

John Barnard
Steven Lund
USPAS
January 19-30, 2015
Hampton, Virginia

II. Envelope Equations

Paraxial Ray Equation

Envelope equations for axially
symmetric beams

Cartesian equation of motion

Envelope equations for elliptically
symmetric beams

Roadmap:

Single particle equation with Lorentz force
 $q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$



Make use of:

1. Paraxial (near-axis) approximation
(Small r and r')
2. Conservation of canonical angular momentum
3. Axisymmetry $f(r,z)$



Paraxial Ray Equation for Single Particle

Next take statistical averages over the distribution function

⇒ Moment equations

Express some of the moments in terms of the rms radius and emittance

⇒ Envelope equations (axisymmetric case)

Some focusing systems have quadrupolar symmetry
Rederive envelope equations in cartesian coordinates
(x,y,z) rather than radial (r,z)

Start with Newton's equations with the Lorentz force:

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

In cartesian coordinates this can be written:

$$\frac{d(\gamma m \dot{x})}{dt} = \gamma m \ddot{x} + \dot{\gamma} m \dot{x} = q(E_x + \dot{y}B_z - \dot{z}B_y)$$

$$\frac{d(\gamma m \dot{y})}{dt} = \gamma m \ddot{y} + \dot{\gamma} m \dot{y} = q(E_y + \dot{z}B_x - \dot{x}B_z)$$

$$\frac{d(\gamma m \dot{z})}{dt} = \gamma m \ddot{z} + \dot{\gamma} m \dot{z} = q(E_z + \dot{x}B_y - \dot{y}B_x)$$

In cylindrical coordinates: (use $\frac{d\hat{e}_r}{dt} = \dot{\theta} \hat{e}_\theta$ and $\frac{d\hat{e}_\theta}{dt} = -\dot{\theta} \hat{e}_r$)
(see next page).

$$\frac{d(\gamma m \dot{r})}{dt} - \gamma m r \dot{\theta}^2 = q(E_r + r\dot{\theta}B_z - \dot{z}B_\theta) \quad (I)$$

$$\frac{1}{r} \frac{d(\gamma m r^2 \dot{\theta})}{dt} = q(E_\theta + \dot{z}B_r - \dot{r}B_z) \quad (II)$$

$$\frac{d(\gamma m \dot{z})}{dt} = \gamma q(E_z + \dot{r}B_\theta - r\dot{\theta}B_r) \quad (III)$$

In general $\mathbf{E} = -\nabla\phi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla \times \mathbf{A}$

When $\frac{\partial}{\partial \theta} = 0$:

$$\mathbf{E} = \hat{e}_r \left[\frac{-\partial\phi}{\partial r} - \frac{\partial A_r}{\partial t} \right] + \hat{e}_\theta \left[-\frac{\partial A_\theta}{\partial t} \right] + \hat{e}_z \left[\frac{-\partial\phi}{\partial z} - \frac{\partial A_z}{\partial t} \right]$$

$$\mathbf{B} = \hat{e}_r \left[-\frac{\partial A_\theta}{\partial z} \right] + \hat{e}_\theta \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right] + \hat{e}_z \left[\frac{1}{r} \frac{\partial(rA_\theta)}{\partial r} \right]$$

To calculate the rate of change of the momentum \underline{p} in cylindrical coordinates we must take into account that the unit vectors change directions as the particle moves:

$$\underline{p} = p_r \hat{e}_r + p_\theta^* \hat{e}_\theta + p_z \hat{e}_z = \gamma m \underline{v}$$

where $p_r = \gamma m \dot{r}$

$$p_\theta^* = \gamma m r \dot{\theta}$$

$$p_z = \gamma m \dot{z}$$

Note: on this page p_θ^* = θ -component of mechanical momentum, not to be confused with $p_\theta \equiv \gamma m r^2 \dot{\theta} + q r A_\theta \equiv$ canonical angular momentum.

$$\begin{aligned} \text{So } \frac{d\underline{p}}{dt} &= \dot{p}_r \hat{e}_r + p_r \dot{\hat{e}}_r + \dot{p}_\theta^* \hat{e}_\theta + p_\theta^* \dot{\hat{e}}_\theta + \dot{p}_z \hat{e}_z \\ &= (\dot{p}_r - p_\theta^* \dot{\theta}) \hat{e}_r + (p_r \dot{\theta} + \dot{p}_\theta^*) \hat{e}_\theta + \dot{p}_z \hat{e}_z \end{aligned}$$

where we have used:

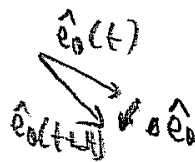
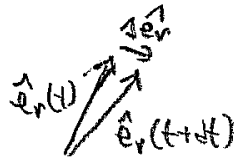
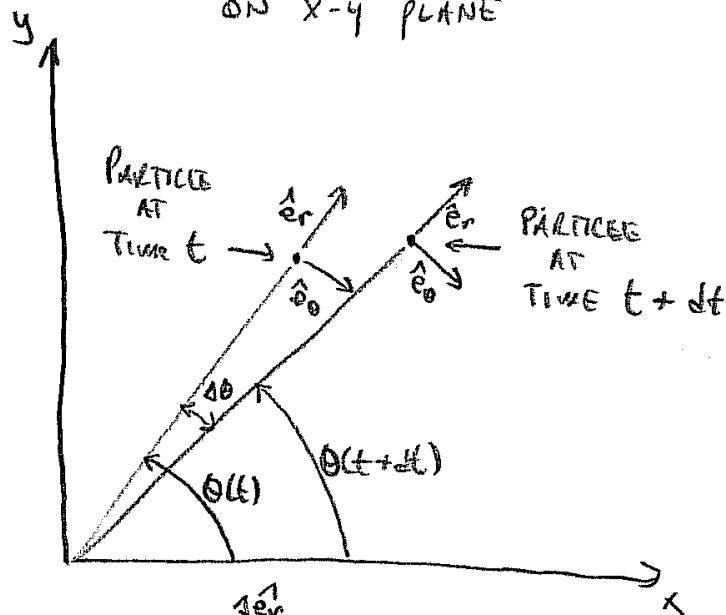
$$\frac{d\hat{e}_r}{dt} = \hat{e}_\theta \dot{\theta} \qquad \frac{d\hat{e}_\theta}{dt} = -\hat{e}_r \dot{\theta}$$

$$\Rightarrow \frac{d\underline{p}}{dt} = \left(\frac{d(\gamma m \dot{r})}{dt} - \gamma m r \dot{\theta}^2 \right) \hat{e}_r + \left(\gamma m \dot{r} \dot{\theta} + \frac{d(\gamma m r \dot{\theta})}{dt} \right) \hat{e}_\theta + \frac{d(\gamma m \dot{z})}{dt} \hat{e}_z$$

Note: second term = $\frac{1}{r} \frac{d}{dt} (\gamma m r^2 \dot{\theta})$

↑
mechanical angular momentum

PROJECTION OF PARTICLE POSITION AT TIMES t & $t+dt$ ON X-Y PLANE (5)



$$\Delta \hat{e}_r = \hat{e}_\theta \Delta \theta$$

$$\Delta \hat{e}_\theta = -\hat{e}_r \Delta \theta$$

$$\frac{d\hat{e}_r}{dt} = \hat{e}_\theta \dot{\theta}$$

$$\frac{d\hat{e}_\theta}{dt} = -\hat{e}_r \dot{\theta}$$

Algebraically

$$\hat{e}_r = \hat{e}_x \cos \theta + \hat{e}_y \sin \theta$$

$$\hat{e}_\theta = -\hat{e}_x \sin \theta + \hat{e}_y \cos \theta$$

$$\Rightarrow \frac{d\hat{e}_r}{dt} = -\hat{e}_x \dot{\theta} \sin \theta + \hat{e}_y \dot{\theta} \cos \theta = \hat{e}_\theta \dot{\theta}$$

$$\text{and } \frac{d\hat{e}_\theta}{dt} = -\hat{e}_x \dot{\theta} \cos \theta - \hat{e}_y \dot{\theta} \sin \theta = -\hat{e}_r \dot{\theta}$$

Conservation of Canonical Angular Momentum

Now the RHS of eq. II multiplied by r can be written:

$$\begin{aligned} qr(E_\theta + \dot{z}B_r - \dot{r}B_z) &= q\left(-\frac{\partial rA_\theta}{\partial t} - \dot{z}\frac{\partial rA_\theta}{\partial z} - \dot{r}\frac{\partial rA_\theta}{\partial r}\right) \\ &= -q\left[\frac{\partial}{\partial t} + \underline{v} \cdot \frac{\partial}{\partial \underline{x}}\right](rA_\theta) \\ &= -q\frac{d(rA_\theta)}{dt} \end{aligned} \quad (IV)$$

So eq. II and eq. IV \Rightarrow
$$\frac{d}{dt}(\gamma mr^2\dot{\theta} + qrA_\theta) = 0$$

Define:

$$p_\theta \equiv \gamma mr^2\dot{\theta} + qrA_\theta \equiv \text{canonical angular momentum}$$

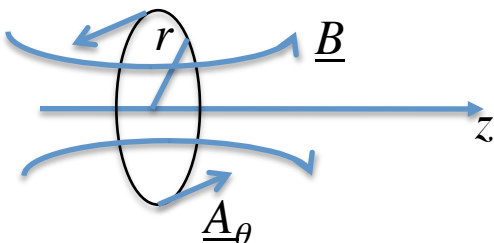
$$\Rightarrow \frac{dp_\theta}{dt} = 0$$

Note that the flux ψ enclosed by a circle of radius r about the origin is given by:

$$\psi = \int \underline{B} \cdot d\underline{S} = \int \underline{\nabla} \times \underline{A} \cdot d\underline{S} = \oint \underline{A} \cdot d\underline{l} = 2\pi rA_\theta$$

$$\text{So } p_\theta = \gamma mr^2\dot{\theta} + \frac{q\psi}{2\pi}$$

is conserved along an orbit in axisymmetric geometries



$d\underline{S}$ =element of area spanning circle; $d\underline{l}$ = line element along circle

"External" electric and magnetic field with azimuthal symmetry ($\partial/\partial\theta = 0$) (cf. Reiser section 5.3)

Consider the field \underline{E}_{ext} and \underline{B}_{ext} created by external sources (time steady, vacuum fields):

$$\nabla \times \underline{B}_{ext} = 0 \quad \nabla \times \underline{E}_{ext} = 0 \quad (\Rightarrow E_{ext}, B_{ext} \sim \nabla\phi)$$

$$\nabla \cdot \underline{B}_{ext} = 0 \quad \nabla \cdot \underline{E}_{ext} = 0 \quad (\Rightarrow \nabla^2\phi = 0)$$

In cylindrical coordinates:

$$\nabla^2\phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial\phi}{\partial r} \right) + \left(\frac{\partial^2\phi}{\partial z^2} \right)$$

$$\text{Let } \phi(r,z) = \sum_{\nu=0}^{\infty} f_{2\nu}(z) r^{2\nu} = f_0(z) + f_2(z)r^2 + f_4(z)r^4 + \dots$$

$$\nabla^2\phi = 0 \quad \Rightarrow \quad \sum_{\nu=1}^{\infty} (2\nu)^2 f_{2\nu}(z) r^{2\nu-2} + \sum_{\nu=0}^{\infty} f_{2\nu}''(z) r^{2\nu} = 0$$

Let $B_z(0,z) = B(z) = -f_0'(z)$ and let $\phi(0,z) = V(z) = f_0(z)$

$$\begin{aligned} B_z(r,z) &= -\frac{\partial\phi(r,z)}{\partial z} = -f_0'(z) + \frac{1}{4} f_0'''(z) r^2 - \frac{1}{64} f_0''''(z) r^4 + \dots \\ &= B(z) - \frac{r^2}{4} \frac{d^2 B(z)}{dz^2} + \frac{r^4}{64} \frac{d^4 B(z)}{dz^4} + \dots \end{aligned}$$

$$\begin{aligned} B_r(r,z) &= -\frac{\partial\phi(r,z)}{\partial r} = \frac{1}{2} f_0''(z) r - \frac{1}{16} f_0''''(z) r^3 + \dots \\ &= -\frac{r}{2} \frac{dB(z)}{dz} + \frac{r^3}{16} \frac{d^3 B(z)}{dz^3} + \dots \end{aligned}$$

Similarly, for the electric field define

Let $\phi(0,z) = V(z) = f_0(z)$

$$\phi(r,z) = V(z) - \frac{r^2}{4} \frac{d^2V(z)}{dz^2} + \frac{r^4}{64} \frac{d^4V(z)}{dz^4} + \dots$$

$$\begin{aligned} E_r(r,z) &= -\frac{\partial\phi(r,z)}{\partial r} = \frac{1}{2} f_0''(z)r - \frac{1}{16} f_0''''(z)r^3 + \dots \\ &= \frac{r}{2} V_0''(z) + \frac{r^3}{16} \frac{d^4V(z)}{dz^4} + \dots \end{aligned}$$

$$\begin{aligned} E_z(r,z) &= -\frac{\partial\phi(r,z)}{\partial z} = -f_0'(z) + \frac{1}{4} f_0'''(z)r^2 - \frac{1}{64} f_0'''''(z)r^4 + \dots \\ &= -V_0'(z) + \frac{r^2}{4} \frac{d^3V(z)}{dz^3} - \frac{r^4}{64} \frac{d^5V(z)}{dz^5} + \dots \end{aligned}$$

(9)

RETURNING TO THE RADIAL COMPONENT OF THE
MOMENTUM EQUATION IN CYLINDRICAL COORDINATES (EQ I):

$$\frac{d}{dt}(\gamma m \dot{r}) - \gamma m r \dot{\theta}^2 = q(E_r + r \dot{\theta} B_z - \dot{z} B_\theta) \quad (I)$$

for the external field use (keeping only terms through
linear order in r)

$$E_{r \text{ ext}} = \frac{r}{z} V'' + O(r^3)$$

$$B_{z \text{ ext}} = B_z(z) + O(r^3)$$

$$B_{\theta \text{ ext}} = 0 \quad [\text{since } \frac{\partial \Phi_{\text{ext}}}{\partial \theta} = 0]$$

for the self field use:

$$E_{r \text{ self}} = \text{non-zero (to be shown)}$$

$$B_{z \text{ self}} = 0 \text{ in paraxial approx. (} v_\theta B_{z \text{ self}} \sim (\omega_c r_b/c)^2 E_{r \text{ self}} \text{)}$$

$$B_{\theta \text{ self}} = \text{non-zero (to be shown)}$$

We let:

$$\underline{B} = \underline{B}_{\text{ext}} + \underline{B}_{\text{self}}$$

$$\underline{E} = \underline{E}_{\text{ext}} + \underline{E}_{\text{self}}$$

Paraxial ray equation:

$$\frac{d(\gamma m \dot{r})}{dt} - \gamma m r \dot{\theta}^2 = q(E_r + r \dot{\theta} B_z - \dot{z} B_\theta)$$
$$\equiv q\left(\frac{V''}{2} r + r \dot{\theta} B(z)\right) + q(E_r^{self} + r \dot{\theta} B_z^{self} - \dot{z} B_\theta^{self})$$

Now use s as the independent variable: $v_z dt = ds$

$$v_z \frac{d(\gamma m v_z r')}{ds} - \gamma m v_z^2 r \theta'^2 = q\left(\frac{V''}{2} r + r v_z \theta' B(z)\right) + q(E_r^{self} - v_z B_\theta^{self})$$

Expanding 1st term, using $v_z \cong v$ and dividing by $\gamma m v^2 (= \gamma m \beta^2 c^2)$:

$$r'' - r \theta'^2 + \frac{(\gamma \beta)'}{\gamma \beta} r' = \frac{q}{\gamma m \beta^2 c^2} \left(\frac{V''}{2} r + r \beta c \theta' B + E_r^{self} - v_z B_\theta^{self} \right) \quad (\text{PI})$$

Define $\omega_c \equiv qB/m$. Using definition of p_θ eliminate θ' via:

$$\theta' = \frac{p_\theta - q\psi/(2\pi)}{\gamma m r^2 \beta c} = \frac{p_\theta}{\gamma m r^2 \beta c} - \frac{qB}{2\gamma m \beta c} = \frac{p_\theta}{\gamma m r^2 \beta c} - \frac{\omega_c}{2\gamma \beta c}$$

Adding the two θ' terms in equation (PI):

$$\begin{aligned} -r \theta'^2 - \frac{r \omega_c \theta'}{\gamma \beta c} &= \frac{-p_\theta^2}{\gamma^2 m^2 \beta^2 c^2 r^3} + \frac{p_\theta \omega_c}{\gamma^2 m \beta^2 c^2 r} - \frac{r \omega_c^2}{4\gamma^2 \beta^2 c^2} \\ &\quad - \frac{p_\theta \omega_c}{\gamma^2 m \beta^2 c^2 r} + \frac{r \omega_c^2}{2\gamma^2 \beta^2 c^2} \\ &= \frac{-p_\theta^2}{\gamma^2 m^2 \beta^2 c^2 r^3} + \frac{r \omega_c^2}{4\gamma^2 \beta^2 c^2} \end{aligned}$$

So eq. P1 becomes:

$$r'' + \frac{(\gamma\beta)'}{\gamma\beta} r' = \frac{q}{\gamma m \beta^2 c^2} \left(\frac{V''}{2} r \right) - \frac{r \omega_c^2}{4 \gamma^2 \beta^2 c^2} + \frac{p_\theta^2}{\gamma^2 m^2 \beta^2 c^2 r^3} + \frac{q}{\gamma m \beta^2 c^2} (E_r^{self} - v_z B_\theta^{self}) \quad (P2)$$

Now

$$\frac{d\gamma mc^2}{dt} = q \underline{E} \cdot \underline{v} \Rightarrow \gamma' mc^2 = q \frac{\underline{E} \cdot \underline{v}}{v_z} \cong q E_z \quad \text{so } \gamma'' = -\frac{q}{mc^2} \left(V'' + \frac{\partial^2 \phi^{self}}{\partial z^2} \right)$$

How do we calculate $\frac{q}{\gamma m \beta^2 c^2} \left(\frac{V''}{2} r + E_r^{self} - v_z B_\theta^{self} \right)$?

$$\nabla^2 \phi^{self} = -\frac{\rho}{\epsilon_0} \Rightarrow \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi^{self}}{\partial r} \right) = -\frac{\rho}{\epsilon_0} - \frac{\partial^2 \phi^{self}}{\partial z^2}$$

$$\Rightarrow \frac{\partial}{\partial r} \left(r \frac{\partial \phi^{self}}{\partial r} \right) = -\frac{r \rho}{\epsilon_0} - \frac{r \partial^2 \phi^{self}}{\partial z^2}$$

$$r \frac{\partial \phi^{self}}{\partial r} = -\frac{1}{2\pi\epsilon_0} \int_0^r 2\pi \tilde{r} \rho(\tilde{r}) d\tilde{r} - \int_0^r \tilde{r} \frac{\partial^2 \phi^{self}}{\partial z^2} d\tilde{r}$$

$$= -\frac{\lambda(r)}{2\pi\epsilon_0} - \frac{r^2}{2} \frac{\partial^2 \phi^{self}}{\partial z^2}$$

(Here we have included only the lowest order term for $\frac{\partial^2 \phi^{self}}{\partial z^2}$).

$$\Rightarrow E_r^{self} \cong \frac{\lambda(r)}{2\pi\epsilon_0 r} + \frac{r}{2} \frac{\partial^2 \phi^{self}}{\partial z^2}$$

$$\underline{\nabla} \times \underline{B}^{self} = \mu_0 \underline{J} \Rightarrow 2\pi r B_\theta^{self} = \mu_0 \int_0^r 2\pi \tilde{r} J_z(\tilde{r}) d\tilde{r} = \mu_0 v_z \lambda(r)$$

$$B_\theta^{self} = \frac{\mu_0 v_z \lambda(r)}{2\pi r} = \frac{v_z}{c^2} \frac{\lambda(r)}{2\pi\epsilon_0 r}$$

$$\left(\frac{V''}{2} r + E_r^{self} - v_z B_\theta^{self} \right) = \frac{r}{2} \left(V'' + \frac{\partial^2 \phi^{self}}{\partial z^2} \right) + \left(1 - \frac{v_z^2}{c^2} \right) \frac{\lambda(r)}{2\pi\epsilon_0 r}$$

$$= -\frac{\gamma'' mc^2}{2q} r + \frac{1}{\gamma^2} \frac{\lambda(r)}{2\pi\epsilon_0 r}$$

Leading to the "Paraxial Ray Equation:"

$$r'' + \frac{(\gamma\beta)'}{\gamma\beta} r' + \frac{\gamma''}{2\gamma\beta^2} r + \left(\frac{\omega_c}{2\gamma\beta c}\right)^2 r + \left(\frac{p_\theta}{\gamma\beta mc}\right)^2 \frac{1}{r^3} - \frac{q}{\gamma^3 m \beta^2 c^2} \frac{\lambda(r)}{2\pi\epsilon_0 r} = 0$$

↑
↑
↑
↑
↑
↑

Inertial	E_r from converging field lines	Part of centrifugal term	Self-field ($E_r^{self} - v_z B_\theta^{self}$)
Accelerative damping (of angle r')	Solenoidal focusing ($v_\theta B_z -$ part of centrifugal term)		

which together with the conservation of canonical angular momentum,

$$p_\theta \equiv \gamma\beta mcr^2\theta' + \frac{m\omega_c r^2}{2}$$

and initial conditions, specifies the orbit a particle an axisymmetric field.

MOMENT EQUATIONS

Vlasov eqn: $\frac{\partial f}{\partial s} + x' \frac{\partial f}{\partial x} + x'' \frac{\partial f}{\partial x'} + y' \frac{\partial f}{\partial y} + y'' \frac{\partial f}{\partial y'} = 0$

Let $g = g(x, x', y, y')$; $N = \iiint f dx dx' dy dy'$

MULTIPLY Vlasov equation by g & $\frac{1}{N} \iiint dx dx' dy dy'$

$\int dx dx' dy dy' \left[g \frac{\partial f}{\partial s} + g x' \frac{\partial f}{\partial x} + g x'' \frac{\partial f}{\partial x'} + g y' \frac{\partial f}{\partial y} + g y'' \frac{\partial f}{\partial y'} \right] = 0$

$\Rightarrow \frac{d}{ds} \langle g \rangle + \underbrace{\iiint g f \left[- \frac{\partial g}{\partial x} f x' + \dots \right]}_{\text{INTEGRATE BY PARTS}} = 0$

$\rightarrow 0 = \langle x' \frac{\partial g}{\partial x} \rangle$

$\Rightarrow \frac{d}{ds} \langle g \rangle = \langle x' \frac{\partial g}{\partial x} \rangle + \langle x'' \frac{\partial g}{\partial x'} \rangle + \langle y' \frac{\partial g}{\partial y} \rangle + \langle y'' \frac{\partial g}{\partial y'} \rangle$

But $\frac{dg}{ds} = \frac{\partial g}{\partial x} x' + \frac{\partial g}{\partial x'} x'' + \frac{\partial g}{\partial y} y' + \frac{\partial g}{\partial y'} y''$

$\Rightarrow \frac{d}{ds} \langle g \rangle = \langle g' \rangle$

So $\frac{d}{ds} \langle x^2 \rangle = 2 \langle x x' \rangle$

$\frac{d}{ds} \langle x'^2 \rangle = 2 \langle x' x'' \rangle$ etc...

$\frac{d}{ds} \langle x x' \rangle = \langle x x'' \rangle + \langle x'^2 \rangle$

ENVELOPE EQUATION FOR AXISYMMETRIC BEAMS

$$\text{LET } r_b^2 = 2 \langle r^2 \rangle = 2(\langle x^2 \rangle + \langle y^2 \rangle) = 4 \langle x^2 \rangle$$

for an
axisymmetric
beam

$$2r_b r_b' = 4 \langle r r' \rangle \quad \Rightarrow \quad r_b' = \frac{2 \langle r r' \rangle}{r_b}$$

$$\begin{aligned} r_b'' &= \frac{2 \langle r r'' \rangle + 2 \langle r'^2 \rangle}{r_b} - \frac{2 \langle r r' \rangle}{r_b^2} \left(\frac{2 \langle r r' \rangle}{r_b} \right) \\ &= 2 \frac{\langle r r'' \rangle}{r_b} + \frac{4 \langle r'^2 \rangle}{r_b} - 4 \frac{\langle r r' \rangle^2}{r_b^3} \end{aligned}$$

WHAT IS $\langle r r'' \rangle$?

RECALL EQUATION P1 (ON PATH TO MAXWELL EQUATION):

$$r'' - r\theta'^2 + \frac{\gamma'}{\beta^2 \gamma} r' = \frac{q}{\gamma m \beta^2 c^2} \left(\frac{V''}{2} r + r \rho c \theta' B + E_r^{self} - v_z B_\theta^{self} \right)$$

P1 may be rewritten:

$$r'' - r\theta'^2 + \frac{\gamma'}{\beta^2 \gamma} r' = \frac{q}{\gamma m \beta^2 c^2} \left[\frac{-m c^2}{q} \gamma'' \frac{r}{2} + \frac{\lambda(r)}{\gamma^2 2 \pi \epsilon_0 r} + r \rho c \theta' B \right]$$

$$r'' + \frac{\gamma'}{\beta^2 \gamma} r' + \frac{\gamma''}{2 \beta^2 \gamma} r - \frac{q}{\gamma^3 m v_z^2} \frac{\lambda(r)}{2 \pi \epsilon_0 r} - \frac{\omega_c}{\gamma \rho c} \theta' r - r\theta'^2 = 0$$

What is $\langle r r'' \rangle$?

$$\langle r r'' \rangle + \frac{-\omega_c}{\gamma \rho c} \langle \theta' r^2 \rangle - \langle r^2 \theta'^2 \rangle + \dots = 0$$

$$\langle r^2 \rangle = \gamma^2 m^2 \beta^2 c^2 \langle r^2 \theta'^2 \rangle + \frac{\omega_c^2}{4} m^2 \langle r^2 \rangle + \omega_c \gamma m^2 \beta c \langle r \theta' \rangle \langle r^2 \rangle$$

$$\Rightarrow \frac{-\omega_c}{\gamma \rho c} \langle \theta' r^2 \rangle = \frac{-\omega_c}{\gamma \rho c} \left[\frac{\langle r^2 \rangle^2}{\omega_c \gamma m^2 \beta c \langle r^2 \rangle} - \frac{\omega_c \langle r^2 \rangle}{4 \gamma m \rho c} - \frac{\gamma \rho c \langle r^2 \theta'^2 \rangle}{\omega_c \langle r^2 \rangle} \right]$$

$$\Rightarrow \langle r r'' \rangle = \frac{\langle r^2 \rangle^2}{\gamma^2 m^2 \beta^2 c^2 \langle r^2 \rangle} - \frac{\omega_c^2 \langle r^2 \rangle}{4 \gamma m \rho c} - \frac{\langle r^2 \theta'^2 \rangle}{\langle r^2 \rangle} + \langle r^2 \theta'^2 \rangle + \dots = 0$$

⇒

$$\langle r r'' \rangle = \frac{\gamma'}{\beta^2 \gamma} \langle r r' \rangle + \frac{\gamma''}{2\beta^2 \gamma} \langle r^2 \rangle - \frac{q}{\gamma^3 m v_e^2} \frac{\langle \lambda(r) \rangle}{2\pi \epsilon_0} + \frac{\langle p_0 \rangle^2}{(\gamma m \beta c)^2 \langle r^2 \rangle} - \frac{\omega_c^2 \langle r^2 \rangle}{4(\gamma^2 \beta c)^2} - \frac{\langle r^2 \theta'^2 \rangle}{\langle r^2 \rangle} + \langle r^2 \theta''^2 \rangle$$

$$r_b'' = \frac{2 \langle r r'' \rangle}{r_b} + \frac{4 \langle r^2 \rangle \langle r''^2 \rangle - 4 \langle r r' \rangle^2}{r_b^3}$$

$$\begin{aligned} &= \frac{\gamma'}{\beta^2 \gamma} \frac{2 \langle r r' \rangle}{r_b} + \frac{\gamma''}{2\beta^2 \gamma} \frac{2 \langle r^2 \rangle}{r_b} - \frac{2q}{\gamma^3 m v_e^2} \frac{\langle \lambda(r) \rangle}{2\pi \epsilon_0} \frac{1}{r_b} \\ &+ \frac{\langle p_0 \rangle^2}{(\gamma m \beta c)^2} \frac{2}{\langle r^2 \rangle r_b} - \frac{\omega_c^2}{4(\gamma \beta c)^2} \frac{2 \langle r^2 \rangle}{r_b} - \frac{2 \langle r^2 \theta'^2 \rangle}{r_b \langle r^2 \rangle} \\ &+ \frac{2 \langle r^2 \theta''^2 \rangle}{r_b} + \frac{4 \langle r^2 \rangle \langle r''^2 \rangle - 4 \langle r r' \rangle^2}{r_b^3} \end{aligned}$$

Using $r_b^2 \equiv 2 \langle r^2 \rangle$ & $r_b' = \frac{2 \langle r r' \rangle}{r_b}$

ENVELOPE EQUATION

$$\Rightarrow \left[r_b'' + \frac{\gamma'}{\beta^2 \gamma} r_b' + \frac{\gamma''}{2\beta^2 \gamma} r_b + \left(\frac{\omega_c}{2\gamma \beta c} \right)^2 r_b + \frac{-4 \langle p_0 \rangle^2}{(\gamma m \beta c)^2 r_b^3} - \frac{E_r^2}{r_b^3} - \frac{Q}{r_b} = 0 \right]$$

WHERE $E_r^2 = 4(\langle r^2 \rangle \langle r''^2 \rangle - \langle r r' \rangle^2) + \langle r^2 \rangle \langle r^2 \theta''^2 \rangle - \langle r^2 \theta'^2 \rangle^2$

ENVELOPE EQUATION -- CONTINUED

$$r_b'' + \frac{\gamma'}{\beta^2 \gamma} r_b' + \frac{\gamma''}{2\beta^2 \gamma} r_b + \left(\frac{\omega_c}{2\gamma \beta c} \right)^2 r_b - \frac{4 \langle p_0 \rangle^2}{(\gamma m \beta c)^2} r_b^3 - \frac{E_r^2}{r_b^3} - \frac{Q}{r_b} = 0$$

COMPARE WITH THE SINGLE PARTICLE PARAXIAL RAY EQUATION:

$$r'' + \underbrace{\frac{\gamma'}{\beta \gamma}}_{\text{INERTIAL}} r' + \underbrace{\frac{\gamma''}{2\beta \gamma}}_{E_r} r + \underbrace{\left(\frac{\omega_c}{2\gamma \beta c} \right)^2}_{V_0 B_z - \text{CENTRIFUGAL}} r - \underbrace{\left(\frac{p_0}{\gamma m \beta c} \right)^2 \frac{1}{r^3}}_{\text{CENTRIFUGAL}} - \underbrace{\frac{q}{\gamma^3 m v_z^2} \frac{\chi(r)}{2\pi \epsilon_0 r}}_{E_r - V_0 B_z \text{ self field}} = 0$$

$$E_r^2 = 4(\langle r^2 \rangle \langle r'^2 \rangle - \langle r r' \rangle^2) + \langle r^2 \rangle \langle r^2 \theta'^2 \rangle - \langle r^2 \theta' \rangle^2$$

NOTE THAT FOR AXISYMMETRIC BEAMS ($\rho = \rho(r)$ ONLY)

$$\langle r^2 \rangle = \langle x^2 \rangle + \langle y^2 \rangle = 2 \langle x^2 \rangle$$

$$\Rightarrow 2 \langle r r' \rangle = 4 \langle x x' \rangle$$

$$\& \langle x'^2 \rangle + \langle y'^2 \rangle = 2 \langle x'^2 \rangle = \langle r'^2 \rangle + \langle r^2 \theta'^2 \rangle$$

DEFINE $E_x^2 = 16(\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2)$

$$\Rightarrow \boxed{E_r^2 = E_x^2 - 4 \langle r^2 \theta' \rangle^2}$$

EXAMPLES OF
SYSTEMS WITH AXIAL SYMMETRY

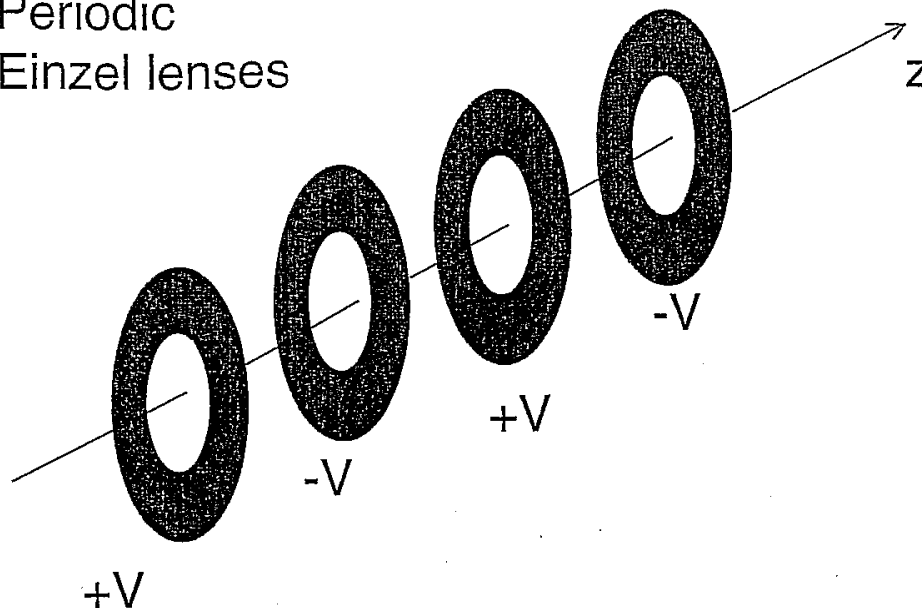
- PERIODIC SOLENOIDS
- EINZEL LENSES
- CONTINUOUS FOCUSING

EXAMPLES OF
SYSTEMS WITHOUT AXIAL SYMMETRY

- ELECTRIC OR MAGNETIC QUADRUPOLE
- ⇒ USE CARTESIAN COORDINATES WITH
ELLIPTICAL SPACE CHARGE SYMMETRY

EXAMPLES OF AXISYMMETRIC SYSTEMS

Periodic Einzel lenses



PERIODIC SOLENOIDS

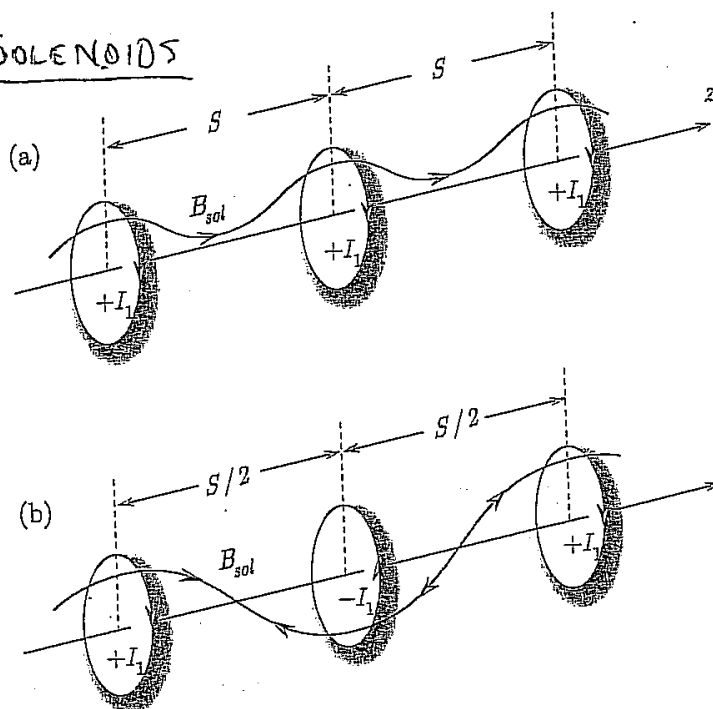


Figure 3.2. Schematic of magnet sets producing a periodic focusing solenoidal field with axial periodicity length S . In Fig. 3.2 (a), successive coils are spaced by S and have the same current polarity $+I_1, +I_1, \dots$. In Fig. 3.2 (b), successive coils are spaced by $S/2$ and have alternating current polarities $+I_1, -I_1, +I_1, \dots$.

(FIGURE FROM
DAVIDSON & QIN
2003) P. 55
"PHYSICS OF
INTENSE CHARGED
PARTICLE BEAMS
IN HIGH ENERGY
ACCELERATORS"

EXAMPLE OF NON-AXISYMMETRIC SYSTEM

Figure from
Davidson & Qin, 2003.

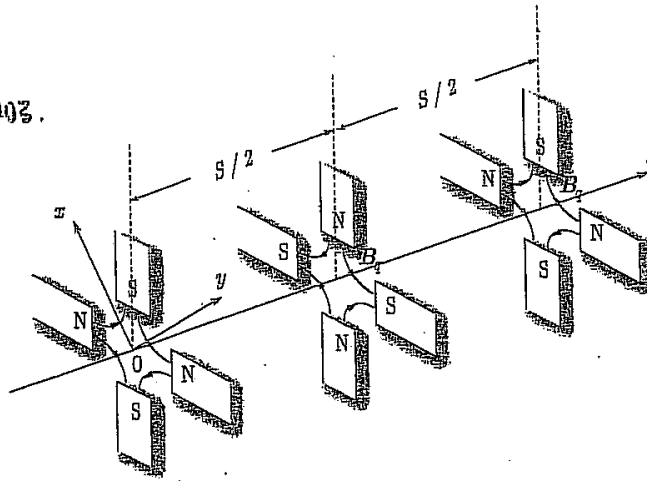


Figure 3.1. Schematic of magnet sets producing an alternating-gradient quadrupole field with axial periodicity length S .

NON-AXISYMMETRIC SYSTEM

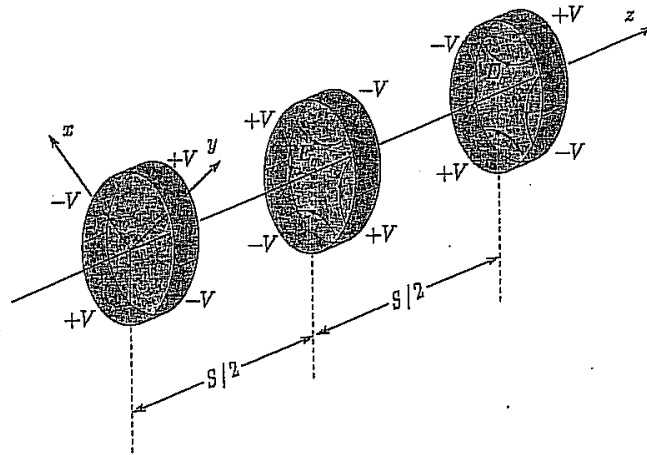


Figure 3.3. Schematic of conductor configuration with applied voltages producing an alternating-gradient quadrupole electric field with axial periodicity length S .

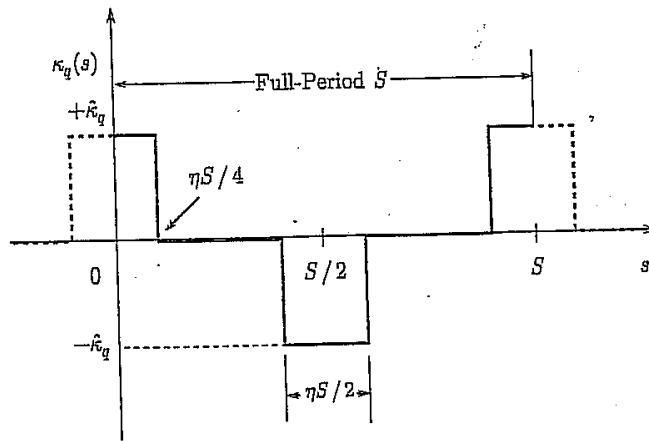


Figure 3.7. Alternating step-function model of a periodic quadrupole lattice with filling factor η for the lens elements. The figure shows a plot of the quadrupole coupling coefficient $\kappa_q(s)$ versus s for one full period (S) of the lattice. Such a configuration is often called a FODO transport lattice (acronym for focusing-off-defocusing-off).

FIGURES FROM DAVIDSON & QIN, 2003

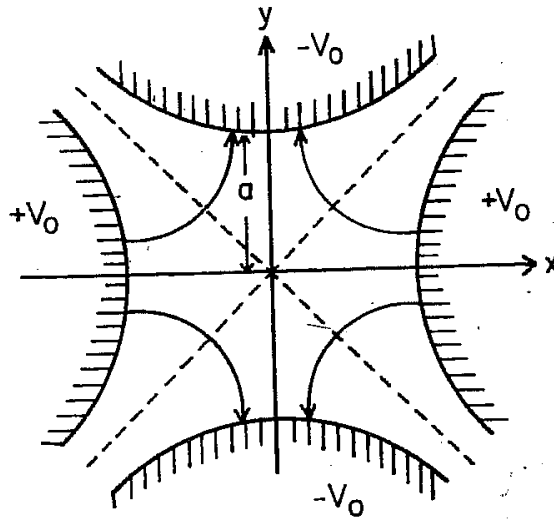
J. BANWANI
 (22)

2 ≡ BEAM OPTICS AND FOCUSING SYSTEMS WITHOUT SPACE CH

FROM
 REISER, p. 112

$$E_x = -E'x$$

$$E_y = E'y$$



$$F_x = -qE'x$$

$$F_y = qE'y$$

ELECTROSTATIC
 QUADS

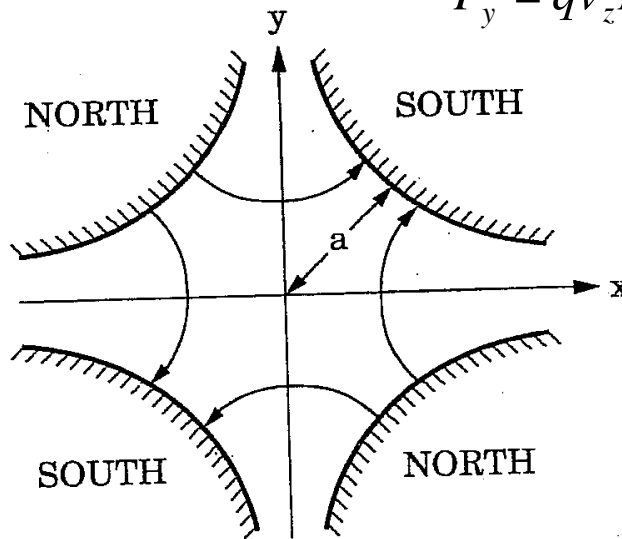
Figure 3.15. Electrodes and force lines in an electrostatic quadrupole.

$$B_x = B'y$$

$$B_y = B'x$$

$$F_x = -qv_z B'x$$

$$F_y = qv_z B'y$$



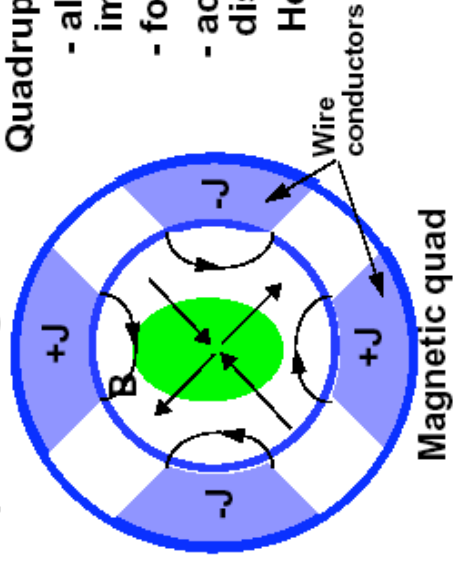
MAGNETIC
 QUADS

Heavy ion accelerators use alternating gradient quadrupoles to confine the beams

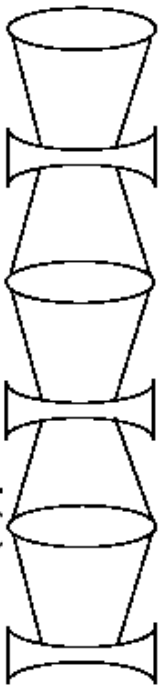
Space-charge forces and thermal forces act to expand beam

Quadrupoles (magnetic or electric):

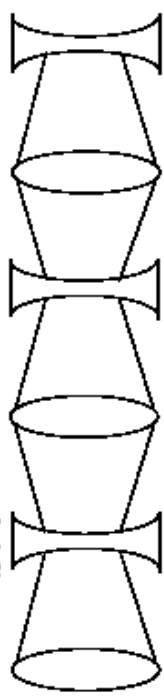
- alternately provide inward then outward impulse
- focus in one plane and defocus in other
- act as linear lenses. (Force proportional to distance from axis).



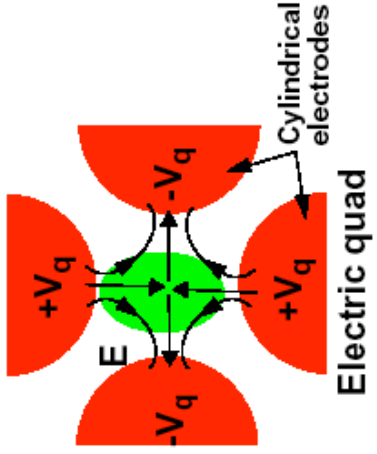
Horizontal (x) plane:



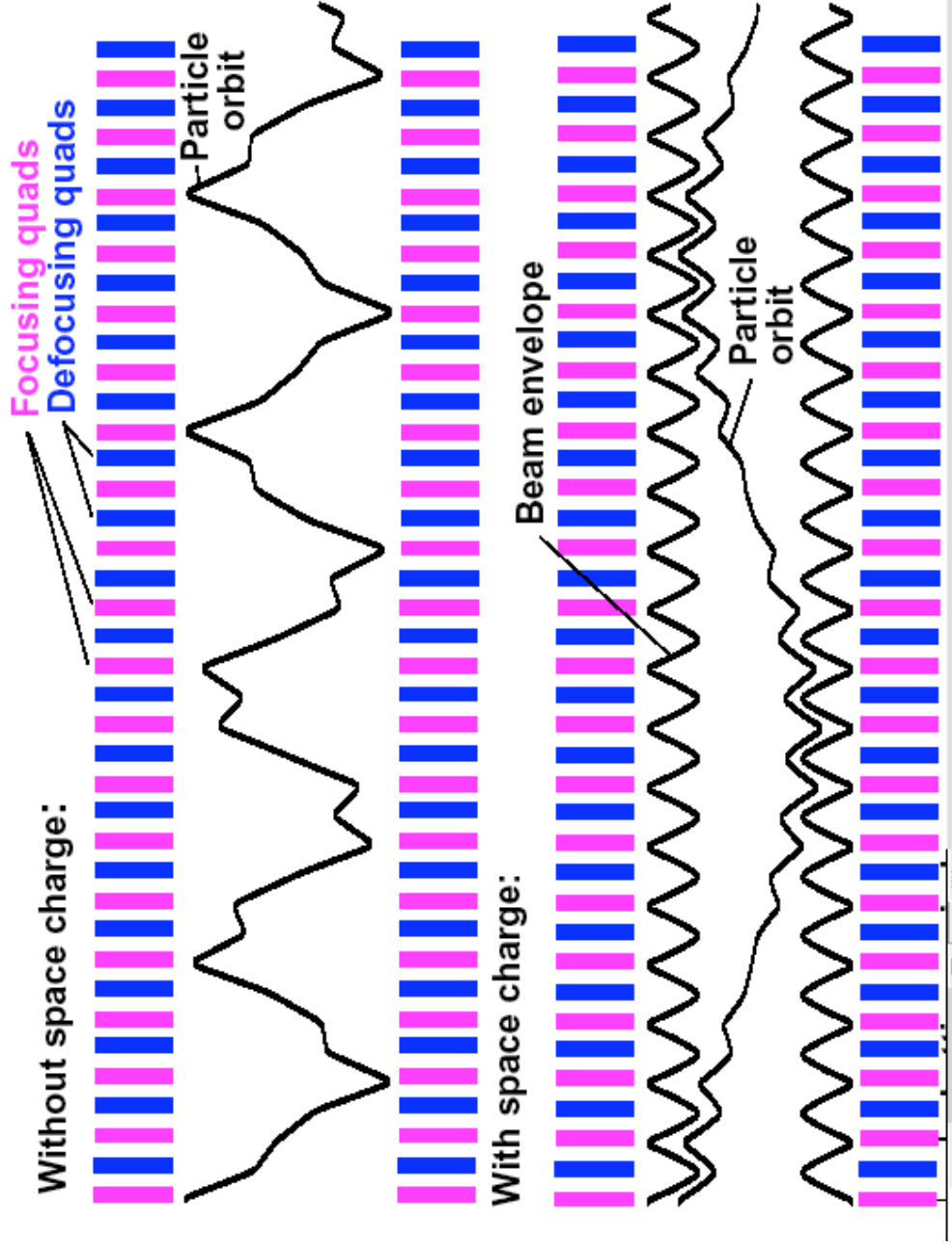
Vertical (y) plane:



Average displacement is larger in focusing lenses so the net effect is focusing.



Space charge reduces betatron phase advance



ENVELOPE EQUATIONS FOR NON-AXISYMMETRIC SYSTEMS

(25)

$$r_x^2 \equiv 4 \langle x^2 \rangle \quad r_y^2 \equiv 4 \langle y^2 \rangle$$

$$2 r_x r_x' = 8 \langle x x' \rangle$$

$$r_x' = \frac{4 \langle x x' \rangle}{r_x}$$

$$\begin{aligned} r_x'' &= \frac{4 \langle x x'' \rangle}{r_x} + \frac{4 \langle x'^2 \rangle}{r_x} - \frac{4 \langle x x' \rangle}{r_x^2} r_x' \\ &= \frac{4 \langle x x'' \rangle}{r_x} + \frac{16 \langle x'^2 \rangle \langle x^0 \rangle}{r_x^2} - \frac{16 \langle x x' \rangle^2}{r_x^3} \end{aligned}$$

DEFINE $E_x^2 = 16 (\langle x'^2 \rangle \langle x^0 \rangle - \langle x x' \rangle^2)$

$$\Rightarrow \boxed{r_x'' = \frac{4 \langle x x'' \rangle}{r_x} + \frac{E_x^2}{r_x^3}}$$

SO HOW DO WE CALCULATE $\langle x x'' \rangle$?

RETURN TO SINGLE PARTICLE EQUATION (IN CARTESIAN COORDINATES)

$$\frac{d}{dt} (\gamma m \dot{x}) = \gamma m \ddot{x} = q (E_x + \dot{y} B_z - \dot{z} B_y)$$

\downarrow
 x''
& similarly
 y''

\downarrow
QUADRUPOLE FOCUSING
SPACE-CHARGE OF ELLIPTICAL
BEAMS

TO BE CONTINUED ...

J. BARWAD

QUADRUPOLE FOCUSING

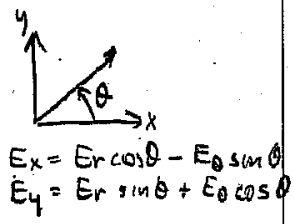
Now, relax radial symmetry:

For $\nabla \cdot \mathbf{E} = 0$ or $\nabla \times \mathbf{B} = 0$

EXPAND FIELD IN CYLINDRICAL "MULTIPOLES":

$$E_r, B_r = \sum_{n=1}^{\infty} f_n r^{n-1} \cos(n\theta)$$

$$E_\theta, B_\theta = \sum_{n=1}^{\infty} f_n r^{n-1} \sin(n\theta)$$



$n=1 \Rightarrow$ dipole $\begin{cases} E_r = f_1 \cos \theta \\ E_\theta = -f_1 \sin \theta \end{cases} \Rightarrow \begin{cases} E_x = f_1 \\ E_y = 0 \end{cases}$

$n=2 \Rightarrow$ quadrupole $\begin{cases} E_r = f_2 r \cos 2\theta \\ E_\theta = -f_2 r \sin 2\theta \end{cases} \Rightarrow \begin{cases} E_x = f_2 x \\ E_y = -f_2 y \end{cases}$

NOTE: ABOVE EXPANSION IS VALID WHEN E or $B \neq$ function(z).
 FOR MAGNETS OF FINITE AXIAL EXTENT, FOR EACH FUNDAMENTAL n -pole, A SET OF HIGHER ORDER MULTIPOLES WITH SAME AZIMUTHAL SYMMETRY ARE REQUIRED TO SATISFY $\nabla^2 \phi = 0$.

FOR EXAMPLE FOR A FUNDAMENTAL QUADRUPOLE THE FIELD MAY BE EXPANDED:

$$E_r = \sum_{\nu=0}^{\infty} f_{2,\nu}(z) [1+\nu] r^{1+2\nu} \cos[2\theta]$$

$$E_\theta = \sum_{\nu=0}^{\infty} -f_{2,\nu}(z) r^{1+2\nu} \sin[2\theta]$$

$$E_z = \sum_{\nu=0}^{\infty} \frac{1}{2} \frac{df_{2,\nu}}{dz} r^{2+2\nu} \cos 2\theta$$

with $f_{2,\nu+1}(z) = \frac{-1}{4(\nu+1)(\nu+3)} \frac{d^2 f_{2,\nu}(z)}{dz^2}$

SEE LUND, S. M. (1996)
 FOR EXAMPLE. HIF note 96-1
 LLNL.

42-182 100 SHEETS
 National Grand
 Made in U.S.A.