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I. Introduction

(related reading in parentheses)

Particle motion (Reiser 2.1)

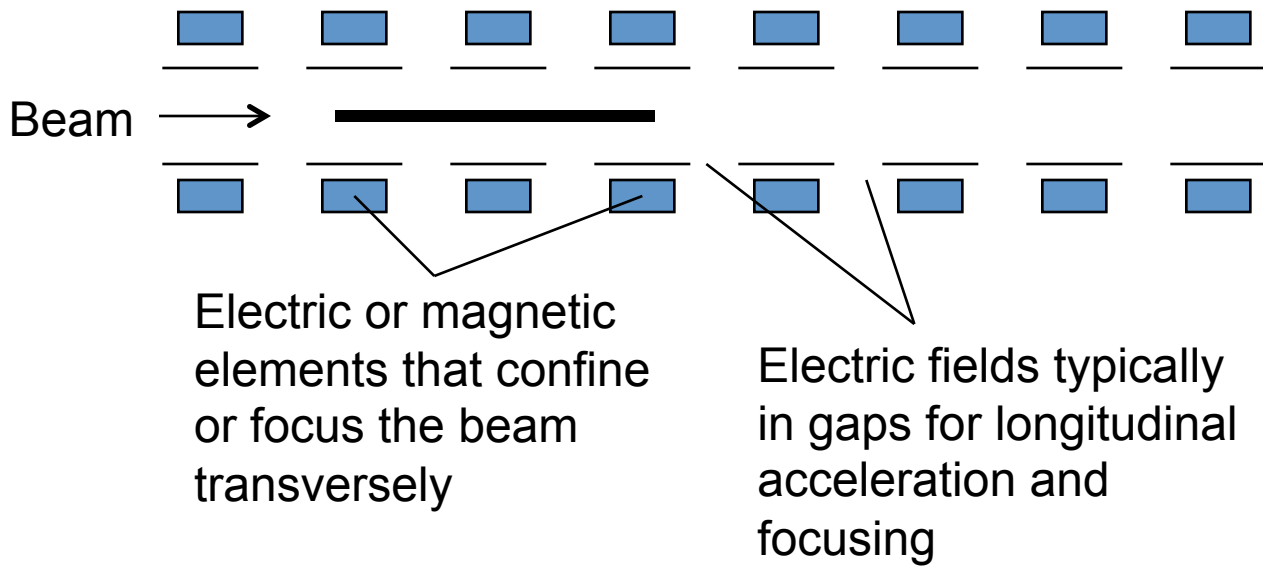
Equation of motion (Reiser 2.1)

Dimensionless quantities (Reiser 4.2)

Plasma physics of beams (Reiser 3.2, 4.1)

Emittance and brightness (Reiser 3.1 - 3.2)

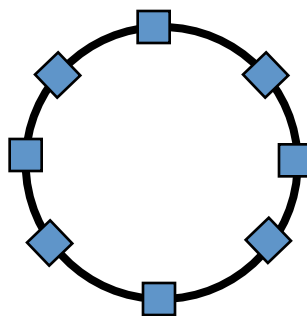
How do we describe and calculate the evolution of a collection of particles under the EM forces in an accelerator?



This array or "lattice" of focusing elements may be arranged in a linac or circular accelerator



or

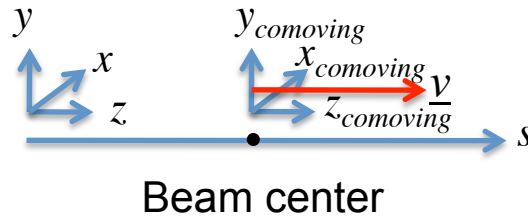


Particle equations of motion and dimensionless quantities

Consider the Lorentz force on a particle (mass m , charge q , momentum \underline{p} , velocity $\underline{v} = c\underline{\beta}$, Lorentz factor γ) under the influence of an electric (\underline{E}) and magnetic field (\underline{B}):

$$\frac{d\underline{p}}{dt} = q(\underline{E} + \underline{v} \times \underline{B}) \quad (\text{SI units employed throughout})$$

$$\underline{p} = \gamma m \underline{v} \quad \gamma^2 = \frac{1}{1 - \beta^2} \quad \underline{\beta} = \underline{v}/c$$



Consider the x -component of the motion (transverse to the streaming direction). s is the coordinate of the "design" (ideal) orbit (equivalent to z for a linear accelerator) and subscripts "comoving" indicate coordinates comoving with the design particle.

We may transform to s as the independent variable:

$$dt = \frac{ds}{v_z}; \quad v_x = \frac{dx}{dt} = v_z x' \quad \text{where prime ' = } \frac{d}{ds}$$

$$v_z \frac{d}{ds} (\gamma m v_z x') = q(\underline{E} + \underline{v} \times \underline{B})_x$$

$$\gamma m v_z^2 x'' + x' m v_z \frac{d(\gamma v_z)}{ds} = q(\underline{E} + \underline{v} \times \underline{B})_x$$

$$\Rightarrow x'' + \left[\frac{1}{\gamma v_z} \frac{d(\gamma v_z)}{ds} \right] x' = \frac{q}{\gamma m v_z^2} (\underline{E} + \underline{v} \times \underline{B})_x$$

Now consider an unbunched beam of uniform charge density ρ and circular cross section, with radius r_b

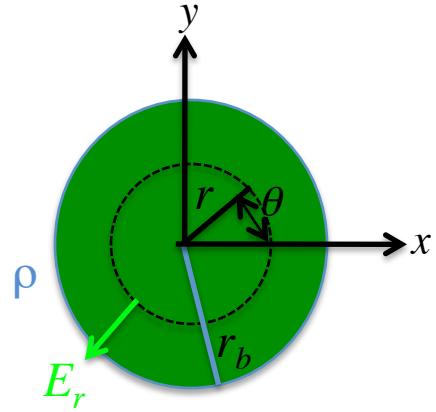
Line charge density $\lambda = \pi r_b^2 \rho$

First calculate electric field:

$$\nabla \cdot \underline{E} = \frac{\rho}{\epsilon_0}$$

$$2\pi r E_r = \pi r^2 \frac{\rho}{\epsilon_0} \quad (\text{Gauss theorem})$$

$$\Rightarrow E_r = \frac{\rho}{2\epsilon_0} r = \frac{\lambda}{2\pi\epsilon_0} \frac{r}{r_b^2} \quad E_x = E_r \cos\theta = E_r \frac{x}{r} = \frac{\lambda}{2\pi\epsilon_0} \frac{x}{r_b^2} y$$



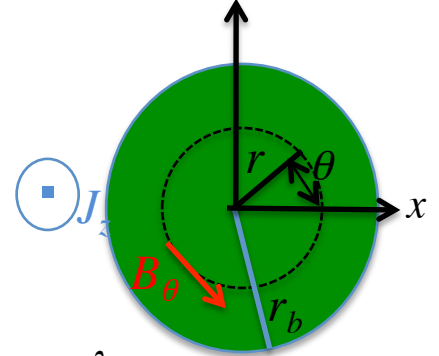
Similarly, calculate the magnetic field:

$$\nabla \times \underline{B} = \mu_0 \underline{J}$$

$$2\pi r B_\theta = \mu_0 \pi r^2 \rho v_z \quad (\text{Stokes theorem})$$

$$\Rightarrow B_\theta = \mu_0 \frac{\lambda v_z}{2\pi\epsilon_0} \frac{r}{r_b^2} \quad B_y = B_\theta \cos\theta = B_\theta \frac{x}{r} = \mu_0 \frac{\lambda v_z}{2\pi} \frac{x}{r_b^2}$$

$$(B_z = 0)$$



Let $(\underline{E} + \underline{v} \times \underline{B})_x = (E_x - v_z B_y)^{self} + (E_x + v_y B_z - v_z B_y)^{ext}$

$$\Rightarrow x'' + \left[\frac{1}{\gamma v_z} \frac{d(\gamma v_z)}{ds} \right] x' = \frac{q}{\gamma m v_z^2} \frac{\lambda}{2\pi\epsilon_0} \frac{x}{r_b^2} [1 - \mu_0 \epsilon_0 v_z^2] + \frac{q}{\gamma m v_z^2} (E_x + v_y B_z - v_z B_y)^{ext}$$

Now $\mu_0 \epsilon_0 = \frac{1}{c^2}$; Assuming $\beta_x^2 + \beta_y^2 \ll \frac{1}{\gamma^2} \Rightarrow \gamma^2 \cong \frac{1}{1 - v_z^2/c^2}$ (Paraxial approximation)

($\gamma^2 \cong 1/(1 - v_z^2/c^2)$) equivalent to assuming $\beta_x^{comoving}, \beta_y^{comoving} \ll 1$).

$$\Rightarrow x'' + \left[\frac{1}{\gamma v_z} \frac{d(\gamma v_z)}{ds} \right] x' = \frac{q}{\gamma^3 m v_z^2} \frac{\lambda}{2\pi\epsilon_0} \frac{x}{r_b^2} + \frac{q}{\gamma m v_z^2} (E_x + v_y B_z - v_z B_y)^{ext}$$

First consider the self-field.

$$x'' + \left[\frac{1}{\gamma v_z} \frac{d(\gamma v_z)}{ds} \right] x' = \frac{q}{\gamma^3 m v_z^2} \frac{\lambda}{2\pi\epsilon_0 r_b^2} x + \frac{q}{\gamma m v_z^2} (E_x + v_y B_z - v_z B_y)^{ext}$$

$$= Q \frac{x}{r_b^2} + \frac{q}{\gamma m v_z^2} (E_x + v_y B_z - v_z B_y)^{ext}$$

$$Q \equiv \frac{q}{\gamma^3 m v_z^2} \frac{\lambda}{2\pi\epsilon_0} \equiv \text{Generalized Perveance} \rightarrow \begin{cases} \frac{\lambda}{4\pi\epsilon_0 V} & \text{for } \gamma^2 v_z^2 \ll c^2 \\ \frac{\lambda}{2\pi\epsilon_0 V \left(\frac{qV}{mc^2} \right)^2} & \text{for } \gamma^2 v_z^2 \gg c^2 \end{cases}$$

$$\equiv \frac{(q/e)}{m/m_{amu}} \frac{2I}{I_0} \frac{1}{\gamma^3 \beta^3} \quad \text{where } I_0 \equiv \frac{4\pi\epsilon_0 m_{amu} c^3}{e} \approx 31 \text{ MA}$$

Here $qV = (\gamma - 1)mc^2 =$ ion kinetic energy,
 e is the proton charge, and m_{amu} is the atomic mass unit.
 Also note in the non-relativistic limit:

$$Q \equiv \frac{1}{4\pi\epsilon_0} \left(\frac{m}{2q} \right)^{1/2} \left(\frac{I}{V^{3/2}} \right) \quad (\text{non - relativistic})$$

(same scaling as original term "perveance" characterizing injectors)

$$Q \equiv \frac{\phi_{self}}{V} = \frac{\int_0^{r_b} (E_r - v_z B_\theta) dr}{V} = \frac{\text{Potential energy of beam particle}}{\text{Kinetic energy of beam particle}}$$

Now consider the external field. We often try to create focusing forces that are linear in x (examples are: electric or magnetic quadrupoles, solenoids, Einzel lenses.) So let this focusing force be represented by $K(s)$.

$$\frac{q}{\gamma m v_z^2} (E_x + v_y B_z - v_z B_y)^{ext} = K(s)x$$

$$x'' + \left[\frac{1}{\gamma v_z} \frac{d(\gamma v_z)}{ds} \right] x' = Q \frac{x}{r_b^2} + \frac{q}{\gamma m v_z^2} (E_x + v_y B_z - v_z B_y)^{ext}$$

$$= Q \frac{x}{r_b^2} + K(s)x$$

The focusing forces are often periodic:

$K(s) = K(s + L_p)$ where L_p = period of focusing element
(when $dv_z/ds = 0$, and Q is periodic with period L_p , then:

$$x'' = f(s)x \text{ where } f(s) \text{ is periodic. (Hill's equation).}$$

For some purposes a suitable constant can be found which captures the "average" variation (over several periods) of the particle motion (continuous focusing approximation)

Then we replace the effects of the periodic lattice with a single focusing parameter $k_{\beta 0}^2$

$$x'' + \left[\frac{1}{\gamma v_z} \frac{d(\gamma v_z)}{ds} \right] x' = Q \frac{x}{r_b^2} - k_{\beta 0}^2 x$$

$k_{\beta 0}$ is defined as the "undepressed" betatron frequency

$$x'' + \left[\frac{1}{\gamma v_z} \frac{d(\gamma v_z)}{ds} \right] x' = Q \frac{x}{r_b^2} - k_{\beta 0}^2 x$$

Consider a drifting beam ($dv_z/ds = 0$). The particle equation becomes:

$$\begin{aligned} x'' &= Q \frac{x}{r_b^2} - k_{\beta 0}^2 x \\ &= -k_{\beta 0}^2 \left(1 - \frac{Q}{k_{\beta 0}^2 r_b^2} \right) x \end{aligned}$$

This is simple harmonic oscillator equation.

Note some frequently encountered definitions:

$$k_{\beta 0}^2 \left(1 - \frac{Q}{k_{\beta 0}^2 r_b^2} \right) \equiv k_{\beta}^2 \equiv \text{depressed betatron frequency}$$

Define also

$\sigma_0 \equiv k_{\beta 0} L_p \equiv$ undepressed phase advance (per period)

and

$\sigma \equiv k_{\beta} L_p \equiv$ depressed phase advance (per period) (includes space charge)

$$\frac{\sigma}{\sigma_0} \equiv \frac{k_{\beta}}{k_{\beta 0}} = \left(1 - \frac{Q}{k_{\beta 0}^2 r_b^2} \right) = \text{tune depression}$$

Examples: $\frac{\sigma}{\sigma_0} = 0 \Rightarrow$ Fully tune depressed

$\frac{\sigma}{\sigma_0} = 1 \Rightarrow$ No space charge depression

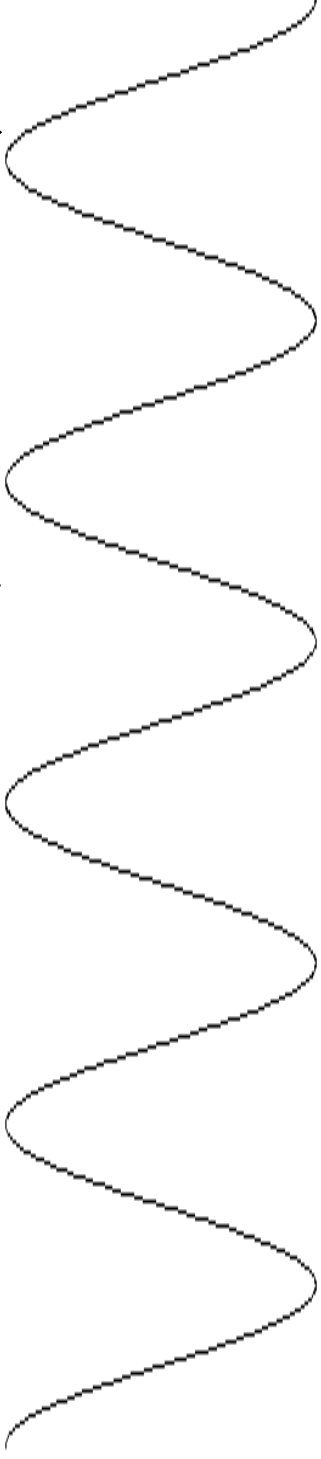
(so two dimensionless parameters: Q characterizes space charge relative to ion kinetic energy, σ/σ_0 characterizes space charge force relative to focusing force)

Space charge reduces betatron phase advance

Without space charge:

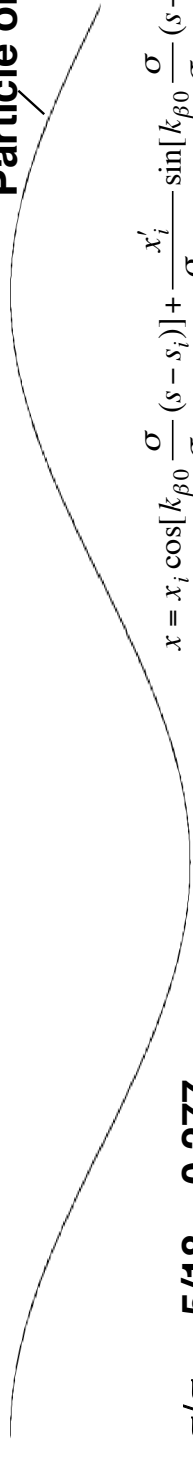
$$x = x_i \cos[k_{\beta 0}(s - s_i)] + \frac{x'_i}{k_{\beta 0}} \sin[k_{\beta 0}(s - s_i)]$$

Particle orbit



With space charge:

Particle orbit



$$x = x_i \cos\left[k_{\beta 0} \frac{\sigma}{\sigma_0} (s - s_i)\right] + \frac{x'_i}{\frac{\sigma}{\sigma_0} k_{\beta 0}} \sin\left[k_{\beta 0} \frac{\sigma}{\sigma_0} (s - s_i)\right]$$

Beam envelope

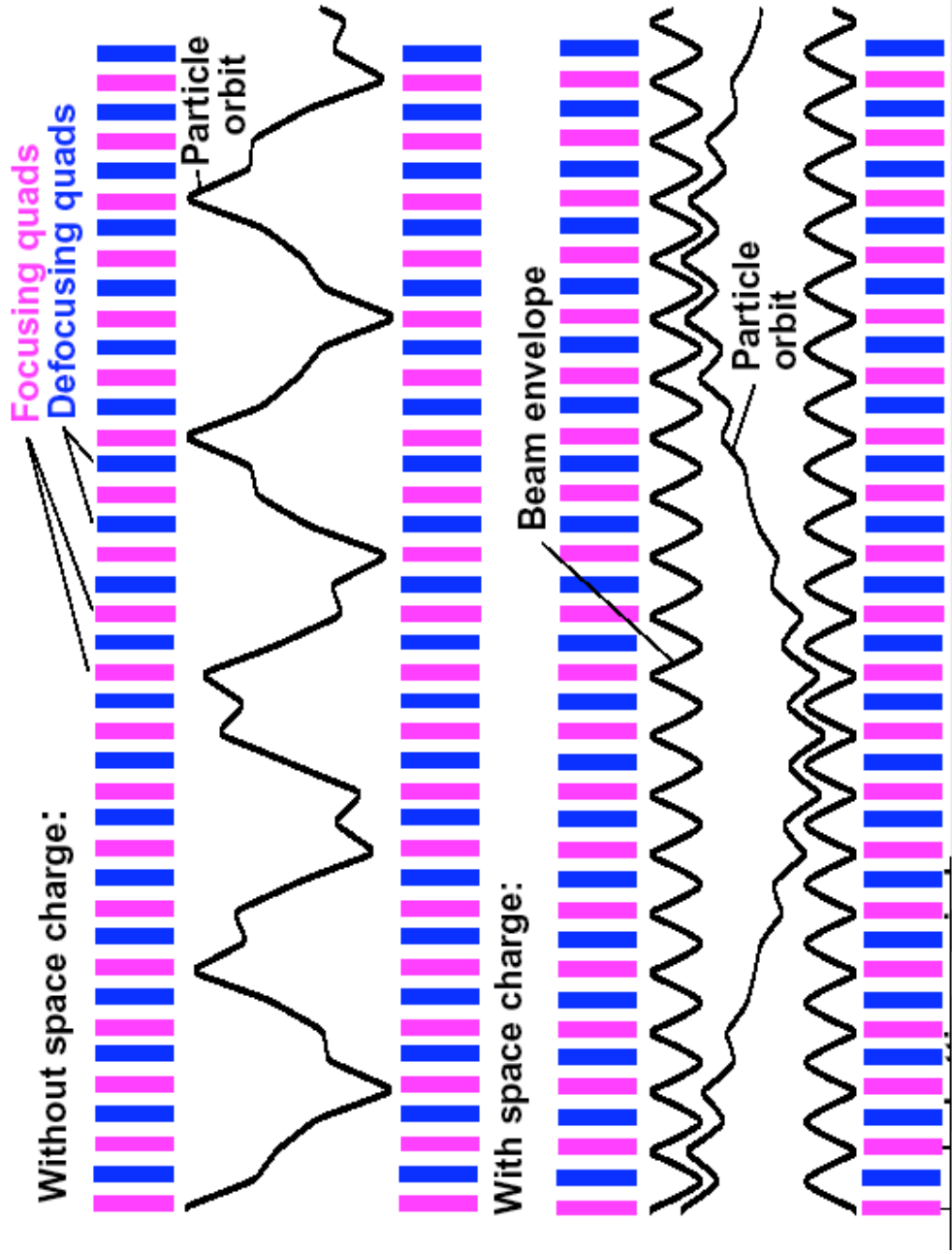
$$\sigma/\sigma_0 \sim 5/18 \sim 0.277$$



The Heavy Ion Fusion Virtual National Laboratory



Space charge reduces betatron phase advance



BENDING BEAMS

RETURNING TO PARTICLE EQUATION WITH ARBITRARILY $\underline{E}, \underline{B}$:

$$x'' + \left[\frac{1}{\gamma m v_z} \frac{d}{dz} (\gamma m v_z) \right] x' = \frac{q}{\gamma m v_z^2} (\underline{E} + \underline{v} \times \underline{B})_x$$

IF EXTERNAL FORCE IS PROPORTIONAL TO $-x$
 \Rightarrow FOCUSING (HARMONIC OSCILLATIONS)

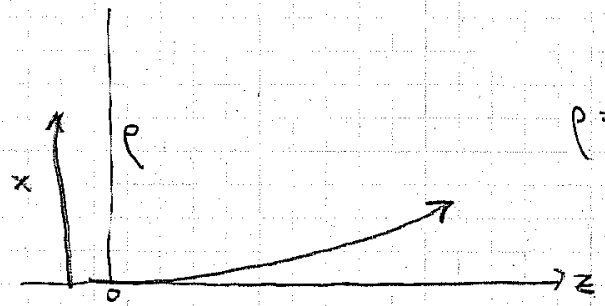
HOWEVER, IF $\underline{E} + \underline{v} \times \underline{B} = \text{CONSTANT}$
 \Rightarrow BENDING

EXAMPLE: If $\underline{B} = B_y \hat{e}_y$
 $\underline{v} = v_0 \hat{e}_z + v_x \hat{e}_x$ where $v_0 \gg v_x$

$$\Rightarrow x'' = \frac{q B_y}{\gamma m v_z^2} = \frac{B_y}{[B\rho]} \quad [B\rho] \equiv \text{RIGIDITY} = \frac{\gamma m v_z}{q} = \frac{p}{q}$$

$$x' = \frac{B_y}{[B\rho]} z + x_0'$$

$$x = \frac{B_y}{[B\rho]} \frac{z^2}{2} + x_0' z + x_0$$



$p = \text{RADIUS OF CURVATURE OF ARC} = \frac{[B\rho]}{B_y}$

(BENDING CAN ALSO BE CARRIED OUT WITH ELECTRIC FIELDS $\underline{E} = \dots$)

Plasma physics of beams

Physics of space charge = physics of space charge

= plasma physics of particle beams

Plasma parameter Λ :

$$\begin{aligned} q\phi_{IP} &= \frac{1}{4\pi\epsilon_0} \frac{q^2}{r_{IP}} \\ &= \frac{1}{4\pi\epsilon_0} q^2 n_0^{1/3} \end{aligned}$$

Average potential energy $q\phi_{IP}$ of particle due to its nearest neighbor a distance $r_{IP} = n_0^{-1/3}$ (q = charge of particle; n_0 = number density)

If $q\phi_{IP} \ll k_B T \Rightarrow$ "Weakly coupled plasma" or simply "plasma"

Define $\lambda_D \equiv \frac{(k_B T / m)^{1/2}}{(q^2 n_0 / (\epsilon_0 m))^{1/2}} \equiv \frac{v_t}{\omega_p} = \left(\frac{k_B T \epsilon_0}{q^2 n_0} \right)^{1/2} =$ Debye Length

= characteristic distance whereby charges are shielded in plasma

Define $\Lambda \equiv \frac{4\pi}{3} n_0 \lambda_D^3 \equiv$ Plasma Parameter

$$\sim \left(\frac{k_B T}{q\phi_{IP}} \right)^{1/2} \gg 1 \quad [\text{if } q\phi_{IP} \ll k_B T]$$

Klimontovich Equation

Ref.: "Introduction to Plasma Theory," D.R. Nicholson, [Wiley, 1983].

$$N(\underline{x}, \underline{v}, t) = \sum_{i=1}^{N_0} \delta(\underline{x} - \underline{X}_i(t)) \delta(\underline{v} - \underline{V}_i(t))$$

$N(\underline{x}, \underline{v}, t)$ is the density of particles in phase space.

Note there are N_0 particles: $\int N(\underline{x}, \underline{v}, t) d^3x d^3v = N_0$

$X_i(t)$ and $V_i(t)$ are position and velocity of the i^{th} particle.

The (non-relativistic) equations of motion are:

$$\dot{\underline{X}}_i = \underline{V}_i \quad m \dot{\underline{V}}_i = q \underline{E}^m(\underline{X}_i(t), t) + q[\underline{V}_i \times \underline{B}^m(\underline{X}_i(t), t)]$$

$$\text{Let } u = x - X_i(t) \Rightarrow \frac{\partial f(u)}{\partial x} = f'(u) \quad \text{and} \quad \frac{\partial f(u)}{\partial t} = -\dot{X}(t) f'(u) = -\dot{X}(t) \frac{\partial f(u)}{\partial x}$$

So taking the derivative of $N(\underline{x}, \underline{v}, t)$ with respect to t :

$$\begin{aligned} \frac{\partial N}{\partial t}(\underline{x}, \underline{v}, t) &= - \sum_{i=1}^{N_0} \dot{\underline{X}}_i(t) \cdot \underline{\nabla}_x [\delta(\underline{x} - \underline{X}_i(t)) \delta(\underline{v} - \underline{V}_i(t))] \\ &\quad - \sum_{i=1}^{N_0} \dot{\underline{V}}_i(t) \cdot \underline{\nabla}_v [\delta(\underline{x} - \underline{X}_i(t)) \delta(\underline{v} - \underline{V}_i(t))] \end{aligned}$$

Maxwell's equations:

$$\underline{\nabla} \cdot \underline{E}^m = \frac{\rho^m}{\epsilon_0} \equiv \frac{1}{\epsilon_0} q \int N(\underline{x}, \underline{v}, t) d^3v \quad \underline{\nabla} \cdot \underline{B}^m = 0$$

$$\underline{\nabla} \times \underline{E}^m = - \frac{\partial \underline{B}^m}{\partial t} \quad \underline{\nabla} \times \underline{B}^m = \mu_0 \underline{J}^m + \frac{\partial \underline{E}^m}{\partial t} \equiv \mu_0 q \int \underline{v} N(\underline{x}, \underline{v}, t) d^3v + \frac{\partial \underline{E}^m}{\partial t}$$

(Here superscript "m" denotes "microscopic" quantity, not averaged locally over a small volume).

$$\Rightarrow \frac{\partial N}{\partial t}(\underline{x}, \underline{v}, t) = - \sum_{i=1}^{N_0} \underline{V}_i(t) \cdot \underline{\nabla}_x [\delta(\underline{x} - \underline{X}_i(t)) \delta(\underline{v} - \underline{V}_i(t))] \\ - \sum_{i=1}^{N_0} \left(\frac{q}{m} E^m(\underline{X}_i(t), t) + \frac{q}{m} [\underline{V}_i \times \underline{B}^m(\underline{X}_i(t), t)] \right) \cdot \underline{\nabla}_v [\delta(\underline{x} - \underline{X}_i(t)) \delta(\underline{v} - \underline{V}_i(t))]$$

Note that $\underline{V}_i(t) \delta(\underline{v} - \underline{V}_i(t)) = v \delta(\underline{v} - \underline{V}_i(t))$ so

$$\Rightarrow \frac{\partial N}{\partial t}(\underline{x}, \underline{v}, t) = -v \cdot \underline{\nabla}_x \sum_{i=1}^{N_0} \delta(\underline{x} - \underline{X}_i(t)) \delta(\underline{v} - \underline{V}_i(t)) \\ - \left(\frac{q}{m} E^m(\underline{x}, t) + \frac{q}{m} [\underline{v} \times \underline{B}^m(\underline{x}, t)] \right) \cdot \underline{\nabla}_v \sum_{i=1}^{N_0} \delta(\underline{x} - \underline{X}_i(t)) \delta(\underline{v} - \underline{V}_i(t))$$

$$\Rightarrow \frac{\partial N}{\partial t}(\underline{x}, \underline{v}, t) = -v \cdot \underline{\nabla}_x N(\underline{x}, \underline{v}, t) - \left(\frac{q}{m} E^m(\underline{x}, t) + \frac{q}{m} [\underline{v} \times \underline{B}^m(\underline{x}, t)] \right) \cdot \underline{\nabla}_v N(\underline{x}, \underline{v}, t)$$

 Klimontivich Equation

Note that the total derivative of a quantity along an orbit in phase space:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{d\underline{x}}{dt} \Big|_{orbit} \cdot \underline{\nabla}_x + \frac{d\underline{v}}{dt} \Big|_{orbit} \cdot \underline{\nabla}_v$$

$$\Rightarrow \frac{d}{dt} N(\underline{x}, \underline{v}, t) \Big|_{orbit} = 0$$

Note that $N=0$ or infinity, nothing in between!

Average N over some box in phase space. Δx , and Δy are the dimensions of the box. Assume $n^{-1/3} \ll \Delta x \ll \lambda_D$ so that $f(\underline{x}, \underline{v}, t)$ is a smoothly varying function.

Now let
$$f(\underline{x}, \underline{v}, t) = \frac{1}{\Delta x^3 \Delta v^3} \int_{\Delta x^3, \Delta v^3} N(\underline{x}, \underline{v}, t) d^3 x d^3 v \equiv \langle N(\underline{x}, \underline{v}, t) \rangle$$

$$\text{Then } N = f + \delta f \quad f \equiv \langle N \rangle \quad \langle \delta f \rangle = 0$$

$$\underline{E}^m = \underline{E} + \delta \underline{E} \quad \underline{E} = \langle \underline{E}^m \rangle \quad \langle \delta \underline{E} \rangle = 0$$

$$\underline{B}^m = \underline{B} + \delta \underline{B} \quad \underline{B} = \langle \underline{B}^m \rangle \quad \langle \delta \underline{B} \rangle = 0$$

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \underline{\nabla}_x f + \frac{q}{m} (\underline{E} + \underline{v} \times \underline{B}) \cdot \underline{\nabla}_v f = -\frac{q}{m} \langle \delta \underline{E} + \underline{v} \times \delta \underline{B} \rangle \cdot \underline{\nabla}_v f$$

LHS: Smoothly varying part

RHS: Average over
"rapidly fluctuating
quantities", includes
"discrete particle effects"
or "collisions"

If collisions are neglected (so set RHS to zero):
we have the "Vlasov Equation":

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \underline{\nabla}_x f + \frac{q}{m} (\underline{E} + \underline{v} \times \underline{B}) \cdot \underline{\nabla}_v f = 0$$

$$\Rightarrow \frac{d}{dt} f(\underline{x}, \underline{v}, t) \Big|_{orbit} = 0$$

Phase space density on trajectories is constant.
(Liouville's theorem).

The RHS represents the effects of collisions (i.e. interactions with non-smoothly varying fields).

Very heuristically:

$$-\frac{q}{m} \langle \delta \underline{E} + \underline{v} \times \delta \underline{B} \rangle \cdot \underline{\nabla}_v \delta f \sim \nu_c f$$

$$\nu_c \sim \sigma n v$$

$$\sigma \sim \pi r_c^2$$

where σ is the collision cross section.

For a large angle scattering the kinetic energy of a particle will be of order the potential energy at closest approach, defining a collision radius by :

$$k_B T \sim \frac{q^2}{4\pi\epsilon_0 r_c} \Rightarrow r_c \sim \frac{q^2}{4\pi\epsilon_0 k_B T}$$

$$\Rightarrow \nu_c \sim \pi \left(\frac{q^2}{4\pi\epsilon_0 k_B T} \right)^2 n_0 \left(\frac{k_B T}{m} \right)^{1/2}$$

$$\sim \frac{1}{16\pi} \frac{v_{th}}{\lambda_D^3 n_0}$$

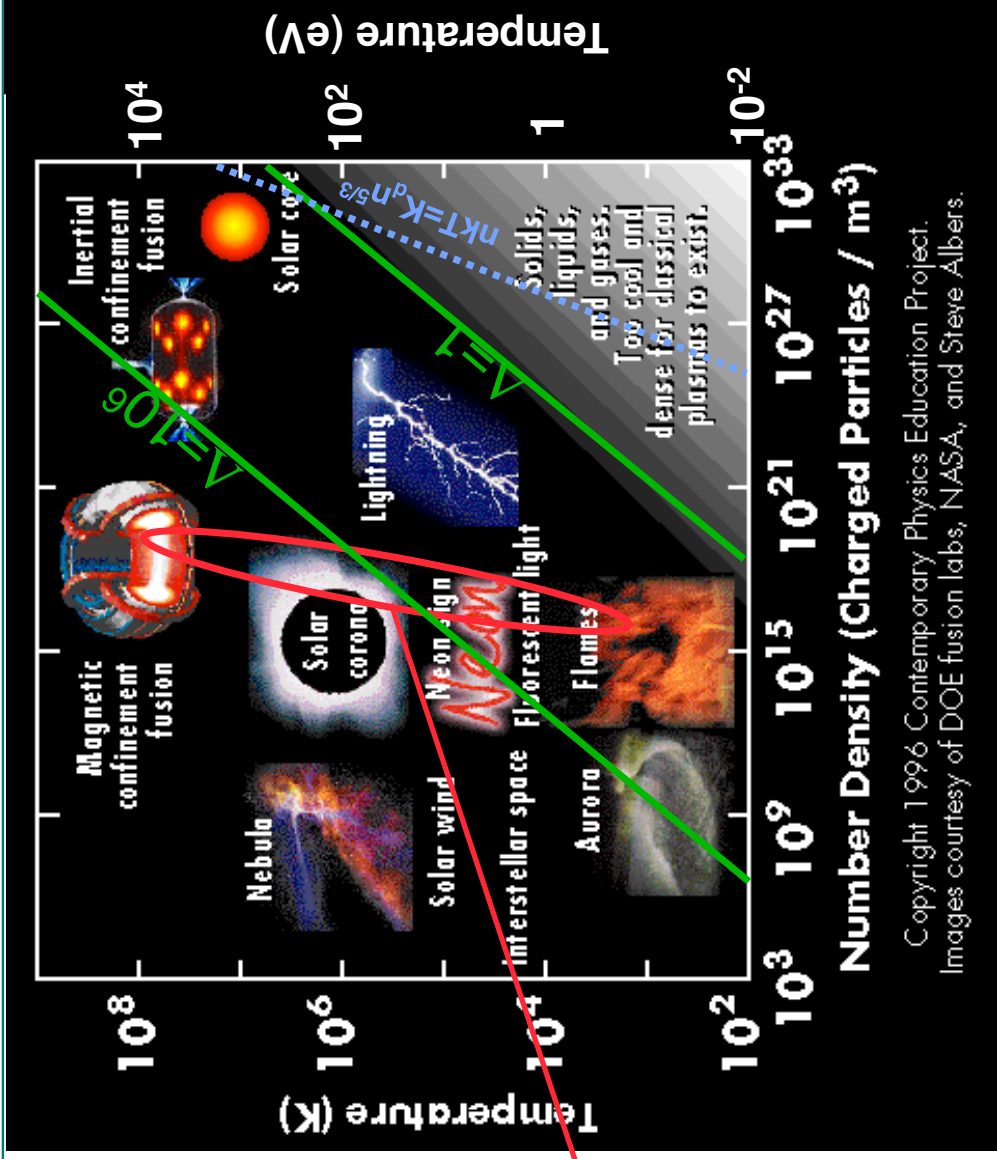
On LHS of Vlasov Equation:

$$\frac{q}{m} \underline{E} \cdot \underline{\nabla}_v f \sim \frac{q}{m} (\lambda_D \underline{\nabla} \cdot \underline{E}) \underline{\nabla}_v f \sim \frac{q}{m} \left(\frac{q \lambda_D n_0}{\epsilon_0} \right) \frac{f}{v_{th}} \sim \frac{\omega_p^2 \lambda_D}{v_{th}} f \quad \text{where } v_t \sim \left(\frac{k_B T}{m} \right)^{1/2}$$

$$\sim \omega_p f$$

$$\frac{\text{Collision term}}{\text{LHS}} \sim \frac{1}{16\pi \lambda_D^3 n_0} \sim \frac{1}{16\Lambda}$$

Accelerator beams are non-neutral plasmas



Accelerator beams
for Heavy Ion Fusion

Copyright 1996 Contemporary Physics Education Project.
Images courtesy of DOE fusion labs, NASA, and Steve Albers.

Phase space density conservation

Liouville's theorem: $\frac{df}{dt} = 0$ along a trajectory in phase space.

Let $dN = f \, dx \, dy \, dz \, dp_x \, dp_y \, dp_z$

The continuity equation in phase space is:

$$\frac{\partial f}{\partial t} + \underline{\nabla}_6 \cdot (f \underline{v}_6) = 0$$

$$\text{where } \underline{v}_6 = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} \text{ and } \underline{\nabla}_6 \cdot \underline{a}_6 = \frac{\partial a_1}{\partial q_1} + \frac{\partial a_2}{\partial q_2} + \frac{\partial a_3}{\partial q_3} + \frac{\partial a_4}{\partial p_1} + \frac{\partial a_5}{\partial p_2} + \frac{\partial a_6}{\partial p_3}$$

If the system is governed by a Hamiltonian $H(\underline{q}, \underline{p}, t)$

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

$$\text{Now } \underline{\nabla}_6 \cdot \underline{v}_6 = \sum_{i=1}^3 \left(\frac{\partial}{\partial q_i} \left(\frac{dq_i}{dt} \right) + \frac{\partial}{\partial p_i} \left(\frac{dp_i}{dt} \right) \right) = \sum_{i=1}^3 \left(\frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} \right) = 0$$

$$\Rightarrow \frac{\partial f}{\partial t} + \underline{\nabla}_6 \cdot (f \underline{v}_6) = \frac{\partial f}{\partial t} + \cancel{f \underline{\nabla}_6 \cdot \underline{v}_6} + \underline{v}_6 \cdot \underline{\nabla}_6 f = 0$$

0

$$\Rightarrow \frac{df}{dt} = 0 \text{ along a 6D trajectory}$$

Emittance & BRIGHTNESS

LIOUVILLE'S EQUATION OR VLAIVU EQUATION $\Rightarrow \frac{dN}{dx dy dz dx dp_y dp_z} = \text{const}$

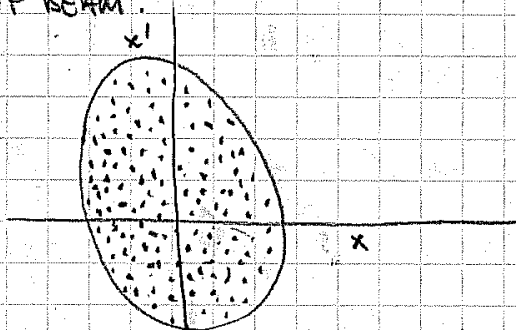
IF $x'' = f(x)$ AND NOT FUNCTIONS (y & z)
 $y'' = f(y) = \dots (x, \& z)$
 $z'' = f(z) = \dots (x, \& y)$

THEN $\frac{dN}{dx dp_x} = \text{const}; \frac{dN}{dy dp_y} = \text{const} \& \frac{dN}{dz dp_z} = \text{const}$

separately.

1st DEFINITION:

EMITTANCE: USE TRACE-SPACE OF ALL PARTICLES IN A GIVEN SLICE OF BEAM.



INSTEAD OF p_x USE $x' = \frac{v_x}{v_z}$ (FOR NON-ACCELERATING PARAXIAL BEAM, x' PROPORTIONAL TO MOMENTUM)

EMITTANCE $\equiv \frac{1}{\pi}$ AREA OF SMALLEST ELLIPSE WHICH ENCLOSES

ALL PARTICLES. (TRACE-SPACE DEFINITION)

(INTUITIVELY, PRODUCT OF WIDTH IN x , TIMES WIDTH IN x' , SO IT IS ESSENTIALLY (WITHIN FACTOR OF π) = PHASE SPACE AREA OF BEAM.

2ND DEFINITION INVOLVES STATISTICAL AVERAGES OF 2ND ORDER QUANTITIES (SUCH AS RMS).

$$\epsilon_x \equiv 4 (\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2)^{1/2}$$

For an upright uniform beam (in phase space): $\langle x^2 \rangle = \frac{x_{\max}^2}{4}$ $\langle x'^2 \rangle = \frac{x'_{\max}^2}{4}$

$\& \langle x x' \rangle = 0$

$$\Rightarrow \epsilon_x = \sqrt{x_{\max} x'_{\max}} = \frac{\text{Area}}{\pi}$$

NORMALIZED EMITTANCE

For a beam that is accelerating, return to x, p_x as definition of phase space area:

$$p_x = \gamma m v_x = \gamma m v_z x' \quad \text{AGAIN, ASSUMING } v \approx v_z$$

$$\Rightarrow \epsilon_{Nx} \equiv \frac{4}{m c} \left(\langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2 \right)^{1/2} = \gamma \beta \epsilon_x$$

$$= \frac{4}{m c} \left(\langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2 \right)^{1/2}$$

SINCE EMITTANCE IS THE AVERAGE PHASE SPACE AREA OF BEAM (AVERAGING OVER EMITTANCE SIZE) THE EMITTANCE IN GENERAL GROWS AS A BEAM FILMENTS (ENGULFING EMITTANCE).

BRIGHTNESS

THE DENSITY OF PARTICLES IN 6-D PHASE SPACE IS:

$$\frac{dN}{dx dy dz dx' dy' dz'} = f \quad \leftarrow \text{MICROSCOPIC DENSITY}$$

DEFINE A QUANTITY \bar{f} WHICH IS THE PHASE-SPACE DENSITY IN AN AVERAGE SENSE

$$\bar{f} = \left\langle \frac{dN}{dx dx' dy dy' dz dz'} \right\rangle = \frac{(I dt)/q}{\pi^3 \epsilon_{Nx} \epsilon_{Ny} \epsilon_{Nz}}$$

Note $f(x, p) = \text{constant}$ along a trajectory, whereas \bar{f} usually is a decreasing function of z .

NORMALIZED BRIGHTNESS $B_N \equiv \frac{I}{\epsilon_{Nx} \epsilon_{Ny}}$

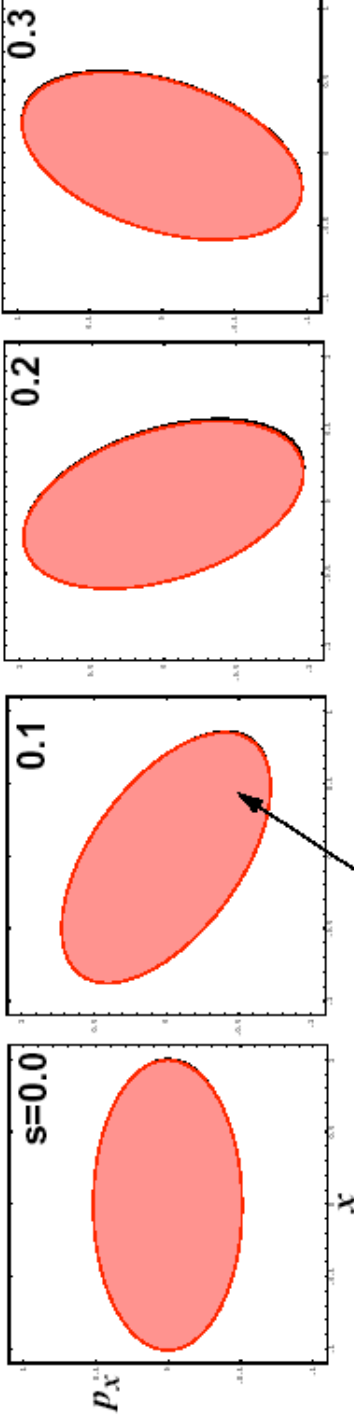
IS A USEFUL MEASURE OF 4D AVERAGE PHASE SPACE DENSITY, (if $dt = \text{constant}$, & all motion is uncoupled.)

For non-accelerating beams the unnormalized brightness B (also if $dt = \text{const.}$ & all motion uncoupled):

$$\Rightarrow B \equiv \frac{I}{\epsilon_x \epsilon_y} \quad \text{MEASURES PHASE SPACE DENSITY.}$$

Emittance is constant for linear force profiles and matched beams

Linear force profile ($x'' = -k^2 x$) \Rightarrow Phase space area preserved, ellipse stays elliptical.



Emittance = phase space area
 Emittance constant if forces linear
 Here, width of beam is oscillating or "mismatched."

Non-linear forces (e.g. $x'' = -k^2 x + \epsilon x^3$) \Rightarrow position-dependent frequency
 \Rightarrow phase mixing, increasing effective area \Rightarrow Emittance increases if forces non-linear

