Corrections to the Fermi Matrix Element for Superallowed $\beta$ Decay

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(Received 10 February 1988; revised manuscript received 5 December 1988)

Corrections to the Fermi matrix element for superallowed transitions due to isospin nonconservation are reexamined. The sources of theoretical uncertainty and the possibility of a previously neglected correction based on the mismatch between the spectator nucleons are investigated. The consequences to the conserved-vector-current hypothesis and the Kobayashi-Maskawa mixing angles are discussed.

PACS numbers: 23.40.--s

Superallowed Fermi $\beta$ decay has been the subject of intense study for several decades (cf. Refs. 1-5 and references therein). According to the conserved-vector-current (CVC) hypothesis, their $f_I$ values should be constant for all nuclei, and given by

$$ f_I = \frac{K}{G_F^2 |M_F|^2}, \tag{1} $$

where $K = 8.1201 \times 10^{-7}$ is a product of fundamental constants, $G_F$ is the vector coupling constant for nucleon $\beta$ decay [measured in units of $(\hbar c)^3$], and $M_F$ is the Fermi matrix element, $M_F = \langle \psi_f | T \pm | \psi_i \rangle$. By comparing the vector coupling constant for nucleon $\beta$ decay to that of muon $\beta$ decay, the Kobayashi-Maskawa mixing angle between $u$ and $d$ quarks ($\psi_{ud}$) can be determined, and a test of the three-generation standard model of the electroweak interaction is possible.

Two classes of nucleus-dependent corrections, however, must be applied to Eq. (1). The first is radiative corrections to the statistical rate function $f_I$, denoted by $\delta_R$, giving $f_{IR} = f_I(1 + \delta_R)$. The second is corrections to the nuclear matrix element due to the presence of isospin-nonconserving (INC) forces in nuclei, and is denoted by $\delta_C$; it is $|M_F|^2 = |M_{F_0}|^2(1 - \delta_C)$, where $M_{F_0} = [T(T+1) - Z T Z_f]^{1/2} \delta_I$. With $\delta_R$ and $\delta_C$, the "nucleus-independent" $f_I$ value for $(0^+, T = 1) \to (0^+, T = 1)$ transitions is

$$ f_I = f_I(1 + \delta_R)(1 - \delta_C) = \frac{K}{2G_F^2}. \tag{2} $$

From these $f_I$ values, it is then possible to determine empirical values of $G_V$.

Until recently, the experimental $f_I$ values, together with the calculated corrections $\delta_R$ and $\delta_C$ for the eight most accurately measured $f_I$ values ($^{14}$O, $^{26}$Al, $^{34}$Cl, $^{38}$K, $^{42}$Sc, $^{46}$V, $^{50}$Mn, $^{54}$Co), failed to yield constant $f_I$ values. In fact, the data gave two $f_I$ values, one consistent for $Z < 21$, and another for $Z \geq 21$. This failure to give constant $f_I$ values persisted even after an intense effort to measure the $f_I$ values as accurately as possible.

It appears that the primary reason for the lack of agreement with the CVC hypothesis was due to an incorrect evaluation of the $Za^2$ contribution to $\delta_R$. The study of superallowed Fermi transitions, however, cannot be satisfactorily concluded because the values of $\delta_C$ calculated previously by Towner, Hardy, and Harvey (THH) and Wilkinson (W) yield considerably different $f_I$ values than those of the more recent reevaluation of Ormand and Brown (OB).

In this Letter, we reexamine the nuclear corrections $\delta_C$, also investigating the possibility of a correction based on the mismatch between the spectator nucleons.

The formalism needed to perform microscopic calculations of $\delta_C$ is given in Refs. 3 and 11. Conventionally, $\delta_C$ has been factored into two components, i.e., $\delta_C = \delta_{IM} + \delta_{RO}$. The first, $\delta_{IM}$, is due to isospin mixing between the different shell-model configuration states, while $\delta_{RO}$ is due to the deviation from unity of the radial overlap between the converted proton and corresponding neutron (i.e., mixing between states that lie outside of the shell-model configuration space). The OB study found that $\delta_{RO} \geq \delta_{IM}$. The essential ingredient of a calculation of $\delta_{IM}$ is an INC interaction that is added onto the standard shell-model Hamiltonian, while the calculation of $\delta_{RO}$ is based on radial wave functions that are obtained from a suitable parametrization of the mean field.

The present problem with superallowed Fermi $\beta$ decay is that in the recent reevaluation both $\delta_{IM}$ and $\delta_{RO}$ were found to be considerably smaller than those obtained previously. Although both sets of $\delta_C$ agree with the CVC hypothesis almost equally well, they yield inconsistent averaged $f_I$ values. The principal differences in these two calculations are as follows: The THH INC interaction was obtained by (i) adding Coulomb matrix elements onto the proton-proton Hamiltonian; (ii) increasing the $T=1$ part of the proton-neutron interaction by 2%; and (iii) determining the single-particle energies from closed-core plus proton and neutron nuclei, whereas
the OB INC interaction was determined empirically by requiring that the parameters of a Coulomb plus phenomenological isovector and isotorus potential reproduce experimental isotopic mass splittings.\textsuperscript{11,14} It was found that this procedure better determined the single-particle energies for nuclei away from a closed major shell, and was responsible for most of the decrease in $\delta_{IM}$.\textsuperscript{11} The THH radial wave functions were obtained with a Woods-Saxon plus Coulomb potential, while those of the OB study were determined with a self-consistent Hartree-Fock (HF) calculation using a Skyrme-type interaction that also included the Coulomb exchange term. The advantage of the HF procedure is that since the mean field is proportional to the nucleon densities, the Coulomb force induces a one-body isovector potential that tends to counter Coulomb repulsion, thereby reducing $\delta_{RO}$.

In addition to these corrections, we should also account for the fact that all the parent and daughter nucleons feel the effects of different mean fields before and after the transition. With this in mind, we expect a correction to Eq. (1) of the form

$$\delta_S = 2 \sum_{\mu, i} \Omega_{\mu i} (n_{\mu i} - 1) \Omega_{\mu i} + n_{\nu i} \Omega_{\nu i},$$

where $\mu$ and $i$ represent a sum over proton ($p$) and neutron ($n$) core orbits, $n_{\mu i}$ represents the number of protons or neutrons occupying the $i$th orbit, and $n_{\nu i}$ denotes the valence orbit in the parent nucleus. The overlaps $\Omega$ denote the deviation from unity of the parent (P) and daughter (D) radial wave functions, i.e., $\Omega_{\mu i} = 1 - \int dr r^2 R_{\mu i}^p R_{\mu i}^D$.

Inserting radial wave functions obtained from a Hartree-Fock calculation utilizing a Skyrme-type force into Eq. (3), one finds $\delta_S$ to be appreciable. Using the SG II force,\textsuperscript{15} we find $\delta_S = 0.18\%$ for $^{54}$Co, as compared to $\delta_{RO} = 0.38\%$. It is known, however, that Hartree-Fock calculations do not conserve isospin under the interchange of the last nucleon in mirror nuclei even though the two-body force is isoscalar.\textsuperscript{16} To determine the sensitivity of Eq. (3) to this type of spurious mixing, we have evaluated $\delta_S$ while excluding the Coulomb potential. For $^{54}$Co, we find $\delta_S = 0.16\%$, indicating that the spurious mixing is important. In addition, a nonzero $\delta_S$ in the absence of the Coulomb potential violates the Behrend-Sirion-Ademollo-Gatto theorem,\textsuperscript{17} which states that corrections to Eq. (1) must be proportional to the mass difference between the initial and final states.

Given these considerations, an estimate of $\delta_S$ might be obtained in two ways. First, by taking the difference between the results obtained by Eq. (3) with and without the Coulomb potential, leading to corrections of the order $0.02\%-0.05\%$. The second procedure, which avoids the problems encountered in the absence of the Coulomb potential, is to use wave functions obtained from the $T_Z = 0$ mean field. In this case, the effect of the Coulomb potential in the $T_Z = 0$ nucleus can be accounted for by using an effective charge $e' = e(1 + 2T_Z/A)$, where $A$ is the number of nucleons, and $T_Z$ for the proton is taken to be positive. Using these wave functions, we find $\delta_S < 0.01$. Given these considerations, $\delta_S$ is probably negligible: however, we account for it by including a conservative uncertainty of $0.05\%$ in the total correction $\delta_C$.

Other sources of uncertainty in $\delta_C$ lie in the procedures used to evaluate $\delta_{RO}$ and $\delta_{IM}$. In order to determine the sensitivity in $\delta_{RO}$ to the Skyrme parameters, we have evaluated $\delta_{RO}$ using five different Skyrme forces: A,\textsuperscript{18} SG I,\textsuperscript{15} SG II,\textsuperscript{15} Skyrme M,\textsuperscript{19} and Skyrme M*.\textsuperscript{20} The SG II values were found to be very nearly equal to the average obtained with all forces, and since this is the same force used previously\textsuperscript{11,12,21} these values are reported here in Table I (under the heading HF). The variations in $\delta_{RO}$ were typically $\pm 0.03\%$, and are due to the fact that the explicit isovector properties of the Skyrme force cannot be determined unambiguously.\textsuperscript{22} An additional uncertainty of $\approx 0.03\%$ in $\delta_{RO}$ arises from the selection of the shell-model Hamiltonian and truncations on the configuration space (see Ref. 21).

Just as in the case for $\delta_S$ mentioned above, we note that in the limit that the Coulomb potential is switched off, $\delta_{RO}$ is also nonzero. This effect is due to the fact that the HF binding energies of the $T_Z = 0$ and $\pm 1$ nuclei are not equal, and is largest for the lightest nuclei, $\delta_{RO} \approx 0.04\%$ for $^{14}$O, and decreases for the heavier nu-

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>HF</th>
<th>ECHF</th>
<th>AVG</th>
<th>THH</th>
<th>OB</th>
<th>THH</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{14}$O</td>
<td>0.15</td>
<td>0.21</td>
<td>0.18(5)</td>
<td>0.23(3)</td>
<td>0.01</td>
<td>0.05</td>
</tr>
<tr>
<td>$^{26}$Al</td>
<td>0.27</td>
<td>0.18</td>
<td>0.23(6)</td>
<td>0.27(4)</td>
<td>0.01</td>
<td>0.07</td>
</tr>
<tr>
<td>$^{34}$Cl</td>
<td>0.47</td>
<td>0.36</td>
<td>0.42(7)</td>
<td>0.62(7)</td>
<td>0.06</td>
<td>0.23</td>
</tr>
<tr>
<td>$^{38}$K</td>
<td>0.49</td>
<td>0.27</td>
<td>0.38(12)</td>
<td>0.54(7)</td>
<td>0.11</td>
<td>0.16</td>
</tr>
<tr>
<td>$^{42}$Sc</td>
<td>0.29</td>
<td>0.25</td>
<td>0.28(5)</td>
<td>0.35(6)</td>
<td>0.11</td>
<td>0.13</td>
</tr>
<tr>
<td>$^{46}$V</td>
<td>0.24</td>
<td>0.17</td>
<td>0.20(6)</td>
<td>0.36(6)</td>
<td>0.01</td>
<td>0.04</td>
</tr>
<tr>
<td>$^{50}$Mn</td>
<td>0.31</td>
<td>0.24</td>
<td>0.28(6)</td>
<td>0.40(9)</td>
<td>0.004</td>
<td>0.03</td>
</tr>
<tr>
<td>$^{54}$Co</td>
<td>0.38</td>
<td>0.30</td>
<td>0.34(6)</td>
<td>0.56(6)</td>
<td>0.005</td>
<td>0.04</td>
</tr>
</tbody>
</table>
clei, \( \delta_{RO} \approx 0.01\% \) for \(^{34}\text{Cl}\). Further, in this case, if the binding energies of the parent and daughter states are constrained to be equal, \( \delta_{RO} \) reduces to nearly zero. We have also evaluated \( \delta_{RO} \) using the \( T_2=0 \) effective-charge Hartree-Fock (ECHF) mean fields described above, and are also given in Table I. The ECHF values of \( \delta_{RO} \) tend to be somewhat smaller than their HF counterparts. In particular, for \(^{38}\text{K}\) where the ECHF value is \( \approx 0.2\% \) smaller. This reflects an uncertainty in our ability to calculate \( \delta_{RO} \). Here, we use the average of the two values when evaluating the total correction \( \delta_C \), which is also given in Table I along with the THH values of \( \delta_{RO} \).

The configuration-mixing corrections \( \delta_{IM} \) are sensitive to the choice of the shell-model configuration space, the isoscalar Hamiltonian, and the isovector single-particle energies. The uncertainties due to these quantities are somewhat difficult to estimate. However, in this work, the most recent isoscalar shell-model Hamiltonians were used within largest shell-model configuration space possible, and, further, the INC interactions were determined empirically for each shell-model Hamiltonian.\(^{14}\) In all cases, \( \delta_{IM} \) was found to be much smaller than \( \delta_{RO} \), with an upper limit being approximately 0.1\%. In this regard, we have chosen to assign an uncertainty of 0.05\% in \( \delta_{IM} \). For the purpose of comparison, both the THH values of \( \delta_{IM} \) and those of the present work are also given in Table I.

The total uncertainties discussed up to this point add up to approximately 0.9\% in most cases. This arises from the addition in quadrature of 0.05\% in \( \delta_{S} \), 0.05\% in \( \delta_{IM} \), and 0.06\% in \( \delta_{RO} \). The uncertainties in \( \delta_{RO} \) are 0.03\% due to different Skyrme forces, 0.03\% due to model-space truncations, and typically 0.04\% (0.11\% in \(^{38}\text{K}\)) from the difference in the ECHF and HF calculations.

Given in Table II are the experimental \( ft \) values,\(^{23}\) the outer radiative correction \( \delta_R \),\(^{23,24}\) and a comparison between the OB and THH\(^{3,4,24}\) values of \( \delta_C \) and the corresponding “nucleus-independent” \( ft \) values. Both sets of corrections yield essentially constant, but inconsistent averaged \( ft \) values. The THH corrections give \( (ft)_{\text{avg}} = 3071.5 \pm 1.6 \text{ sec with } \chi^2/\nu = 0.64 \), while the corrections reported here yield \( (ft)_{\text{avg}} = 3077.3 \pm 1.9 \text{ sec with } \chi^2/\nu = 0.63 \).

From the averaged \( ft \) values the vector coupling constant for single nucleon \( \beta \) decay can be determined, and the Kobayashi-Maskawa mixing matrix element \( v_{ud} \) is then

\[
v_{ud} = \frac{G_V}{G_\mu} (1 + \Delta_\beta - \Delta_\mu)^{-1/2},
\]

where \( G_V/(hc)^3 = 1.166 \pm 0.05 \text{ GeV}^{-2} \) (Ref. 25) is the vector coupling constant for muon \( \beta \) decay, and \( \Delta_\beta \) and \( \Delta_\mu \) are the “inner” radiative corrections to both nucleon and muon \( \beta \) decay, with \( \Delta_\beta - \Delta_\mu = 0.023(2) \).\(^{26}\) The THH and OB corrections give \( v_{ud} = 0.9746 \pm 0.0010 \) and 0.9737 \pm 0.0010, respectively. With \( v_{ud} \) determined, a test of the three-generation standard model is possible at the level of quantum corrections,\(^{27}\) i.e., the Kobayashi-Maskawa matrix should be unitary \( (v^2 = v_{ud}^2 + v_{ub}^2 + v_{ub}^2 = 1) \). Taking \( v_{ub} = 0.220 \pm 0.002 \) (Ref. 27) and \( v_{ub} < 0.0075 \) (90\% confidence level),\(^{28}\) the unitarity condition for the THH and OB values are \( v^2 = 0.9982 \pm 0.0021 \) and 0.9965 \pm 0.0021, respectively. The THH and OB sets of nuclear corrections appear to be in agreement at this level. However, the error is dominated by the uncertainty in the “inner” radiative corrections. If this uncertainty were reduced to the level of the uncertainty in the nuclear corrections, the conclusions might be different. For instance, if the inner radiative correction does not change in value, the \( v^2 \) obtained with the THH nuclear corrections would agree better with the unitarity condition.

At this point, we wish to emphasize that both the OB and THH values of \( \delta_C \) yield \( ft \) values that are essentially in equally good agreement with the CVC hypothesis, i.e., at the level of 0.06\%. However, as is pointed out, they yield inconsistent averaged values, leading to somewhat different conclusions regarding a test of the three-generation standard model. The principal difference be-

### Table II. List of \( ft \) values, corrections, and “nucleus-independent” \( ft \) values.

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>( ft )</th>
<th>( \delta_R ) (%)</th>
<th>OB</th>
<th>THH</th>
<th>OB</th>
<th>THH</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^{14}\text{O})</td>
<td>3038.1(23)</td>
<td>1.53(1)</td>
<td>0.19(13)</td>
<td>0.33(10)</td>
<td>3078.7(47)</td>
<td>3074.4(37)</td>
</tr>
<tr>
<td>(^{26}\text{Al})</td>
<td>3034.5(14)</td>
<td>1.47(2)</td>
<td>0.24(13)</td>
<td>0.34(11)</td>
<td>3071.7(44)</td>
<td>3068.6(37)</td>
</tr>
<tr>
<td>(^{34}\text{Cl})</td>
<td>3052.0(29)</td>
<td>1.45(3)</td>
<td>0.48(14)</td>
<td>0.85(12)</td>
<td>3081.4(53)</td>
<td>3070.0(47)</td>
</tr>
<tr>
<td>(^{38}\text{K})</td>
<td>3045.1(26)</td>
<td>1.44(3)</td>
<td>0.49(17)</td>
<td>0.70(12)</td>
<td>3073.7(59)</td>
<td>3067.3(48)</td>
</tr>
<tr>
<td>(^{42}\text{Sc})</td>
<td>3048.7(63)</td>
<td>1.46(4)</td>
<td>0.39(13)</td>
<td>0.48(12)</td>
<td>3081.1(76)</td>
<td>3078.5(76)</td>
</tr>
<tr>
<td>(^{46}\text{V})</td>
<td>3043.7(22)</td>
<td>1.46(4)</td>
<td>0.21(13)</td>
<td>0.40(12)</td>
<td>3081.6(49)</td>
<td>3075.7(46)</td>
</tr>
<tr>
<td>(^{56}\text{Mn})</td>
<td>3039.9(40)</td>
<td>1.46(5)</td>
<td>0.28(13)</td>
<td>0.43(13)</td>
<td>3075.6(59)</td>
<td>3071.0(59)</td>
</tr>
<tr>
<td>(^{54}\text{Co})</td>
<td>3044.7(23)</td>
<td>1.45(5)</td>
<td>0.35(13)</td>
<td>0.60(12)</td>
<td>3077.1(49)</td>
<td>3070.3(48)</td>
</tr>
</tbody>
</table>

**Average**

| \( \chi^2/\nu \) | 0.63 | 0.64 |

---

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tween these two estimates lies in the radial overlap correction. Here $\delta_{R0}$ is reduced considerably relative to the THH values. This reduction is primarily due to the different treatments of the Coulomb potential and the nuclear mean fields. The THH Coulomb potential was that of a uniformly charged sphere, while in this work it was obtained from the proton densities via a self-consistent Hartree-Fock calculation. Further, the influence of the Coulomb exchange term and the "induced" isovector potential described previously were included. The net effect of these differences is to produce a HF potential for the protons that is both deeper at the origin, and has a higher barrier at the surface than the Woods-Saxon potential, therefore, increasing the overlap between the converted proton and corresponding neutron.\textsuperscript{11}

Discussions with G. F. Bertsch, M. Brack, J. C. Hardy, B. R. Holstein, B. R. Mottelson, and I. S. Towner are gratefully acknowledged. This work was supported in part by the Danish Natural Science Research Council and National Science Foundation Grant No. PHY-83-12245.

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\textsuperscript{3}I. S. Towner, J. C. Hardy, and M. Harvey, Nucl. Phys. \textbf{A284}, 269 (1977).


\textsuperscript{5}A. Sirlin, Rev. Mod. Phys. \textbf{50}, 573 (1978).


\textsuperscript{12}W. E. Ormand and B. A. Brown, in Ref. 7, p. 545.

\textsuperscript{13}Comparisons will be made with the THH values of $\delta_c$ as both the THH and W values are in good agreement.


\textsuperscript{15}S. Köhler, Nucl. Phys. \textbf{A258}, 301 (1976).


\textsuperscript{17}J. Bartel, P. Quentin, M. Brack, C. Guet, and H.-B. Häkansson, Nucl. Phys. \textbf{A386}, 79 (1982).

\textsuperscript{18}The small differences between the values of this work and those in Ref. 12 are due to a different renormalization of the spectroscopic factors in Eq. (2.10a) of Ref. 11.


\textsuperscript{21}The uncertainties quoted in Table II for the THH values of $\delta_c$ are due only to the uncertainty in $\delta_{R0}$. The somewhat worse $\chi^2/N$ for the THH values reflect this somewhat smaller uncertainty.

\textsuperscript{22}Particle Data Group, Phys. Lett. \textbf{B204}, 1 (1988).

