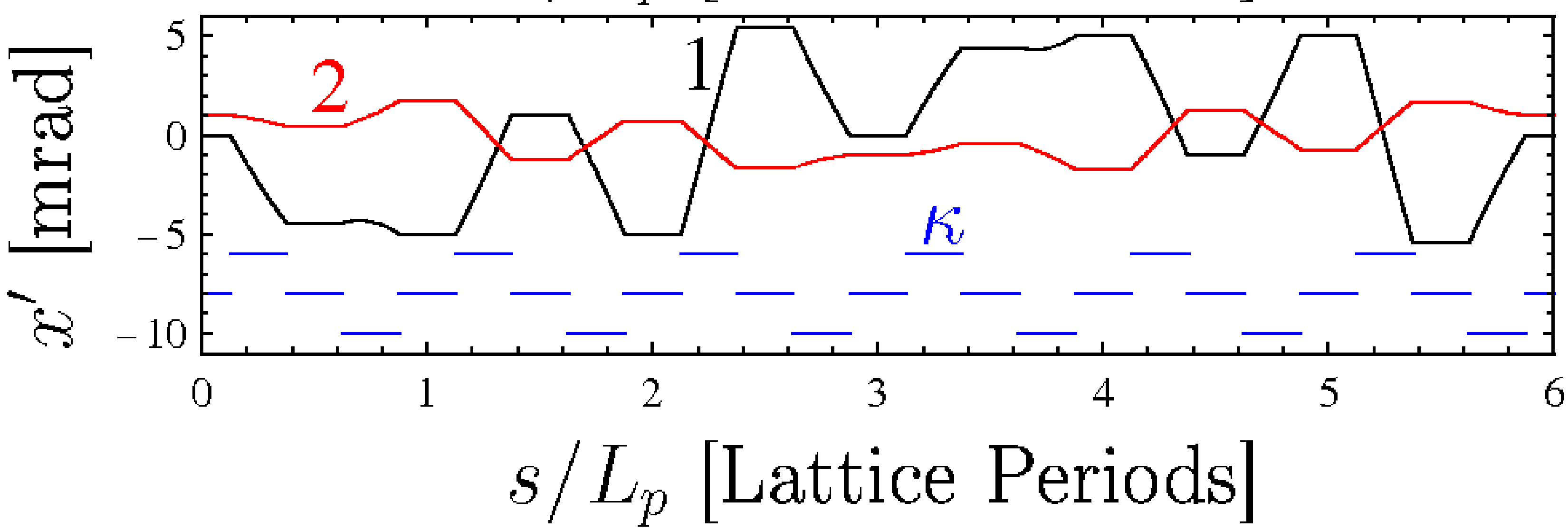
$L_p = 0.5$ m

$\sigma_0 = \pi/3 = 60^\circ$ ($\kappa = 39.24$ m^{-2})

$\eta = 0.5$

	Cosine-Like	Sine-Like
	1: $x(0) = 1$ mm	2: $x(0) = 0$ mm
	$x'(0) = 0$ mrad	$x'(0) = 1$ mrad





8. Beam Phase-Space Area / Emittance

Question:

For Hill's equation:

$$x'' + \kappa(s)x = 0$$

does a quadratic invariant exist that can aid interpretation of the dynamics?

Answer we will find:

Yes, the Courant-Snyder invariant

Comments:

- Very important in accelerator physics
  - Helps interpretation of linear dynamics
- Named in honor of Courant and Snyder who popularized it's use in Accelerator physics while co-discovering alternating gradient (AG) focusing in a single seminal (and very elegant) paper:

  Courant and Snyder, *Theory of the Alternating Gradient Synchrotron*, Annals of Physics **3**, 1 (1958).
- Easily derived using phase-amplitude form of orbit solution
  - Much harder using other methods

Derivation of Courant-Snyder Invariant

The phase amplitude form of the particle orbit makes identification of the invariant elementary:

$$\boxed{\begin{aligned} x(s) &= A_i w(s) \cos\psi(s) \\ x'(s) &= A_i w'(s)\cos\psi(s) - \frac{A_i}{w(s)}\sin\psi(s) \end{aligned}}$$

$A_i, \ \psi_i = \psi(s_i)$ set by initial at $s = s_i$

where

$$w'' + \kappa(s) w - \frac{1}{w^3} = 0$$

Re-arrange the phase-amplitude trajectory equations:

$$\begin{aligned} \frac{x}{w} &= A_i \cos\psi \\ wx' - w'x &= A_i \sin\psi \end{aligned}$$

square and add the equations to obtain the Courant-Snyder invariant:

$$\boxed{\begin{aligned} \left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 &= A_i^2(\cos^2\psi + \sin^2\psi) \\ &= A_i^2 = \text{const} \end{aligned}}$$

Comments on the Courant-Snyder Invariant:

- Simplifies interpretation of dynamics
- Extensively used in accelerator physics
- Quadratic structure in *x*-*x*' defines a rotated ellipse in *x*-*x*' phase space.
- *Cannot* be interpreted as a conserved energy!

// Extra Clarification:

The point that the Courant-Snyder invariant is *not* a conserved energy should be elaborated on. The equation of motion:

$$x'' + \kappa(s)x = 0$$

Is derivable from the Hamiltonian

$$H = \frac{1}{2}x'^2 + \frac{1}{2}\kappa x^2 \quad \Longrightarrow \quad \begin{aligned} \frac{d}{ds}x &= \frac{\partial H}{\partial x'} = x' \\ \frac{d}{ds}x' &= -\frac{\partial H}{\partial x} = -\kappa x \end{aligned} \quad \Longrightarrow \quad x'' + \kappa x = 0$$

H is the energy:

$$H = \frac{1}{2}x'^2 + \frac{1}{2}\kappa x^2 = T + V$$

$$T = \frac{1}{2}x'^2 \quad = \text{Kinetic ''Energy''}$$

$$V = \frac{1}{2}\kappa x^2 \quad = \text{Potential ''Energy''}$$

Apply the chain-Rule with $H = H(x,x';s)$:

$$\frac{dH}{ds} = \frac{\partial H}{\partial s} + \frac{\partial H}{\partial x}\frac{dx}{ds} + \frac{\partial H}{\partial x'}\frac{dx'}{ds}$$

Apply the equation of motion in Hamiltonian form:

$$\frac{d}{ds}x = \frac{\partial H}{\partial x'} \qquad \frac{d}{ds}x' = -\frac{\partial H}{\partial x}$$

0

$$\frac{dH}{ds} = \frac{\partial H}{\partial s} - \frac{dx'}{ds}\frac{dx}{ds} + \frac{dx}{ds}\frac{dx'}{ds} = \frac{\partial H}{\partial s} = \frac{1}{2}\kappa' x^2 \neq 0$$

$$\Longrightarrow \boxed{H \neq \text{const}}$$

- Energy of a "kicked" oscillator with $\kappa(s) \neq \text{const}$ is not conserved
- Energy should not be confused with the Courant-Snyder invariant

End Clarification //

Interpret the Courant-Snyder invariant:

$$\boxed{\left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 = A_i^2 = \text{const}}$$

by expanding and isolating terms quadratic terms in *x*-*x*' phase-space variables:

$$\left[\frac{1}{w^2} + w'^2\right] x^2 + 2[-ww']xx' + [w^2]x'^2 = A_i^2 = \text{const}$$

The three coefficients in [...] are functions of *w* and *w*' only and therefore are *functions of the lattice only* (not particle initial conditions). They are commonly called "Twiss Parameters" and are expressed denoted as:

$$\boxed{\begin{aligned} &\gamma x^2 + 2\alpha x x' + \beta x'^2 = A_i^2 = \text{const} \\ &\qquad \gamma(s) \equiv \frac{1}{w^2(s)} + [w'(s)]^2 = \frac{1+\alpha^2(s)}{\beta(s)} \\ &\qquad \beta(s) \equiv w^2(s) \qquad \text{[Betatron Function]} \\ &\qquad \alpha(s) \equiv -w(s)w'(s) \end{aligned}}$$

$$\gamma\beta = 1 + \alpha^2$$

- All Twiss "parameters" are specified by $w(s)$
- Given w and w' at a point (s) any 2 Twiss parameters give the 3rd

The area of the invariant ellipse is:

- Analytic geometry formulas: $\gamma x^2 + 2\alpha x x' + \beta x'^2 = \pi A_i^2 \rightarrow Area = A_i^2/\sqrt{\gamma\beta - \alpha^2}$
- For Courant-Snyder ellipse: $\gamma\beta = 1 + \alpha^2$

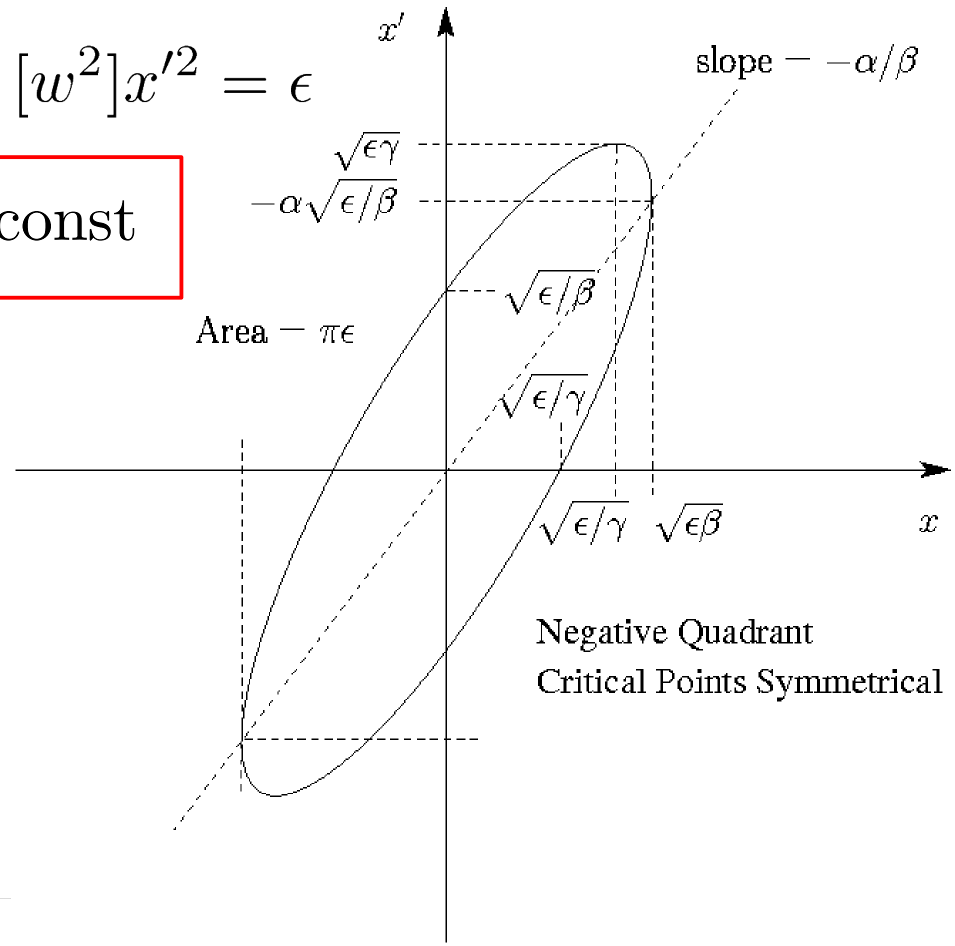
$$\boxed{\text{Phase-Space Area} = \int_{\text{ellipse}} dx dx' \; = \frac{\pi A_i^2}{\sqrt{\gamma\beta - \alpha^2}} = \pi A_i^2 \equiv \pi\epsilon}$$

Where ϵ is the single-particle emittance:

- Emittance is the area of the orbit in *x*-*x*' phase-space divided by π

$$[1/w^2 + w'^2]x^2 + 2[-ww']xx' + [w^2]x'^2 = \epsilon$$

$$\boxed{\gamma x^2 + 2\alpha x x' + \beta x'^2 = \epsilon = \text{const}}$$



Properties of Courant-Snyder Invariant:

- The ellipse will rotate and change shape as the particle advances through the focusing lattice, but the instantaneous area of the ellipse ($\pi\epsilon = \text{const}$) remains constant.
- The location of the particle on the ellipse and the size (area) of the ellipse depends on the initial conditions of the particle.
- The orientation of the ellipse is independent of the particle initial conditions. All particles move on nested ellipses.
- Quadratic in the *x*-*x*' phase-space coordinates, but is *not* the transverse particle energy (which is not conserved)
- Beam edge (envelope) extent is given by that max emittance and betatron func by:

$$\boxed{x_{\text{env}} = \sqrt{\epsilon_x \beta_x} = \sqrt{\epsilon_x} w_x}$$

$$\beta \equiv w^2$$
$$= \text{Betatron Function}$$

Thin Lens FODO Quadrupole Lattice

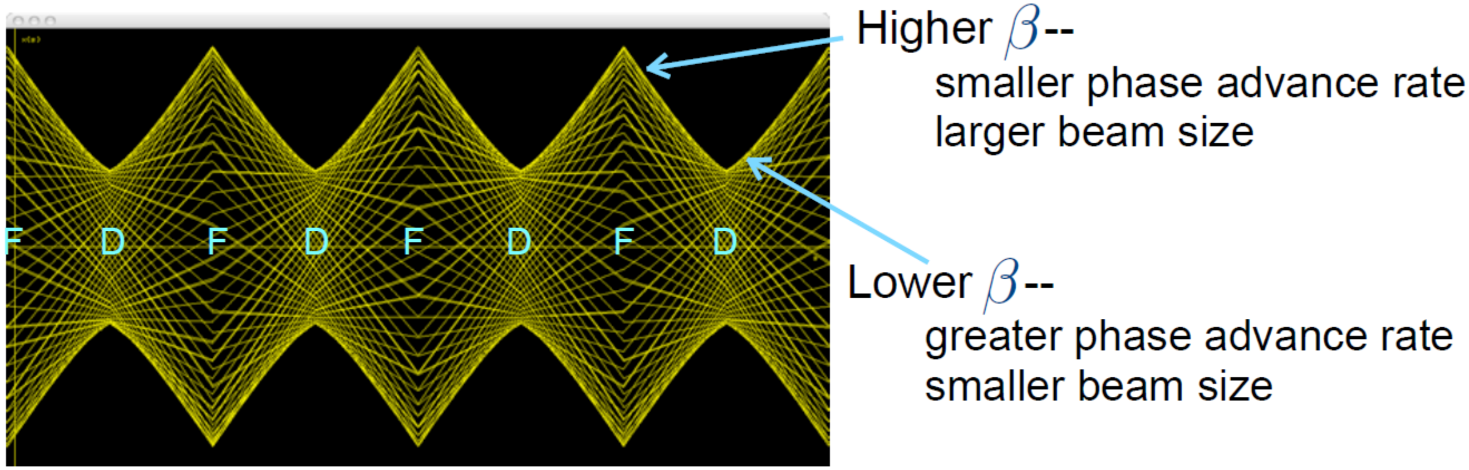


Fig: Syphers USPAS

Emittance is sometimes defined by the largest Courant-Snyder ellipse that will contain a specified fraction of the distribution of beam particles. Common choices are:

- 100%
- 95%
- 90%
- ....
- Depends emphasis

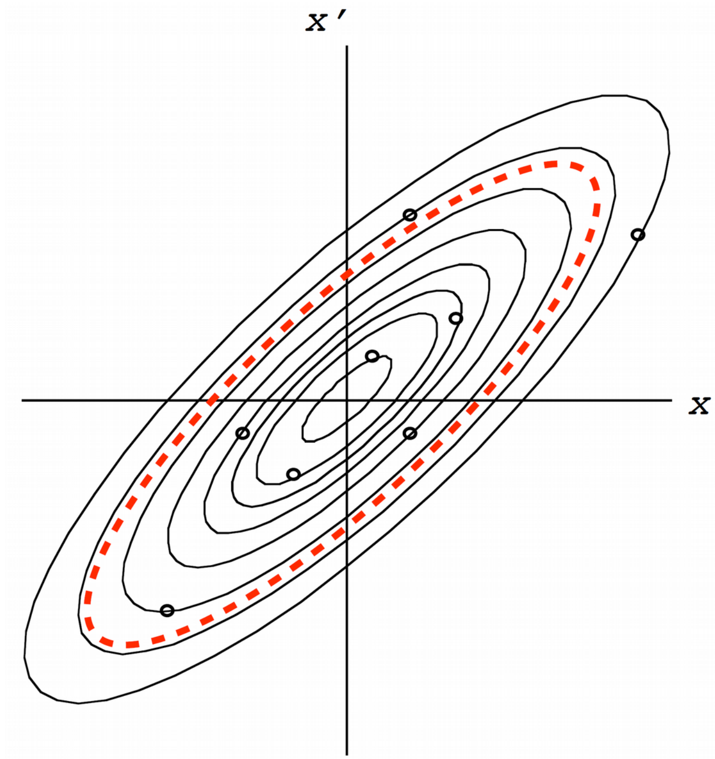


Fig: Syphers, USPAS

One can motivate that the "rms" statistical measure

$$\epsilon_{\rm rms} = \left[\langle \langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2 \right]^{1/2} \qquad \langle \cdots \rangle = \text{Distribution Average}$$
$$= \text{rms Statistical Emittance}$$

provides a distribution weighted average measure of the beam phase-space area. This is commonly used to measure emittance of laboratory beams.

- Can show $4\epsilon_{\rm rms}$ corresponds to the edge particle emittance of a uniformly filled ellipse

/// Aside on Notation: Twiss Parameters and Emittance Units:

Twiss Parameters:

Use of $\alpha, \ \beta, \ \gamma$ should not create confusion with kinematic relativistic factors

- $\beta_b, \ \gamma_b$ are absorbed in the focusing function
- Contextual use of notation unfortunate reality .... not enough symbols!
- Notation originally due to Courant and Snyder, not Twiss, and might be more appropriately called "Courant-Snyder functions" or "lattice functions."

Emittance Units:

x has dimensions of length and x' is a dimensionless angle. So x-x' phase-space area has dimensions [[ϵ]] = length. A common choice of units is millimeters (mm) and milliradians (mrad), e.g.,

$$\epsilon = 10 \text{ mm-mrad}$$

The definition of the emittance employed is not unique and different workers use a wide variety of symbols. Some common notational choices:

$$\pi\epsilon \longrightarrow \epsilon \qquad \epsilon \longrightarrow \varepsilon \qquad \epsilon \longrightarrow E$$

Write the emittance values in units with a π, e.g.,

$$\epsilon = 10.5\,\pi - \text{mm-mrad}$$ (seems falling out of favor but still common)

Use caution! Understand conventions being used before applying results! ///

Illustration: Revisit particle orbits in a periodic FODO quadrupole lattice with aid of Courant Snyder invariant

Reminder: Periodic focusing lattice in x-plane

<u>Parameters:</u>

L_p = Lattice Period

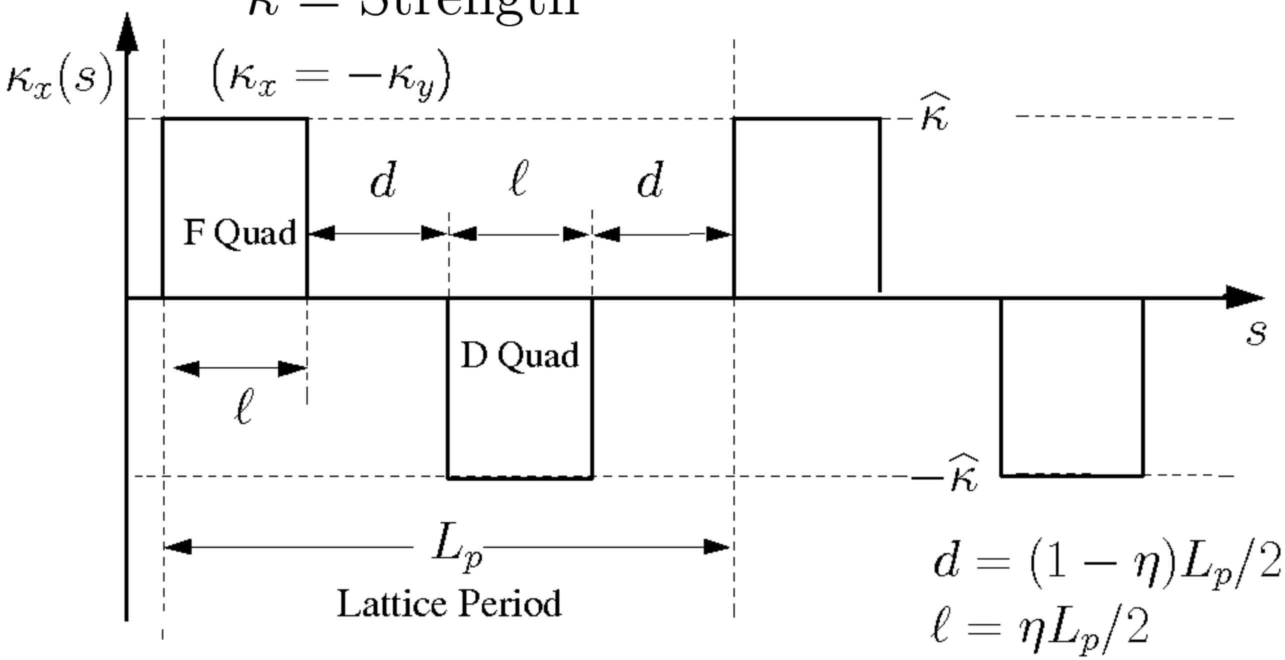
$\eta \in (0, 1]$ = Occupancy

$\hat{\kappa}$ = Strength

<u>Characteristics:</u>

$\eta L_p/2 = \ell$ = F/D Len

$(1-\eta)L_p/2 = d$ = Drift Len



Rescaled Principal Orbit Evolution FODO Quadrupole:

$L_p = 0.5$ m

$\sigma_0 = \pi/3 = 60^\circ$ ($\kappa = 39.24$ m^{-2})

$\eta = 0.5$

Cosine-Like

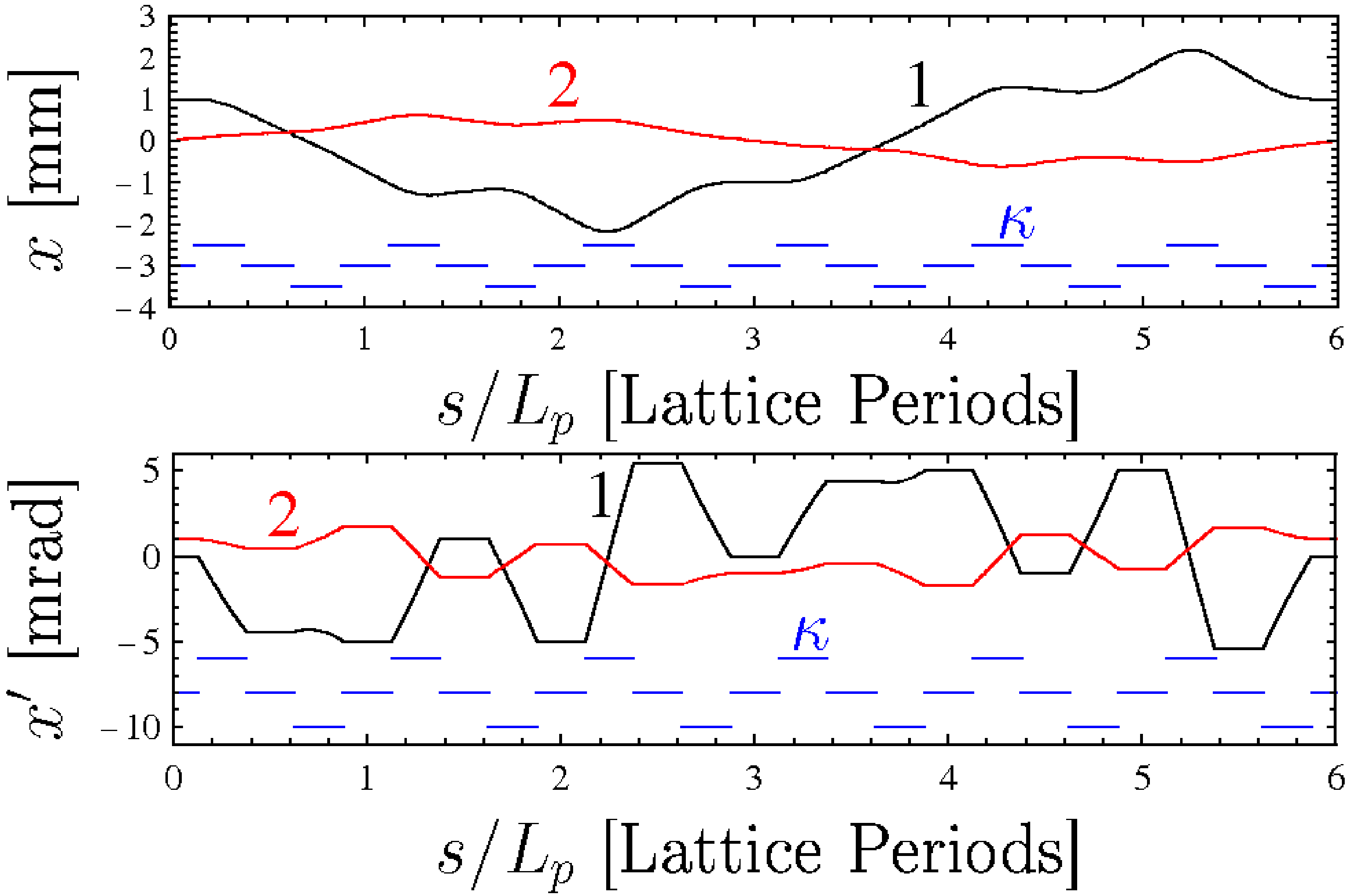
1: $x(0) = 1$ mm

$x'(0) = 0$ mrad

Sine-Like

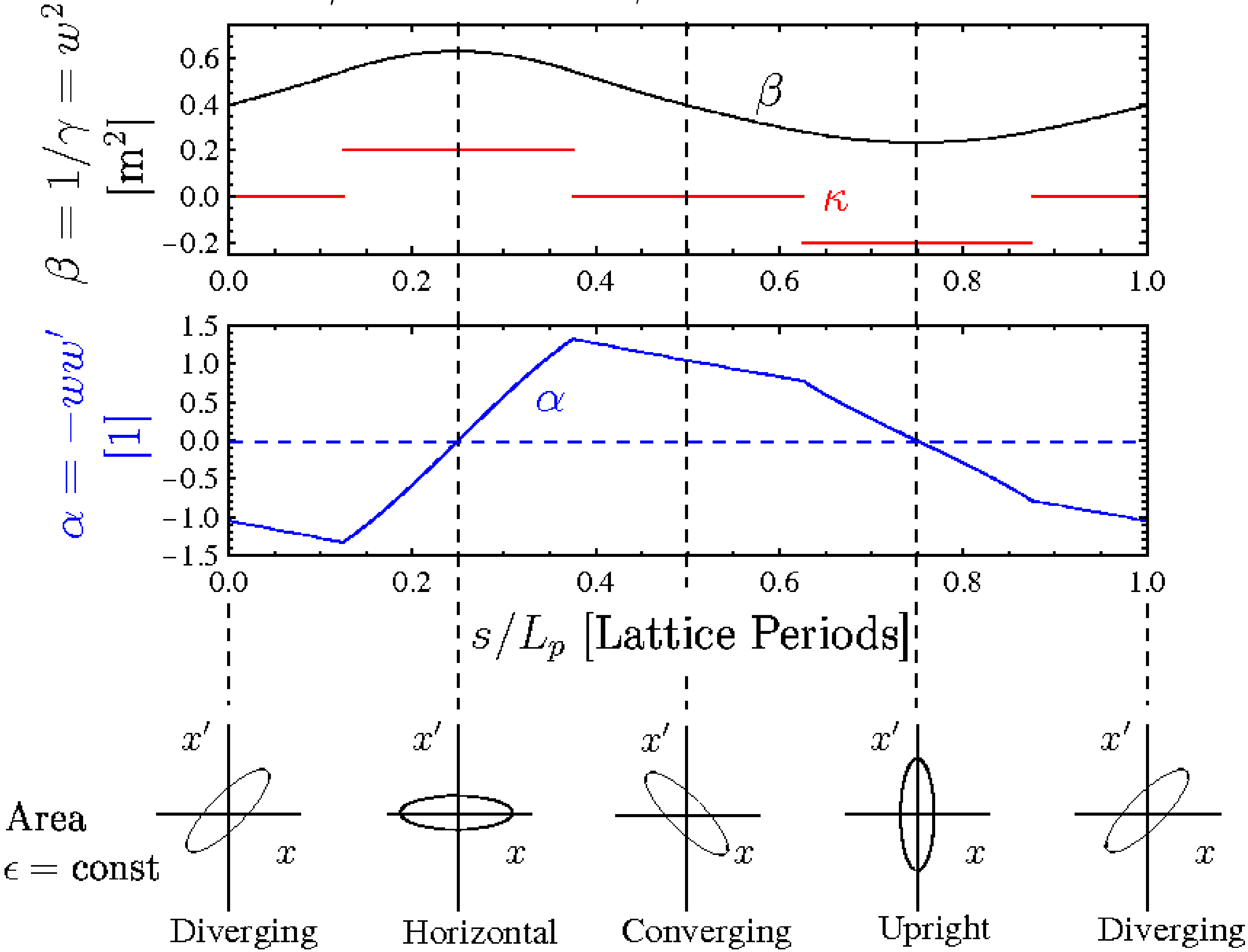
2: $x(0) = 0$ mm

$x'(0) = 1$ mrad



Phase-Space Evolution in terms of Courant-Snyder invariant:

$$\gamma x^2 - 2\alpha x x' + \beta x'^2 = \epsilon = \text{const}$$



9. Effects of Momentum Spread

Until this point we have assumed that all particles have the design longitudinal momentum in the lattice:

$$p_s = m\gamma\beta c = \text{same for every particle}$$

If there is a spread of particle momentum take:

$$p_s = p_0 + \delta p$$

$$p_0 \equiv m\gamma\beta c = \text{Design Momentum}$$

$$\delta p \equiv \text{Off Momentum}$$

Analysis shows:

$$x''(s) + \left[\frac{1}{\rho^2(s)}\frac{1-\delta}{1+\delta} + \frac{\kappa(s)}{1+\delta}\right]x(s) = \frac{\delta}{1+\delta}\frac{1}{\rho(s)}$$

$$y''(s) - \frac{\kappa(s)}{1+\delta}y(s) = 0 \qquad \delta \equiv \frac{\delta p}{p_0}$$

Here: $\rho(s) = \text{Local Bend Radius}$ $\quad \kappa = \text{Quadrupole Focus Function}$

Both defined for design momentum p_0

Terms in the equations of motion associated with momentum spread (δ) can be lumped into two classes:

Dispersive -- Associated with Dipole Bends (ρ)

Chromatic -- Associated with Focusing (κ)

Dispersive terms typically more important and only in the *x*-equation of motion and result from bending. Neglecting chromatic terms and expanding (small $\delta \ll 1$)

$$\boxed{\begin{aligned} x''(s) + \left[\frac{1}{\rho^2(s)} + \kappa(s)x(s)\right] &= \frac{\delta}{\rho(s)} \\ y''(s) - \kappa(s)y(s) &= 0 \end{aligned}}$$

The y-equation is not changed from the usual Hill's Equation

The x-equation

$$x''(s) + \kappa_x(s)x(s) = \frac{\delta}{\rho(s)} \qquad \kappa_x(s) \equiv \frac{1}{\rho^2(s)} + \kappa(s)$$

is typically solved by linearly resolving:

$$\boxed{\begin{aligned} &x(s) = x_h(s) + x_p(s) \\ &x_h \equiv \text{Homogeneous Solution} \\ &x_p \equiv \text{Particular Solution} \end{aligned}}$$

where x_h is the *general* solution to the Hill's Equation:

$$\boxed{x_h''(s) + \kappa_x(s)x_h(s) = 0}$$

and x_p is a solution to the rescaled equation:

$$\boxed{\begin{aligned} &x_p = \delta \cdot D \qquad D''(s) + \kappa_x(s)D(s) = \frac{1}{\rho(s)} \\ &D \equiv \text{Dispersion Function} \end{aligned}}$$

For Ring: D periodic with lattice

$$D(s + L_p) = D(s)$$

Transfer line: D evolved from initial condition

$$D(s_i) = 0 = D'(s_i) \qquad \text{(usually)}$$

This convenient resolution of the orbit $x(s)$ can *always* be made because the homogeneous solution will be adjusted to match any initial condition

Note that x_p provides a measure of the offset of the particle orbit relative to the design orbit resulting from a small deviation of momentum (δ)

$x(s) = 0$ defines the design (centerline) orbit

$[[D]] = \text{meters}$

$\delta \cdot D = \text{Dispersion induced orbit offset in meters}$

The beam edge will have two contributions:

Extent/Emittance (betatron): $x_{\text{edge}} = \sqrt{\epsilon_x \beta_x}$ $\quad y_{\text{edge}} = \sqrt{\epsilon_y \beta_y}$

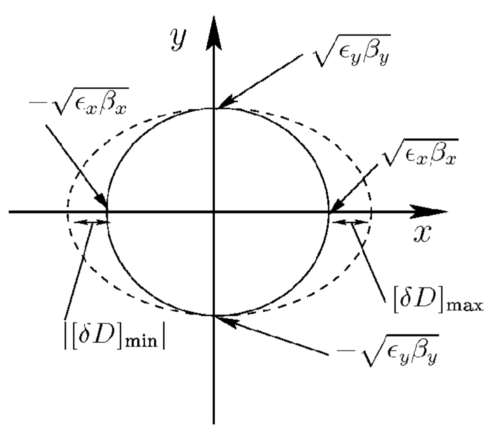
Shift/Dispersive (dispersion D) $x_{\text{shift}} = \delta D$ $\quad y_{\text{shift}} = 0$

Gives two distinct situations:

Dispersion Broaden: distribution of δ

$$x_{\text{edge}} = -\sqrt{\epsilon_x \beta_x} + [\delta D]_{\text{min}}, \quad \sqrt{\epsilon_x \beta_x} + [\delta D]_{\text{max}}$$

$$y_{\text{edge}} = \pm\sqrt{\epsilon_y \beta_y}$$



Dispersion Shift: all particles same δ

$$x_{\text{edge}} = \pm\sqrt{\epsilon_x \beta_x} + \delta D$$

$$y_{\text{edge}} = \pm\sqrt{\epsilon_y \beta_y}$$

The beam edge will have two contributions:

Extent/Emittance (betatron): $x_{\text{edge}} = \sqrt{\epsilon_x \beta_x}$ $\quad y_{\text{edge}} = \sqrt{\epsilon_y \beta_y}$

Shift/Dispersive (dispersion D) $x_{\text{shift}} = \delta D$ $\quad y_{\text{shift}} = 0$

Gives two distinct situations:

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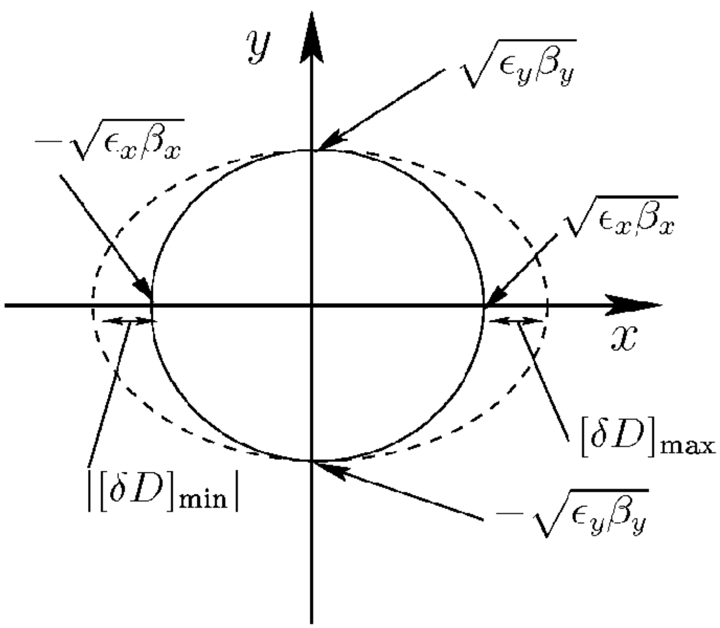
$$x_{\text{edge}} = -\sqrt{\epsilon_x \beta_x} + [\delta D]_{\text{min}}, \quad \sqrt{\epsilon_x \beta_x} + [\delta D]_{\text{max}}$$

$$y_{\text{edge}} = \pm\sqrt{\epsilon_y \beta_y}$$

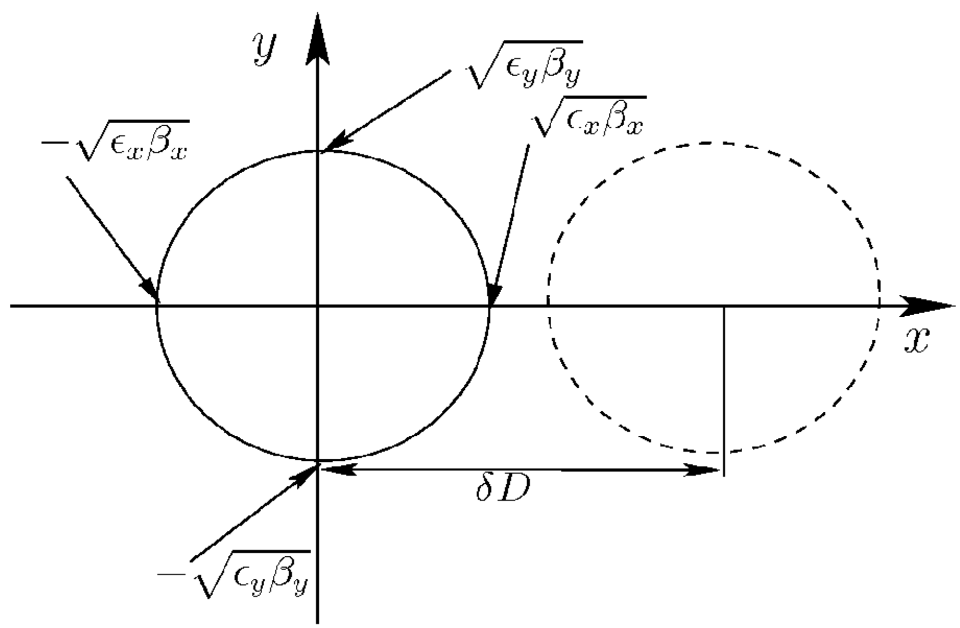


- Usual case for dist of small δ

Dispersion Shift: all particles same δ

$$x_{\text{edge}} = \pm\sqrt{\epsilon_x \beta_x} + \delta D$$

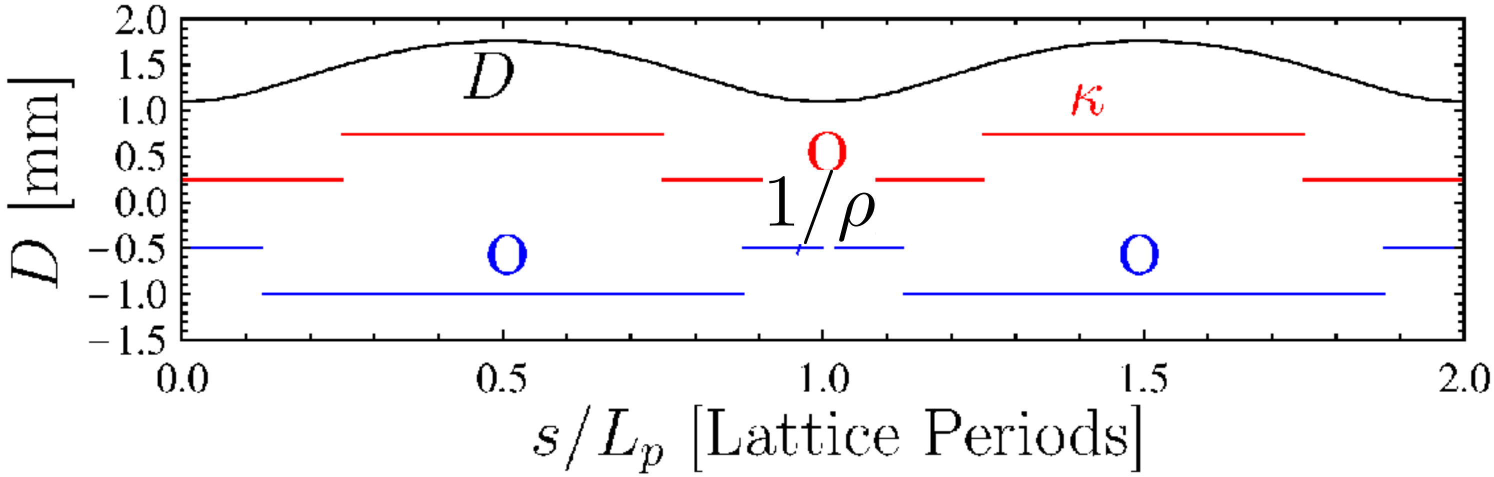
$$y_{\text{edge}} = \pm\sqrt{\epsilon_y \beta_y}$$



- Useful for species separation (Fragment Separator δD large)

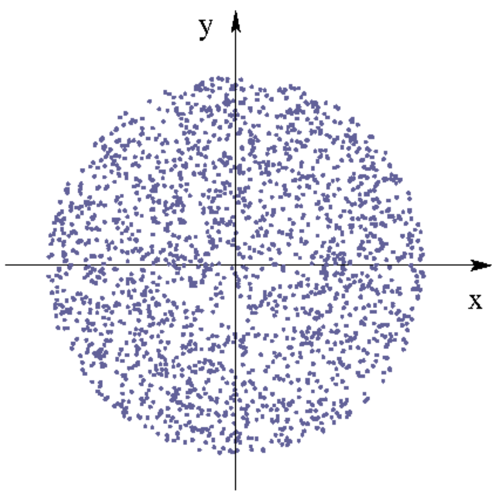
Example: Use an imaginary FO (Focus-Drift) piecewise-constant lattice and a single drift with the bend in the middle of the drift

$$L_p = 0.5 \text{ m} \quad \kappa = 20/\text{m}^2 \;\text{ in Focusing}$$
$$\eta = 0.5 \quad\quad \rho = 15 \text{ m, in bend, } 25\% \text{ Occupancy}$$

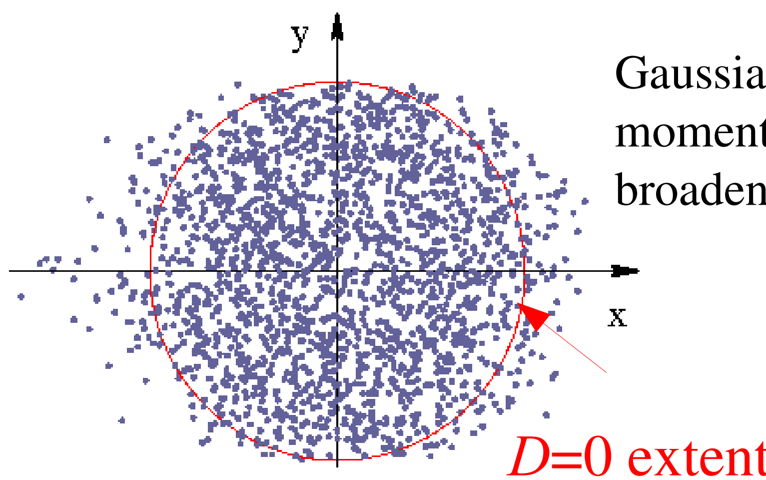


Dispersion broadens the *x*-distribution:

Uniform Bundle of particles D = 0



Same Bundle of particles D nonzero



Gaussian dist of momentum broadens x but not y

10. Illustrative Example: Fragment Separator

(CY Wong, MSU/NSCL Physics and Astronomy Dept.)

Many isotopes are produced when the driver beam impinges on the production target

Fragment separator downstream serves two purposes:

- Eliminate unwanted isotopes
- Select and focus isotope of interest onto a transport line towards experimental area

Different isotopes have different rigidities, which are exploited to achieve isotope selection

Rigidity $$[B\rho] = \frac{p}{q} = \frac{\gamma m v}{q}$$

ref particle (isotope) sets parameters in lattice transfer matrices

$$\delta = \left(\frac{\Delta p}{p}\right)_{\text{eff}} = \frac{\Delta [B\rho]}{[B\rho]_0}$$

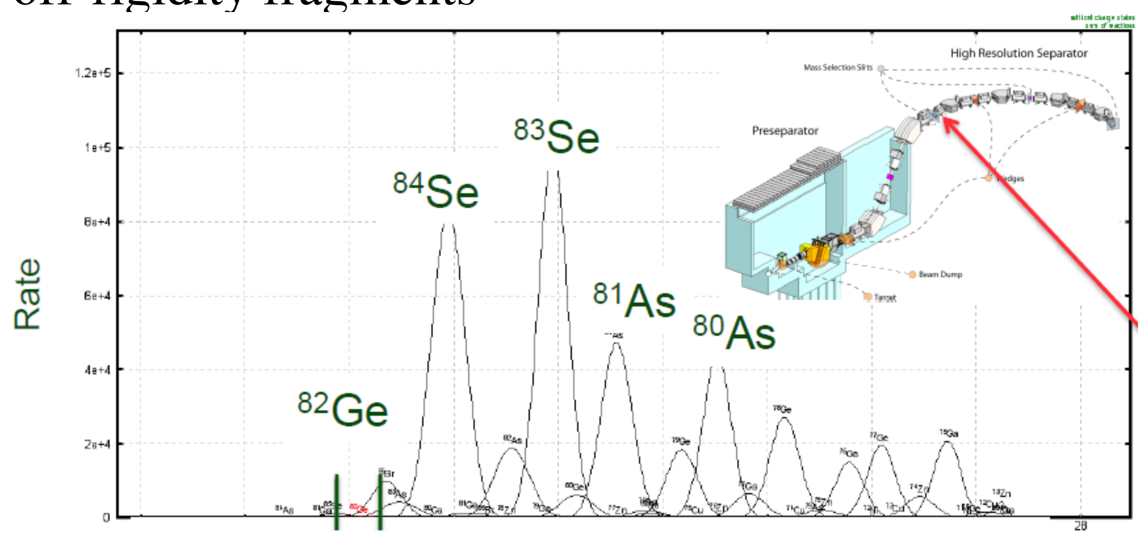
Deviation from the reference rigidity treated as an effective momentum difference

Dispersion exploited to collimate off-rigidity fragments

Example: Isotopes produced by primary ^{86}Kr beam

- LISE++ Code (NSCL)

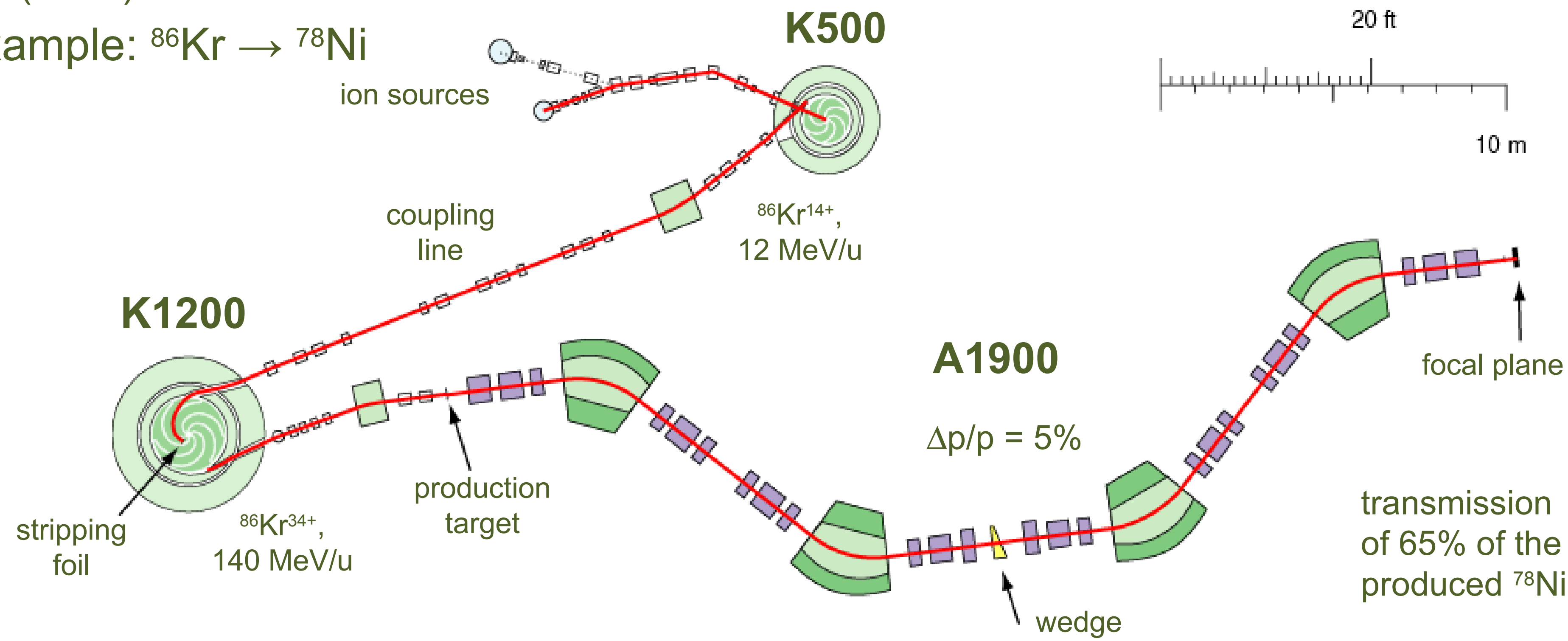
Experiment needs collimated ^{82}Ge secondary beam

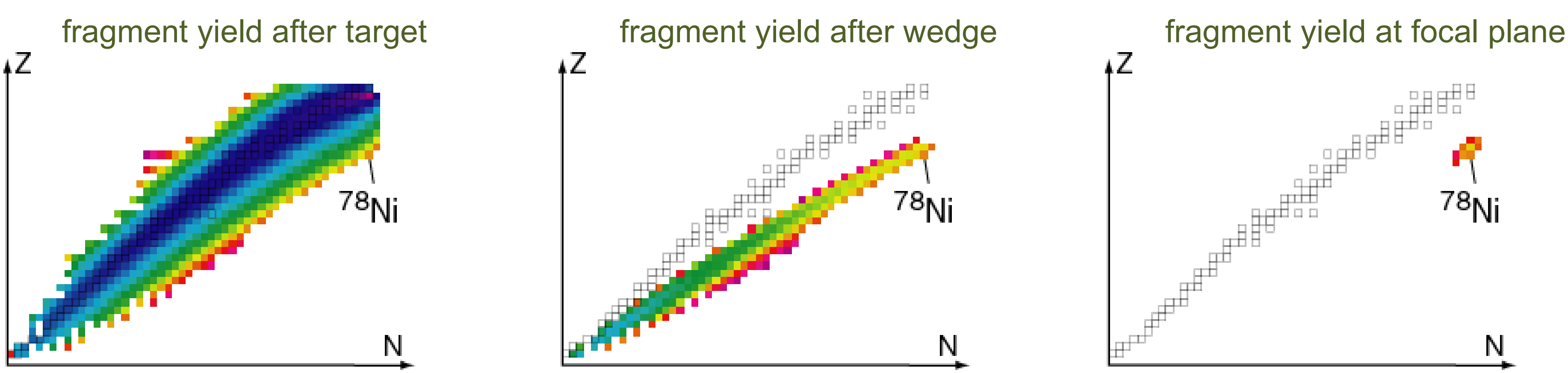


Cyclotron example: NSCL's coupled cyclotron facility

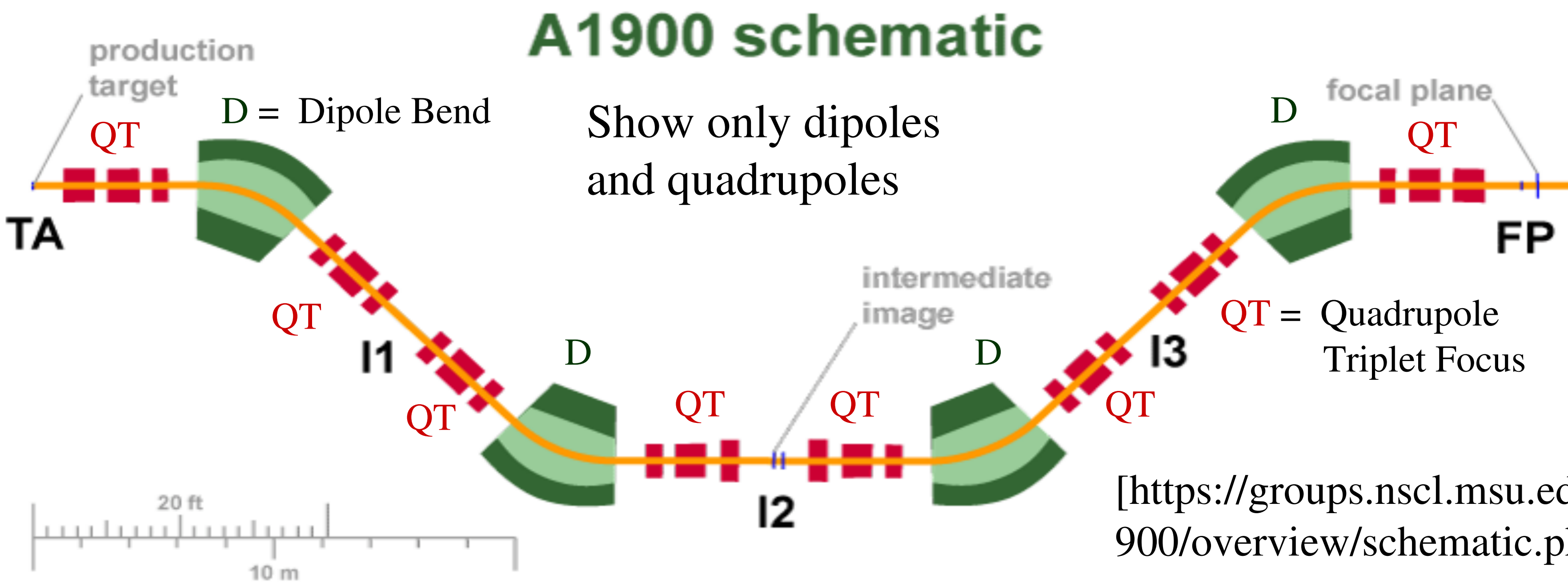
D.J. Morrissey, B.M. Sherrill, Philos. Trans. R. Soc. Lond. Ser. A. Math. Phys. Eng. Sci. 356 (1998) 1985.

Example: ^{86}Kr $\rightarrow$ ^{78}Ni



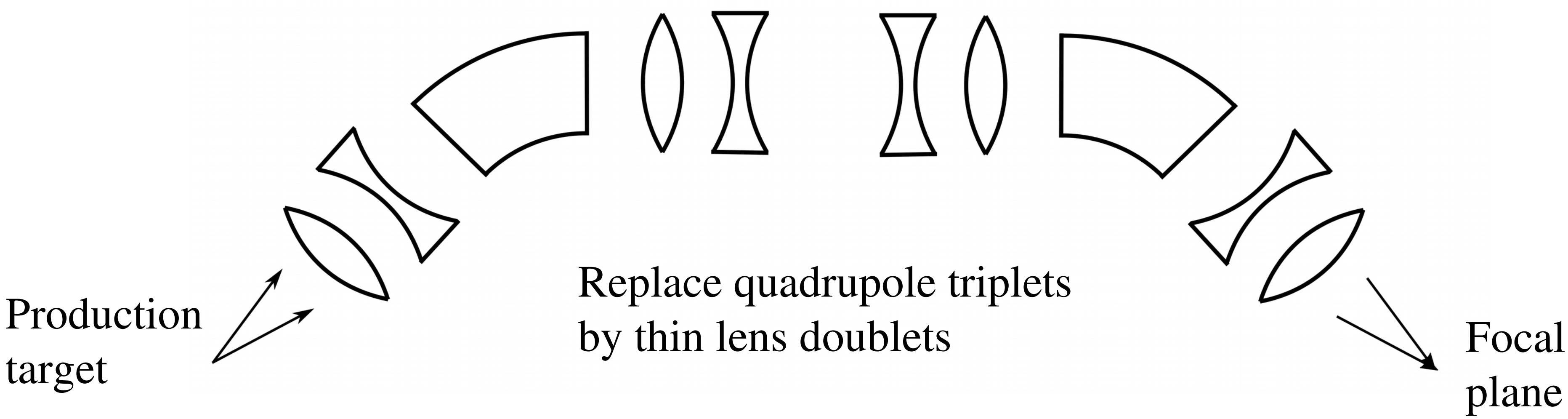


NSCL A1900 Fragment Separator: Simplified Illustration



[https://groups.nscl.msu.edu/a1900/overview/schematic.php]

Further Simplified Example: 2 segment version

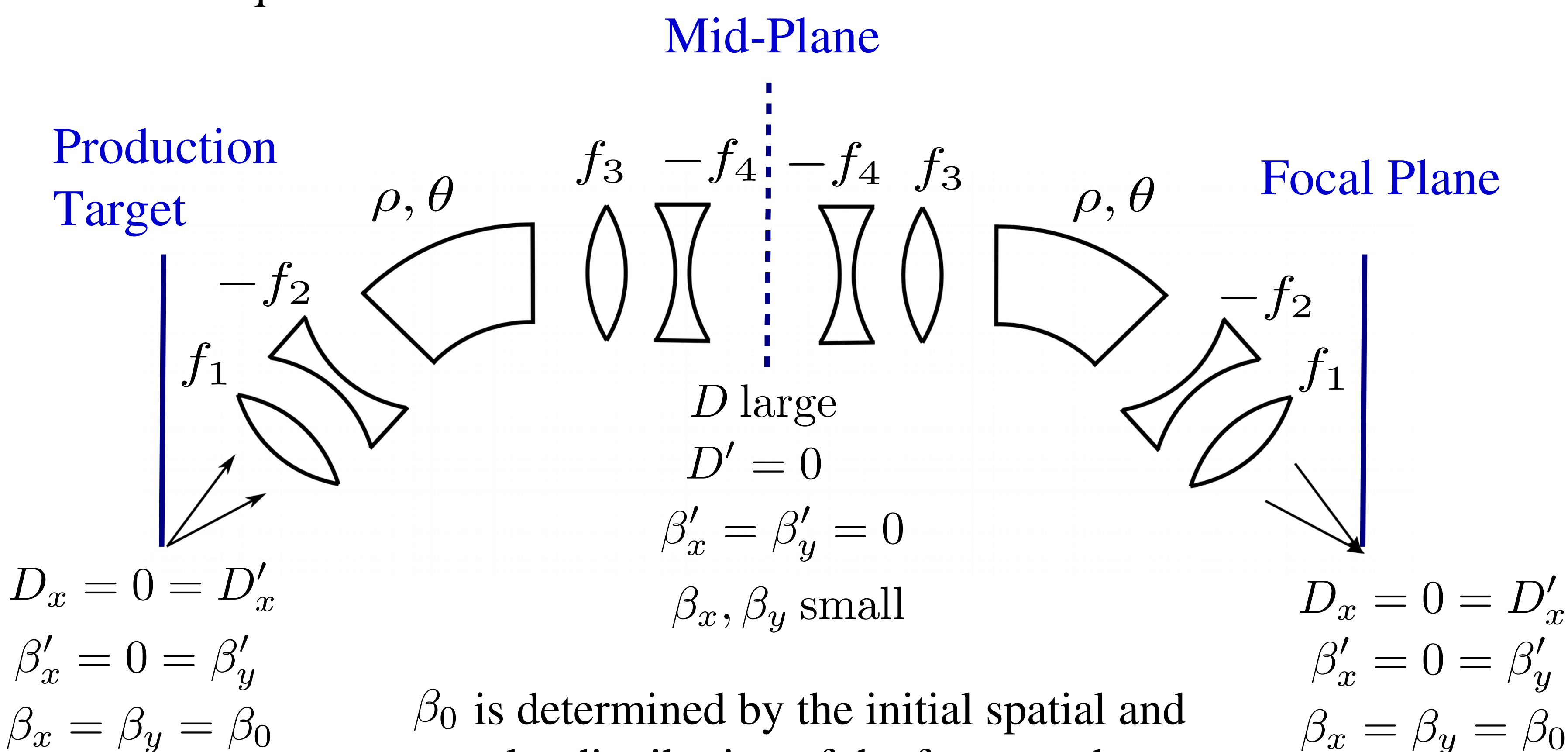


Design Goals:

- Dipoles set so desired isotope traverses center of all elements
- Dispersion function D is: *large at collimation* for rigidity resolution
  small elsewhere to minimize losses
- $\beta_x \quad \beta_y$ should be small at collimation point and focal plane

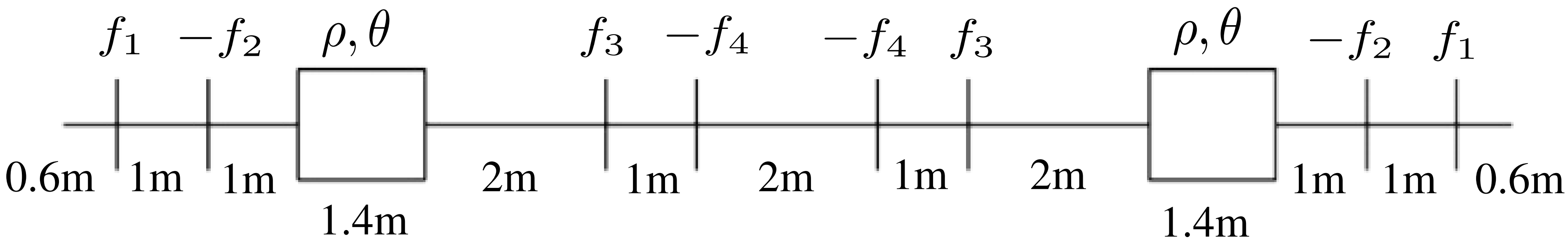
Simplify meeting needs with a lattice with left-right mirror symmetry and adjust for $D' = \beta'_x = \beta'_y = 0$ at mid-plane

- Conditions equivalent to "time reversal" in 2nd half to evolve back to initial condition



β_0 is determined by the initial spatial and angular distribution of the fragment beam

Supplemental Information: Parameters for Simplified Fragment Separator



Desired isotope: $^{31}S^{16+}$ from ^{40}Ar(140 MeV/u) on Be target

Energy: 120 MeV/u

Rigidity: 3.15 Tesla-m

Dipole ρ, θ are fixed

$\rho = 1.78\ \mathrm{m} \quad \theta = \pi/4$

Thus $B_y(0)$ is uniquely determined by $[B\rho]$

$B_y(0) = 1.7\ \mathrm{Tesla}$

Initial conditions at production target:

$$\sqrt{\langle x^2 \rangle} = 1\ \mathrm{mm} \qquad \sqrt{\langle x'^2 \rangle} = 10\ \mathrm{mrad}$$

$$\epsilon_x = 10\ \text{mm-mrad}$$

Impose constraints and solve f's numerically:

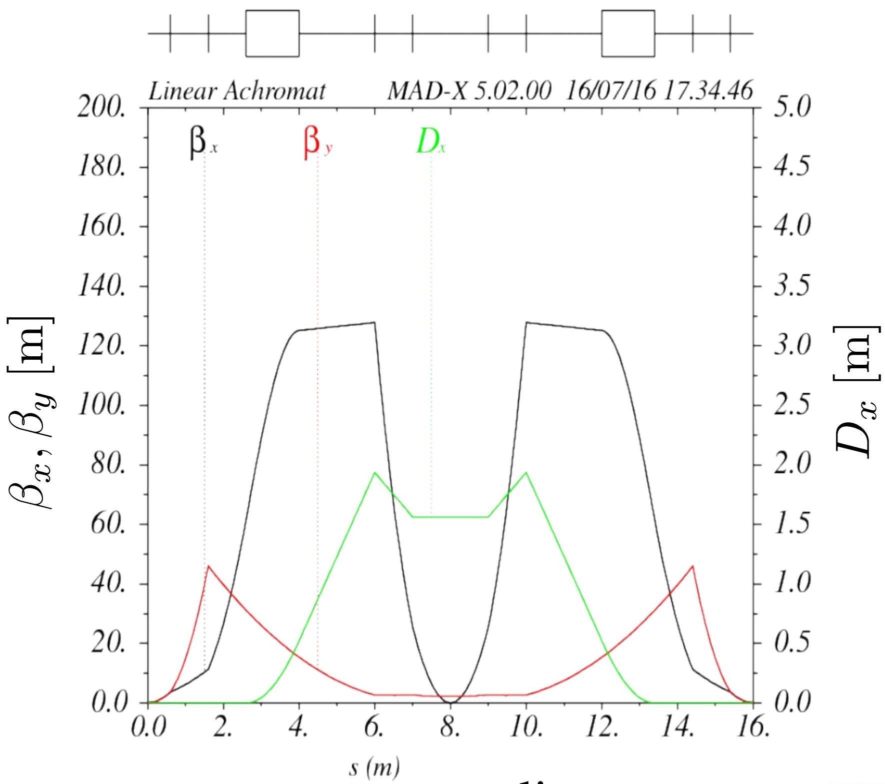
$f_1 = 1.12\ \mathrm{m}$	Quadrupole gradients	$G_1 = 13.9\ \mathrm{T/m}$
$f_2 = f_1$	$\Longleftrightarrow$	$G_2 = 13.9\ \mathrm{T/m}$
$f_3 = 1.79\ \mathrm{m}$	for lenghts	$G_3 = 8.7\ \mathrm{T/m}$
$f_4 = 4.17\ \mathrm{m}$	$\ell = 20\ \mathrm{cm}$	$G_4 = 3.7\ \mathrm{T/m}$

For other isotopes:

If initial $\langle x^2 \rangle, \langle x'^2 \rangle$ are the same, scale all fields to match rigidity $[B\rho]$

If not, the f's also have to be re-tuned to meet the constraints

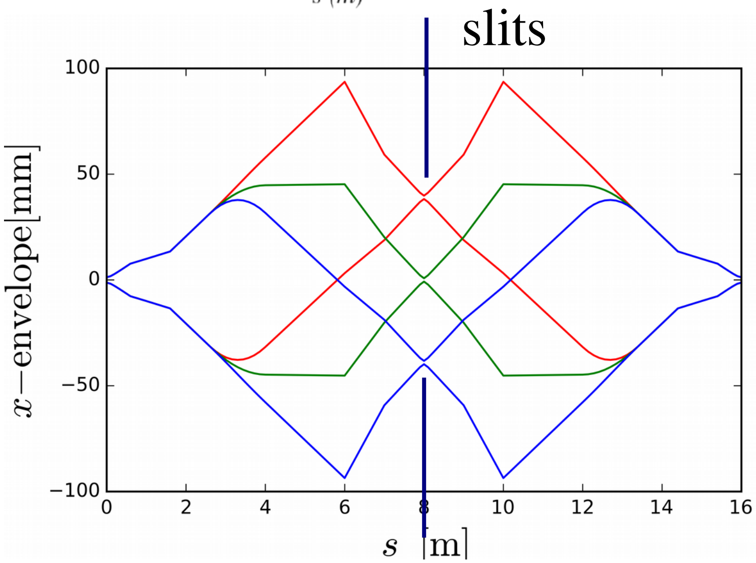
Lattice functions and beam envelope of simplified

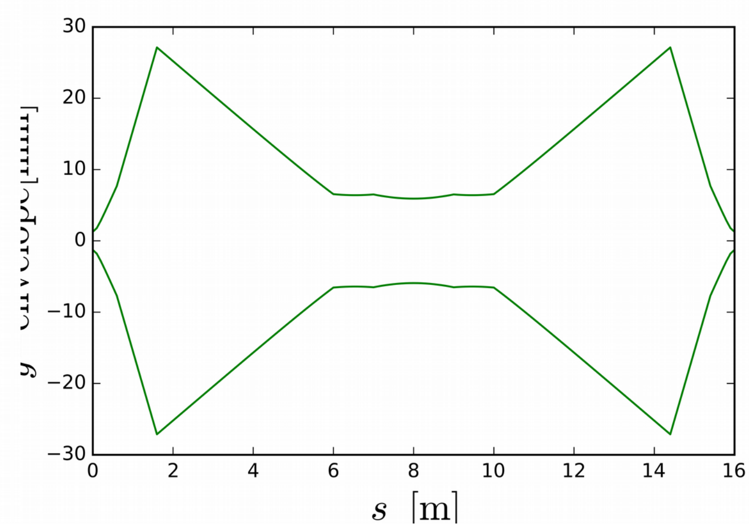


- Slits at mid-plane where dispersion large to collimate unwanted isotopes and discriminate momentum
- x-envelope plotted for 3 momentum values:

$$\boxed{x_{\text{env}} = \pm\sqrt{\beta_x \epsilon_x} + \delta D}$$

- Aperture sizes and D (properties of lattice), determine the angular and momentum acceptance of the fragment separator





$\delta p/p = 2.5\%$

$\delta p/p = 0$

$\delta p/p = -2.5\%$

Discussion

The simplified fragment separator can select and focus the isotope of interest provided:

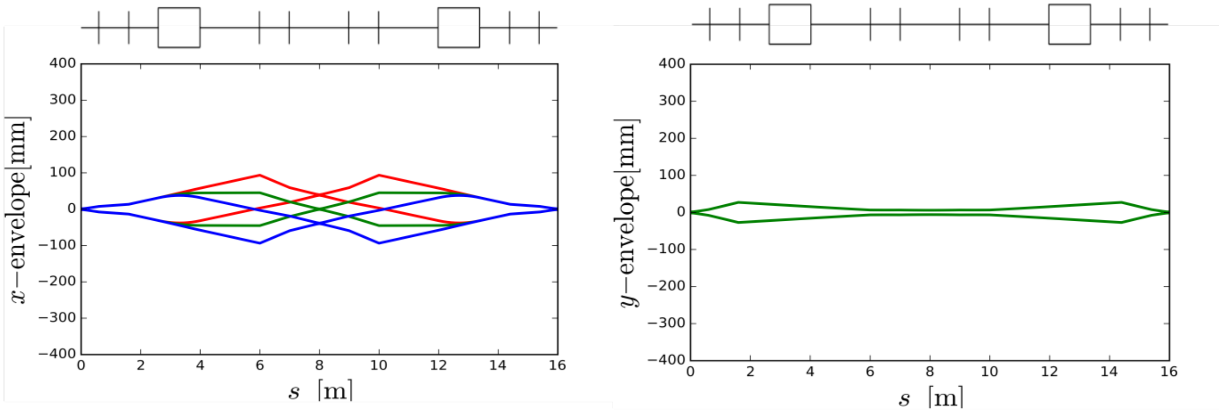
- All ions in each isotope are nearly mono-energetic
- Each isotope has a distinct rigidity

More sophisticated designs are necessary because:

- Want higher momentum and angular acceptance
  - More optics elements (e.g. quadrupole triplets) and stages to better control and optimize discrimination
- Ions of each individual isotope has a momentum spread
  - Nonlinear optics (sextupoles, octupoles) to provide corrections to chromatic aberrations
- Different isotopes may have nearly the same rigidity
  - Further beam manipulation (e.g. wedge degrader at mid-plane of A1900) for particle identification

Extra: Contrast Beam Envelopes in A1900 and Simplified Fragment Separator

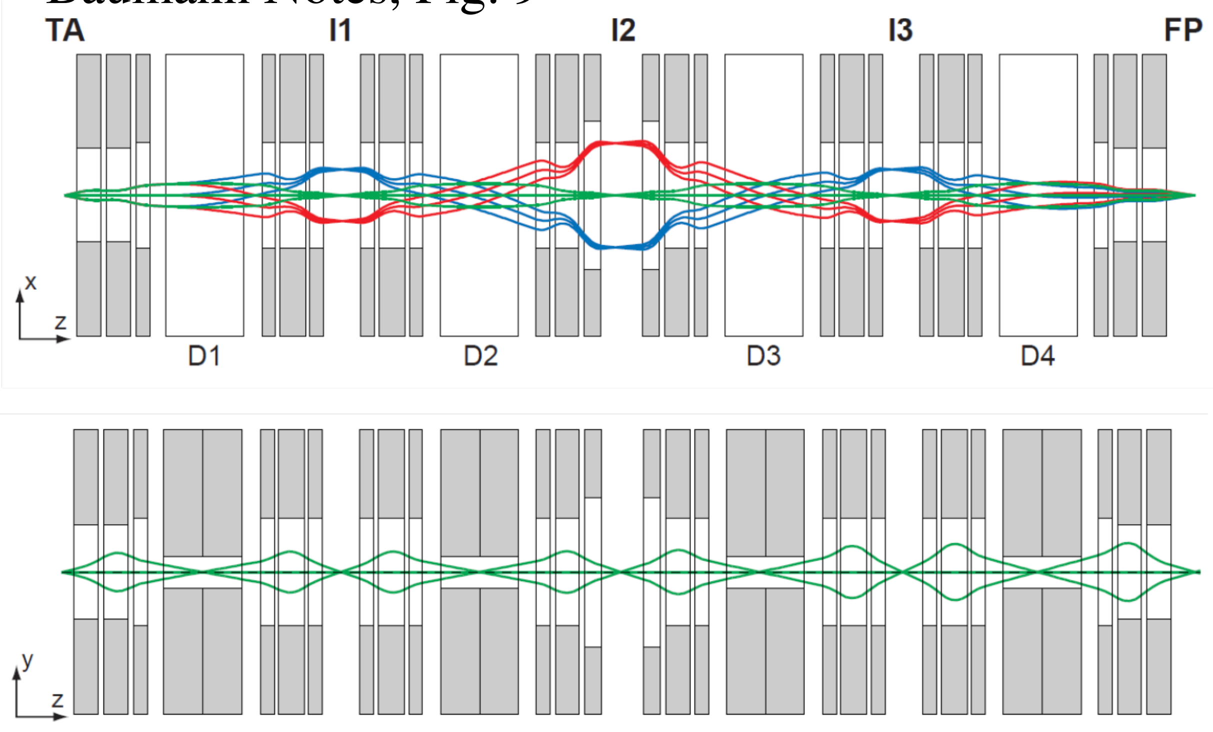
Simplified Separator



(Drawn to the same vertical and horizontal scale)

A1900 Separator: Beam Optics Example

Baumann Notes, Fig. 9



$\delta p/p = 2.5\%$

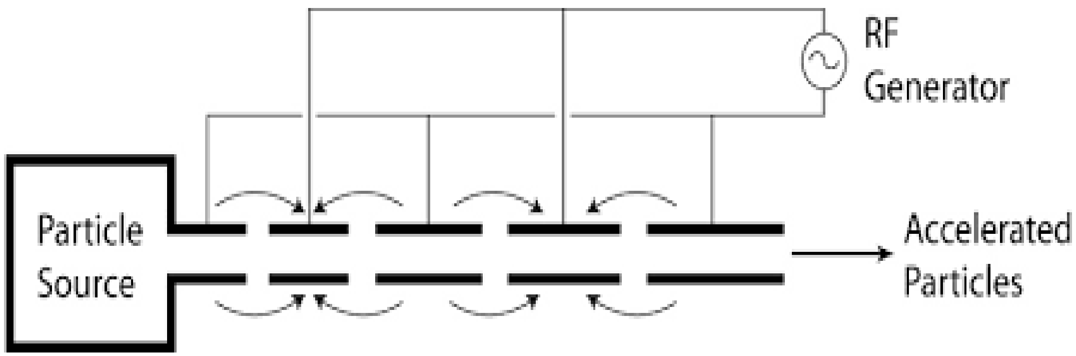
$\delta p/p = 0$

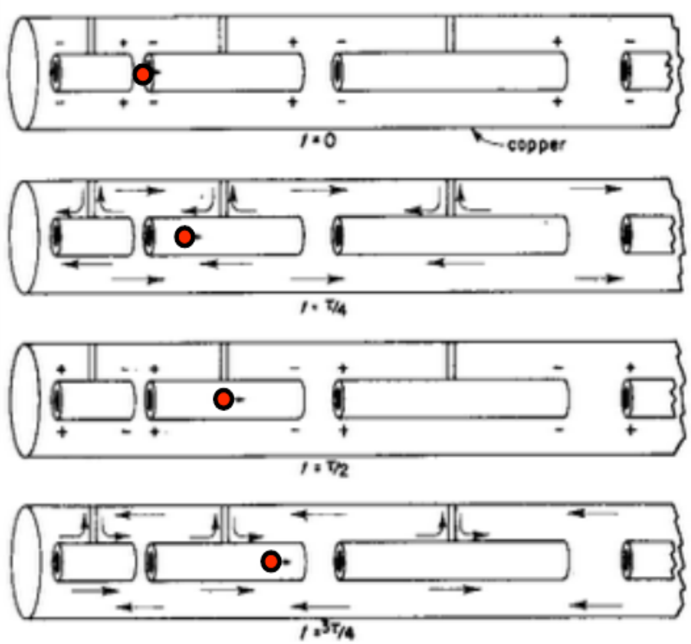
$\delta p/p = -2.5\%$

11. Conclusions

Biggest neglect in these lectures is to not cover acceleration due to a lack of time

Quick sketch of RF acceleration with a Alvarez/Wiedero type linac





Conclusions Continued

Left much out: Just a light overview of limited aspects of linear optics. If you want more, consult the references at the end and/or consider taking courses in the

US Particle Accelerator School: http://uspas.fnal.gov/

- Holds two (Winter and Summer) 2-week intensive (semester equivalent) sessions a year offering graduate credit and student fellowship support
- Courses offered on basic (Accelerator Physics, RF, Magnets, …) as well as many advanced (Space-Charge, Modeling, Superconducting RF, ...) topics
- USPAS programs are highly developed due to inadequate offerings at universities

Thanks for your attention !

Corrections and suggestions for improvements welcome!

These notes will be corrected and expanded for reference and for potential use in future editions of Exotic Beam Summer School (EBSS), the US Particle Accelerator School (USPAS), and Michigan State University (MSU) courses. Contact:

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Please provide corrections with respect to the present archived version at:

https://people.nscl.msu.edu/~lund/ebss/ebss_2016

Redistributions of class material welcome. Please do not remove author credits.

References: For more information see:

These notes will be posted with updates, corrections, and supplemental material at:

https://people.nscl.msu.edu/~lund/ebss/ebss_2016

Materials by the author following a similar format with many extensions can be found at:

SM Lund, *Fundamentals of Accelerator Physics*, Michigan State University, Physics Department, PHY 905, Spring 2016:

https://people.nscl.msu.edu/~lund/msu/phy905_2016

SM Lund and JJ Barnard, *Beam Physics with Intense Space-Charge,* US Particle Accelerator School, latest 2015 verion (taught every 2 years):

https://people.nscl.msu.edu/~lund/uspas/bpisc_2015

SM Lund, J-L Vay, R Lehe, and D. Winklehner, *Self-Consistent Simulation of Beam and Plasma Systems*, US Particle Accelerator School, latest 2016 version (to be taught every 2 years):

https://people.nscl.msu.edu/~lund/uspas/scs_2016

Useful accelerator textbooks include:

H. Wiedemann, *Particle Accelerator Physics*, Third Edition, Springer, 2007

T. Wangler, *RF Linear Accelerators*, 2^{nd} Edition, Wiley, 2008

References Continued

SY Lee, *Accelerator Physics,* 3rd Edition, World Scientific, 2011

D.A. Edwards and M.J. Syphers, *An Introduction to the Physics of High Energy Accelerators*, Wiley, 1993

K Wille, *The Physics of Particle Accelerators an introduction,* Oxford, 2000

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M. Reiser, *Theory and Design of Charged Particle Beams,* Wiley, 1994

J. Lawson, *The Physics of Charged Particle Beams*, Clarendon Press, 1977

M. Berz, K. Makino, and W. Wan, *An Introduction to Accelerator Physics,* CRC Press, 2014

Original, classic paper on strong focusing and Courant-Snyder invariants applied to accelerator physics. Remains one of the best formulated treatments to date:

E.D. Courant and H. S. Snyder, Theory of the Alternating Gradient Synchrotron, Annals Physics **3**, 1 (1958)

Much useful information can also be found in the course note archives of US (USPAS) and European (CERN) accelerator schools:

USPAS: http://uspas.fnal.gov/ CERN: http://cas.web.cern.ch/cas/