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Injectors and longitudinal physics -- II

1. Acceleration - introduction
2. Space charge of short bunches (rf)
3. Space charge of long bunches
4. Longitudinal space charge waves
5. Longitudinal rarefaction waves and bunch ends

Summary of fluid equations

$$\text{Let} \quad n(\underline{x},t) = \int d^3p \; f(\underline{x},\underline{p},t) \qquad \text{PARTICLE DENSITY}$$

$$\bar{\underline{v}}(\underline{x},t) \equiv \frac{1}{n(\underline{x},t)} \int d^3p \; \frac{\underline{p}}{\gamma m} \; f(\underline{x},\underline{p},t) \qquad \text{FLUID VELOCITY}$$

$$\bar{\underline{p}}(\underline{x},t) \equiv \frac{1}{n(\underline{x},t)} \int d^3p \; \underline{p} \; f(\underline{x},\underline{p},t) \qquad \text{FLUID MOMENTUM}$$

$$\underline{\underline{P}}(\underline{x},t) \equiv \int d^3p \; (\underline{p} - \bar{\underline{p}})\left(\frac{\underline{p}}{\gamma m} - \bar{\underline{v}}\right) f(\underline{x},\underline{p},t) \qquad \text{PRESSURE TENSOR}$$

$$\frac{d\underline{x}}{dt} = \frac{\underline{p}}{\gamma m} \qquad \frac{d\underline{p}}{dt} = q\left(\underline{E}(\underline{x},t) + \frac{\underline{p}}{\gamma m} \times \underline{B}(\underline{x},t)\right) \qquad \gamma^2 = \frac{\underline{p} \cdot \underline{p}}{(mc)^2} + 1$$

CONTINUITY EQUATION:
$$\frac{\partial n(\underline{x},t)}{\partial t} + \frac{\partial}{\partial \underline{x}} \cdot n(\underline{x},t)$$

MOMENTUM EQUATION:
$$\frac{\partial \bar{\underline{p}}}{\partial t} + \underline{v} \cdot \frac{\partial \bar{\underline{p}}}{\partial \underline{x}} = q(\underline{E} + \bar{\underline{v}} \times \underline{B}) - \frac{1}{n(\underline{x},t)} \frac{\partial}{\partial \underline{x}} \cdot \underline{\underline{P}}$$

THE ABOVE EQUATIONS ARE RELATIVISTICALLY CORRECT.
IN THE NON-RELATIVISTIC LIMIT THE CONTINUITY EQUATION
REMAINS UNCHANGED & THE MOMENTUM EQUATION MAY BE WRITTEN:

NON RELATIVISTIC →
$$\frac{\partial \bar{\underline{v}}}{\partial t} + \bar{\underline{v}} \cdot \frac{\partial \bar{\underline{v}}}{\partial \underline{x}} = \frac{q}{m}(\underline{E} + \bar{\underline{v}} \times \underline{B}) - \frac{1}{mn} \frac{\partial}{\partial \underline{x}} \cdot \underline{\underline{P}}$$
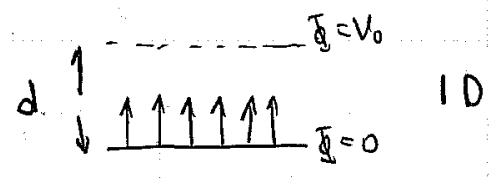
THESE EQUATIONS ARE SUPPLEMENTED WITH MAXWELL'S EQUATIONS!
for $\underline{E}(\underline{x},t)$ & $\underline{B}(\underline{x},t)$

$$\frac{\partial}{\partial \underline{x}} \cdot \underline{E} = \frac{q \, n(\underline{x},t)}{\varepsilon_0} \qquad \frac{\partial}{\partial \underline{x}} \times \underline{E} = \frac{-\partial \underline{B}}{\partial t}$$

$$\frac{\partial}{\partial \underline{x}} \cdot \underline{B} = 0 \qquad \frac{\partial}{\partial \underline{x}} \times \underline{B} = \mu_0 \underline{J} + \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} \qquad \underline{J}(\underline{x},t) = q \, n(\underline{x},t) \bar{\underline{v}}$$

NEED ADDITIONAL EQUATIONS SUCH AS $\underline{\underline{P}} = 0$ OR ENERGY EQUATION
TO TERMINATE SET OF EQUATIONS.

Summary of Child Langmuir Law

$$\tilde{\varphi} = V_0$$

$$d \;\Big\uparrow\uparrow\uparrow\uparrow\uparrow\uparrow \quad \tilde{\varphi} = 0 \qquad 1D \qquad \tilde{\varphi} = -\varphi \; (\text{ACTUAL POTENTIAL})$$

CURRENT DENSITY:
$$J = \frac{4}{9}\varepsilon_0 \left(\frac{2q}{m}\right)^{1/2} \frac{V_0^{3/2}}{d^2}$$

$$\tilde{\varphi}(z) = V_0 \left(\frac{z}{d}\right)^{4/3} \qquad - \text{ELECTROSTATIC POTENTIAL}$$

$$E(z) = \frac{4}{3}\frac{V_0}{d}\left(\frac{z}{d}\right)^{1/3} \qquad \text{ELECTRIC FIELD}$$

$$V(z) = \left(\frac{2q V_0}{m}\right)^{1/2}\left(\frac{z}{d}\right)^{2/3} \qquad \text{LONGITUDINAL VELOCITY}$$

$$\rho(z) = \frac{J}{V(z)} = \left(\frac{J^2 m}{2q V_0}\right)^{1/2}\left(\frac{z}{d}\right)^{-2/3}$$

IF WE MULTIPLY BY πV_b^2 (TO ACCOUNT FOR FINITE RADIUS):

BEAM

$$I = \underbrace{\frac{4}{9}\varepsilon_0 \left(\frac{2q}{m}\right)^{1/2}\left(\frac{V_b}{d}\right)^2 V_0^{3/2}}_{K \equiv \text{GUN perveance} \equiv \dfrac{I}{V_0^{3/2}}} \qquad \left[\begin{array}{c}\text{DIMENSIONS:} \\ \dfrac{\text{CURRENT}}{\text{VOLTAGE}^{3/2}}\end{array}\right]$$

GENERALIZED PERVEANCE: (DIMENSIONLESS)
$$Q(z) = \frac{\lambda}{4\pi\varepsilon_0 \tilde{\varphi}(z)} = \frac{\pi V_b^2 \rho(z)}{4\pi\varepsilon_0 \tilde{\varphi}(z)} = \frac{1}{9}\left(\frac{V_b^2}{z^2}\right)$$

(NOTE THAT CHILD-LANGMUIR LAW ONLY VALID FOR
$z \gg \lambda_D$, WHERE $\frac{1}{2}m V(z)^2 \gg kT$ & $\lambda_D = \frac{V_{th}}{\omega_1} = \frac{\sqrt{kT/m}}{\left(\frac{q N(z)}{\varepsilon_0 m}\right)^{1/2}}$)

ACCELERATION

rf (radio-frequency)
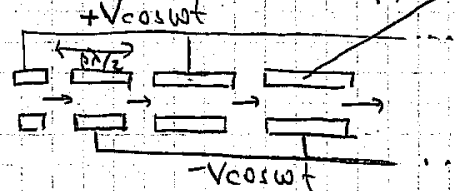
+Vcos ωt ⟶ TUBE SHIELDS BEAM

(Wideröe linac)
LOW FREQUENCIES (< 100 MHz)

−Vcos ωt
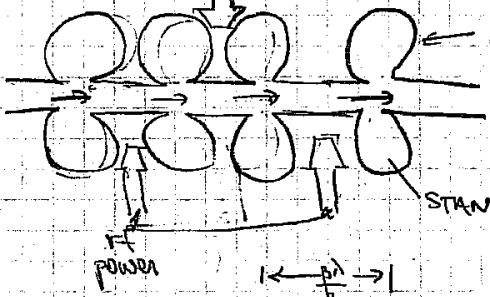
⟶ RESONANT CAVITY

(Coupled Cavity Linac)
$0.4 < \beta < 1.0$

STANDING EM WAVE

FREQUENCIES ~ 100's MHz − ~ 1 GHz

rf power

$|\leftarrow \frac{\beta\lambda}{2} \rightarrow|$
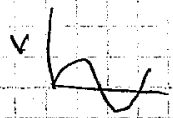
IN EACH GAP $E = E_m \sin \omega t$

Induction acceleration

PULSED POWER
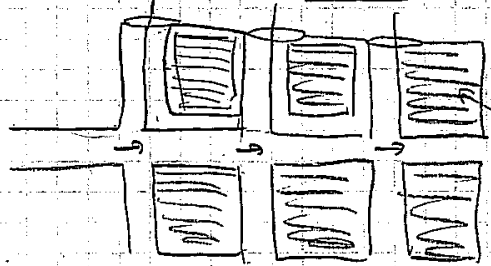
(INDUCTION LINAC)

TOROIDAL
Ferromagnetic
CORE

$\nabla \times E = -\partial B/\partial t$

IN EACH GAP $E = $ CONSTANT
(OR SOME PRESCRIBED
FUNCTION)
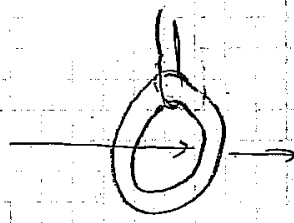
TRANSFORMER

I STRATEGY FOR CALCULATING LONGITUDINAL EQUATIONS OF MOTION

1. rf / short bunches
   - EXTERNAL FIELD: ACCELERATES AND FOCUSES
     - CALCULATE CHANGE IN PHASE AND ENERGY AS PARTICLE MOVES FROM ACCELERATING GAP TO ACCELERATING GAP
     - APPROXIMATE MOTION AS CONTINUOUS
   - SPACE CHARGE FIELD: UNIFORM DENSITY ELLIPSOID

2. INDUCTION / LONG BUNCHES

   - EXTERNAL FIELD: ACCELERATION, FOCUSING, AND COMPRESSION ACCOMPLISHED BY PRESCRIBING VOLTAGE WAVE FORM
   - FOCUSING (CONFINEMENT) IS DONE AT BEAM ENDS

   - SPACE CHARGE FIELD: $E_z \propto \dfrac{\partial \lambda}{\partial z}$

rf longitudinal equation of motion

L_n = center to center distance
between drift tubes

$$E_n = \begin{cases} E_0(-1)^n \cos \omega t & \Leftarrow \pi\text{-mode} \\ E_0 \cos \omega t & \Leftarrow 0\text{-mode} \end{cases}$$
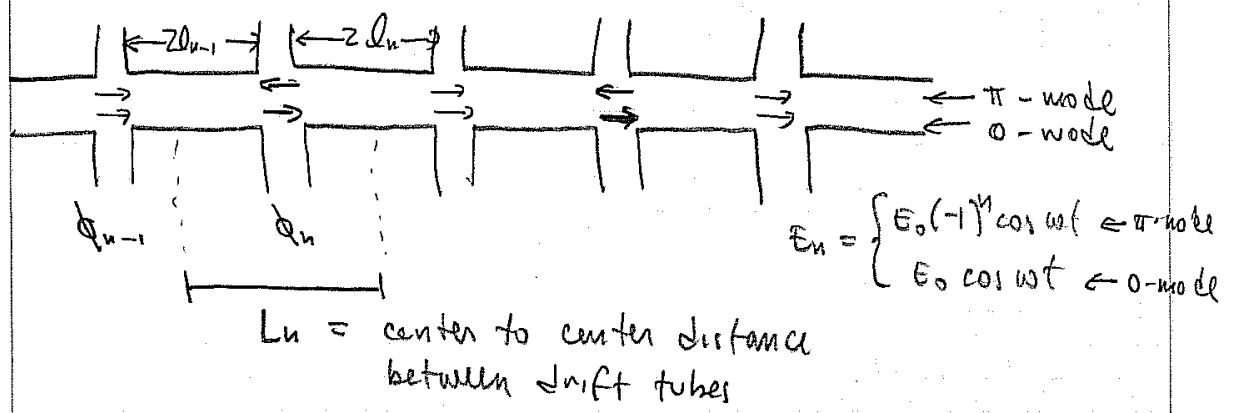
$$E_z = E_0 \cos(\phi_s) \quad \Leftarrow \quad \text{synchronous particle enters}$$
$$\text{each gap at same phase}$$

RESONANCE
CONDITION ON SYNCHRONOUS PARTICLE:

$$\boxed{\ell_{n-1} = \frac{\beta_s \lambda}{2} \begin{cases} \frac{1}{2} \\ 1 \end{cases} \begin{matrix} \pi\text{-mode} \\ \\ 0\text{-mode} \end{matrix}}$$

$$\boxed{\lambda = \frac{2\pi}{\omega} c = \begin{matrix} \text{light travel} \\ \text{distance in one} \\ = c \Delta t \text{ of rf oscillation} \end{matrix}}$$

(IT TAKES $\begin{Bmatrix} 1/2 \\ 1 \end{Bmatrix}$ OSCILLATION PERIOD TO TRAVEL BETWEEN GAPS).

$$\beta_s = \frac{v_s}{c} = \text{velocity of synchronous particle}$$

PARTICLE PHASE RELATIVE TO rf at the nth gap:

$$\phi_n = \phi_{n-1} + \omega \frac{2\ell_{n-1}}{\beta_{n-1} c} + \begin{cases} \pi & \pi\text{-mode} \\ 0 & 0\text{-mode} \end{cases}$$

$$\Delta(\phi - \phi_s)_n = 2\pi \beta_{s,n-1} \left(\frac{1}{\beta_{n-1}} - \frac{1}{\beta_{s,n-1}} \right) \begin{cases} 1/2 \\ 1 \end{cases} \begin{matrix} \pi\text{-mode} \\ 0\text{-mode} \end{matrix}$$

$$\cong -2\pi \frac{\delta\beta}{\beta_{s,n-1}} \begin{cases} 1/2 \\ 1 \end{cases}$$

A VELOCITY
DIFFERENCE
LEADS
TO
A PHASE DIFFERENCE!

$$\Delta(\phi - \phi_s)_n \cong -2\pi \frac{W_{n-1} - W_{s,n-1}}{mc^2 \gamma_{s,n-1}^3 \beta_{s,n-1}^2} \begin{cases} 1/2 \\ 1 \end{cases}$$

$$W = (\gamma - 1)mc^2$$
$$\frac{1}{\beta} - \frac{1}{\beta_s} \cong \frac{-\delta\beta}{\beta_s^2}$$
$$\delta W = \gamma_s^3 \beta_s mc^2 \delta\beta$$

SIMILARLY, A PHASE DIFFERENCE PRODUCES
AN ENERGY CHANGE (RELATIVE TO SYNCHRONOUS PARTICLE)

$$\Delta(W - W_s)_n = q E_0 L_n (\cos\varphi_n - \cos\varphi_{s,n})$$

$$L_n = \frac{(\beta_{s,n-1} + \beta_{s,n})\lambda}{2} \begin{Bmatrix} 1/2 \\ 1 \end{Bmatrix} = \begin{array}{l} \text{CENTER-TO-CENTER} \\ \text{DISTANCE} \\ \text{BETWEEN} \\ \text{DRIFT SECTION} \end{array}$$

$$\left(\Delta W_s = q E_0 L_n \cos\varphi_s \right)$$

ENERGY (VELOCITY) DIFFERENCE $\Rightarrow$
  ARRIVAL TIME DIFFERENCE
    (PHASE DIFFERENCE)

PHASE DIFFERENCE IN rf
FIELD $\Rightarrow$ DIFFERENCE
IN ENERGY GAIN
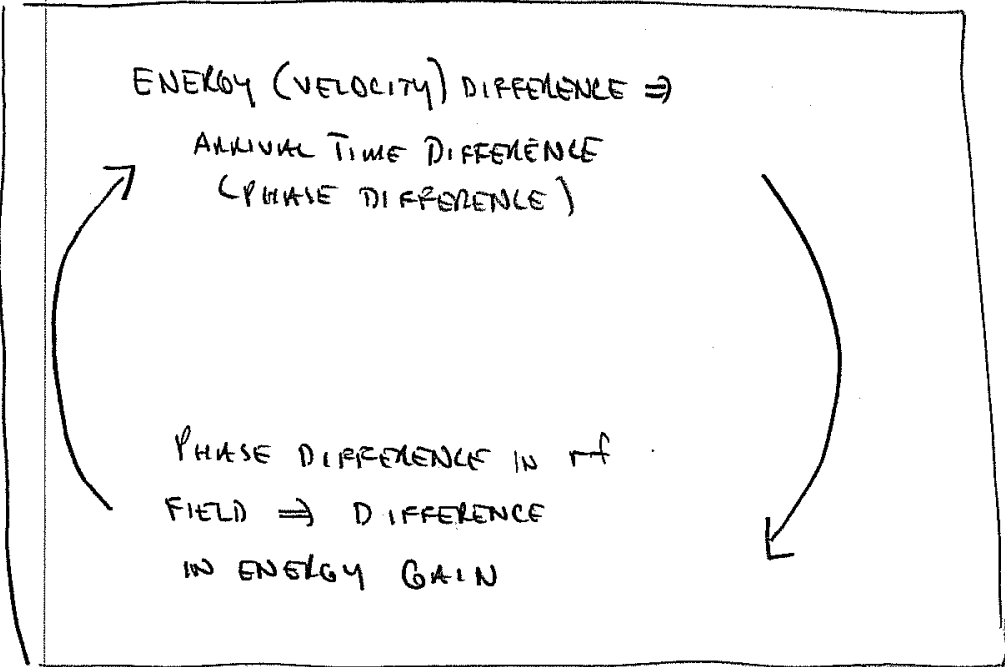
CONVERTING TO A CONTINUOUS VARIABLE:

$$\Delta(\phi - \phi_s) \rightarrow \frac{d}{dn}\Delta\phi \qquad \Delta(W - W_s) \rightarrow \frac{d}{dn}\Delta W$$

$$dn = \frac{ds}{\beta_s \lambda} \cdot \begin{Bmatrix} 2 \\ 1 \end{Bmatrix} = \frac{ds}{2l_{n-1}}$$

$$\gamma_s^3 \beta_s^3 \frac{d\Delta\phi}{ds} = -2\pi \frac{\Delta W}{mc^2 \lambda}$$

$$\frac{d\Delta W}{ds} = qE_0 (\cos\phi - \cos\phi_s)$$

$$\frac{d}{ds}\left[\gamma_s^3 \beta_s^3 \frac{d\Delta\phi}{ds} \right] = -2\pi \frac{qE_0}{mc^2\lambda}\left[\cos\phi - \cos\phi_s \right] \qquad (\text{I})$$

NOW THE SPATIAL SEPARATION IS GIVEN BY:

$$\Delta z \equiv z - z_s = -\frac{\beta_s \lambda}{2\pi}\Delta\phi$$

$\Rightarrow$ ALSO, LET $\cos\phi - \cos\phi_s \simeq -\sin\phi_s \Delta\phi$ $\qquad \left[\text{for } \frac{2\pi \Delta z}{\beta\lambda} = \Delta\phi \ll 1 \right]$

$$\Rightarrow \frac{d}{ds}\left[\gamma_s^3 \beta_s^3 \frac{d}{ds}\left(\frac{\Delta z}{\beta_s} \right) \right] \cong \frac{2\pi}{\lambda}\frac{qE_0}{mc^2}\sin\phi_s \frac{\Delta z}{\beta_s}$$

WHEN THE ACCELERATION RATE IS SMALL

$$\Rightarrow \frac{d^2}{ds^2}\Delta z \cong \frac{2\pi}{\lambda}\frac{qE_0 \sin\phi_s}{\gamma_s^3 \beta_s mc^2}\Delta z$$

$$\equiv -k_{s0}^2 \Delta z \qquad (\text{synchrotron oscillations})$$

RETURNING TO $\Delta W - \phi$ NOTATION

Let $\quad \omega = \dfrac{\Delta W}{mc^2} \qquad A = \dfrac{2\pi}{\beta_s^3 \gamma_s^3 \lambda} \qquad\qquad B = \dfrac{q E_0}{mc^2}$

$$\Rightarrow \quad \omega' = B(\cos\phi - \cos\phi_s)$$

$$\phi' = -A\omega$$

$$\phi'' = -AB(\cos\phi - \cos\phi_s)$$

MULTIPLYING BY ϕ' AND INTEGRATING:

$$\frac{\phi'^2}{2} = -AB(\sin\phi - \phi\cos\phi_s) + \text{const}$$

USING $\phi' = -A\omega$ (& DIVIDING BY A)

$$\Rightarrow \quad \underbrace{\frac{A\omega^2}{2}}_{\text{kinetic energy}} + \underbrace{B(\sin\phi - \phi\cos\phi_s)}_{\text{potential energy}} = \text{CONST.}$$

$$V(\phi) = B(\sin\phi - \phi\cos\phi_s)$$
$$\frac{dV}{d\phi} = B(\cos\phi - \cos\phi_s)$$
$$\frac{d^2V}{d\phi^2} = -B\sin\phi$$
$$> 0 \Rightarrow -\pi < \phi_s < 0$$

$\uparrow$ FOR LONGITUDINAL FOCUSING

$\dfrac{dW_s}{ds} \sim q E_0 \cos\phi_s$

ACCELERATION RANGE

FOCUSING RANGE

OPERATING (OVERLAP REGION)

$\phi_s = -\text{Pi}$

$\phi_s = -\pi$

$\phi_s = -2.35619$

$\phi_s = -3\pi/4$

$\phi_s = -1.5708$

$\phi_s = -\pi/2$

$\phi_s = -0.785398$

$\phi_s = -\pi/4$

$\phi_s = 0$

$\phi_s = 0$

$\phi_s = 1.5708$

$\phi_s = \pi/2$

(no focusing around synchronous phase)
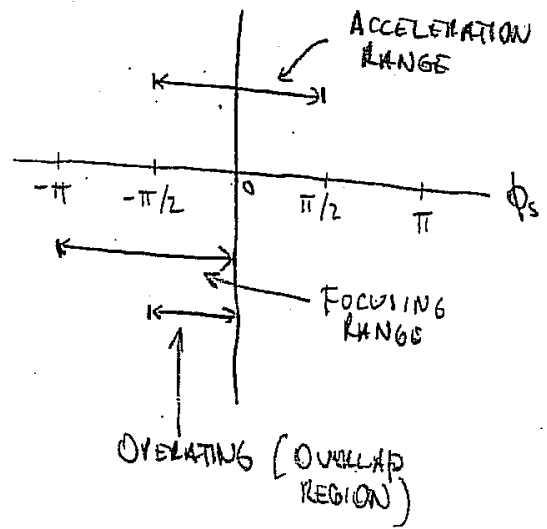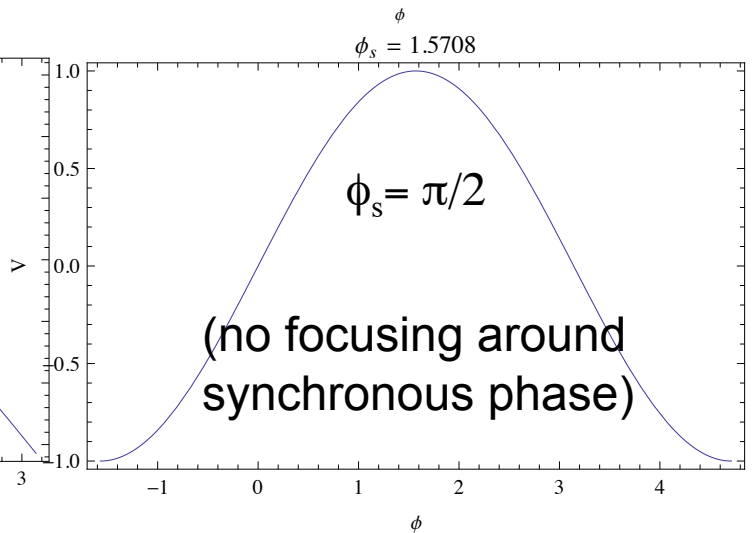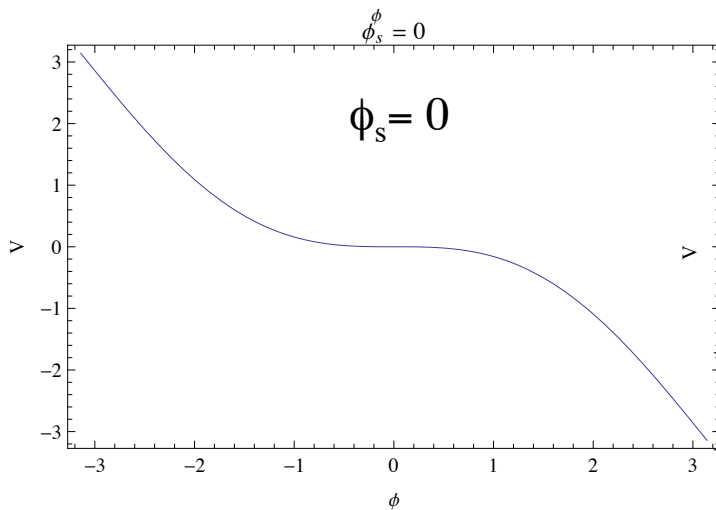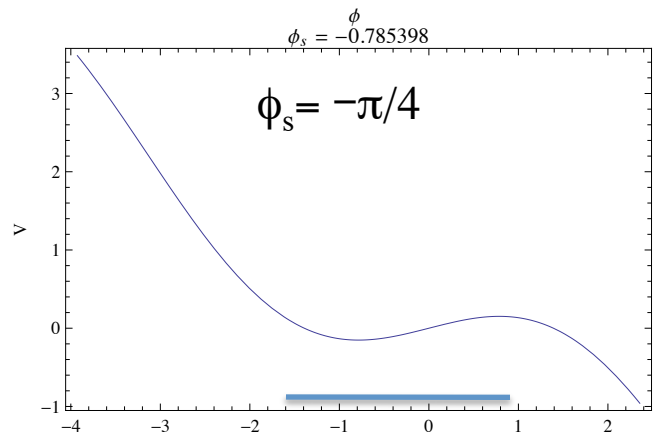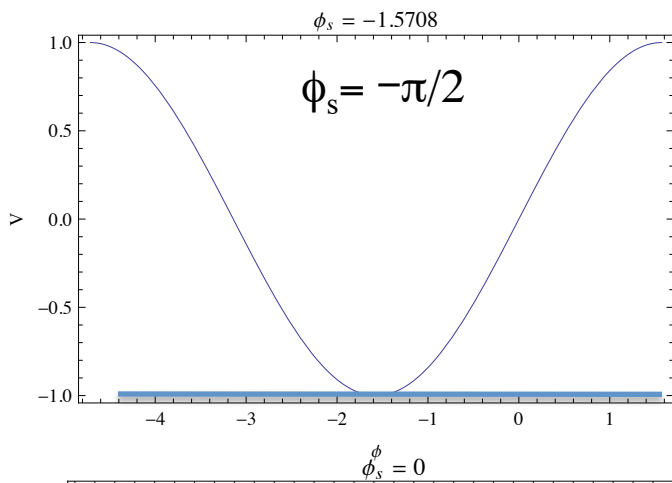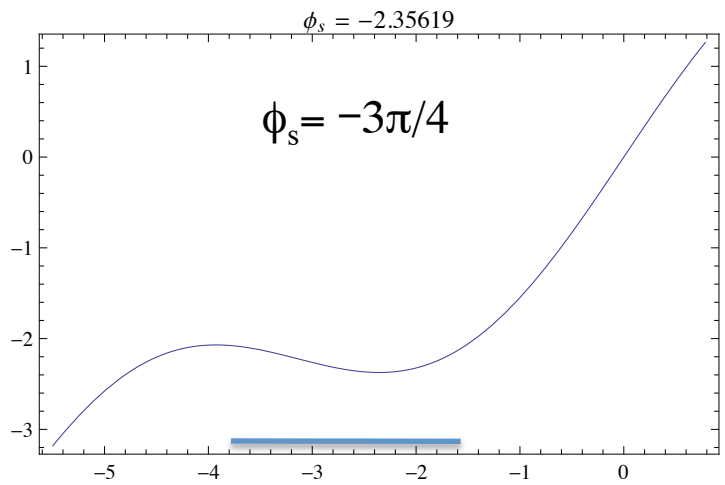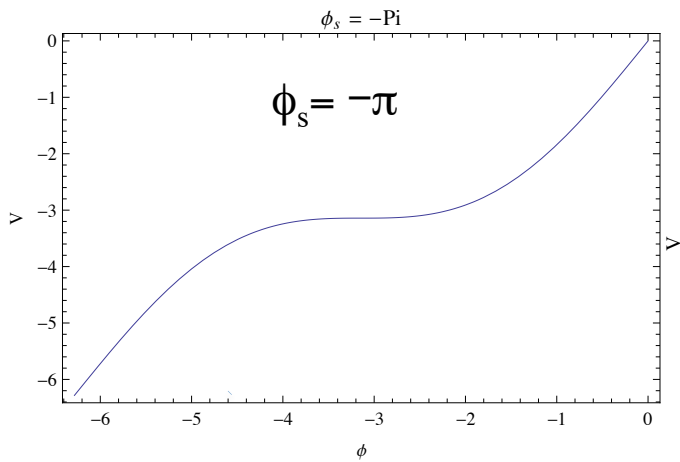
$$V(\phi) \sim B(\sin\phi - \phi\cos\phi_s)$$

For there to be a region of focusing around the synchronous phase, $-\pi < \phi_s < 0$.

simultaneous acceleration and a potential well when $-\pi/2 \leq \phi_s \leq 0$. The stable region for the phase motion extends from $\phi_2 < \phi < -\phi_s$, where the lower phase limit ϕ_2 can be obtained numerically by solving for ϕ_2 using $H_\phi(\phi_2) = H_\phi(-\phi_s)$. Figure 6.3 shows longitudinal phase space and the longitudinal potential well. At the potential maximum, where $\phi = -\phi_s$, we
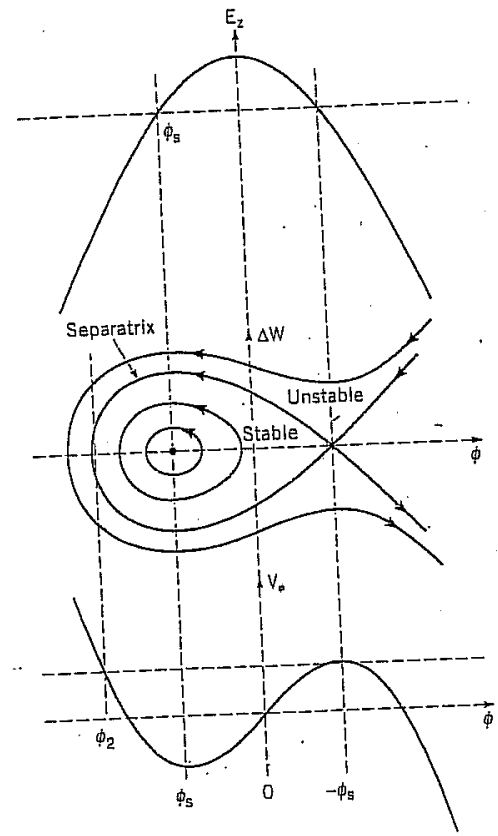
from T. Wangler's "PRINCIPLES of RF LINEAR ACCELERATORS"

← *w - φ PHASE SPACE*

Figure 6.3. At the top, the accelerating field is shown as a cosine function of the phase; the synchronous phase ϕ_s is shown as a negative number, which lies earlier than the crest where the field is rising in time. The middle plot shows some longitudinal phase-space trajectories, including the separatrix, the limiting stable trajectory, which passes through the unstable fixed point at $\Delta W = 0$, and $\phi = -\phi_s$. The stable fixed point lies at $\Delta W = 0$ and $\phi = \phi_s$, where the longitudinal potential well has its minimum, as shown in the bottom plot.

Electrostatic Potential of a uniform density ellipsoid in free space

(cf Landau & Lifshitz, Classical Theory of Fields, p.297)

$$\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2} + \frac{z^2}{r_z^2} = 1$$

It can be shown that:

$$\varphi = \frac{\rho}{4\varepsilon_0} r_x r_y r_z \int_{S_{min}}^{\infty} \left(1 - \frac{x^2}{r_x^2+s} - \frac{y^2}{r_y^2+s} - \frac{z^2}{r_z^2+s}\right) \frac{ds}{R_s}$$

$$R_s = \sqrt{(r_x^2+s)(r_y^2+s)(r_z^2+s)}$$

$$S_{min} = \begin{cases} 0 & \text{if interior point } \left(\frac{x^2}{r_x^2}+\frac{y^2}{r_y^2}+\frac{z^2}{r_z^2} < 1\right) \\ \xi & \text{if exterior point } \left(\frac{x^2}{r_x^2}+\frac{y^2}{r_y^2}+\frac{z^2}{r_z^2} > 1\right) \end{cases}$$

Here ξ is the positive root of

$$\frac{x^2}{r_x^2+\xi} + \frac{y^2}{r_y^2+\xi} + \frac{z^2}{r_z^2+\xi} = 1$$

AXISYMMETRIC

SPACE-CHARGE FIELD OF BUNCHED BEAMS

$$\frac{r^2}{r_\perp^2} + \frac{z^2}{r_z^2} = 1$$

← BOUNDARY OF ELLIPSOID
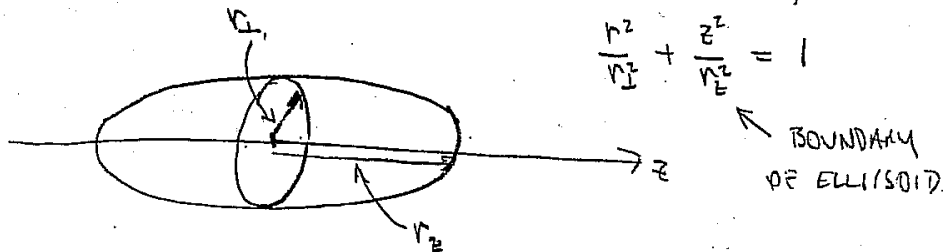
INTERIOR

THE POTENTIAL OF A UNIFORM DENSITY BUNCH IN FREE SPACE (A MACLAURIN SPHEROID) IS GIVEN BY:

(cf Landau & Lifshitz, Classical Theory of Fields, p 247)

$$\phi = \frac{-\rho}{4\epsilon_0}\left(\alpha_\perp r^2 + \alpha_\parallel z^2 - \delta\right)$$

where
$$\alpha_\perp = r_\perp^2 r_z \int_0^\infty \frac{ds}{(r_\perp^2 + s)\,\Delta}$$

$$\alpha_\parallel = r_\perp^2 r_z \int_0^\infty \frac{ds}{(r_z^2 + s)\,\Delta}$$

$$\delta = r_\perp^2 r_z \int_0^\infty \frac{ds}{\Delta}$$

where $\Delta^2 = (r_\perp^2 + s)^2 (r_z^2 + s)$

FOR NON-RELATIVISTIC BEAM:

$$E_z = -\frac{\partial \phi}{\partial z} = f\frac{\rho}{\epsilon_0} z$$

$$E_r = -\frac{\partial \phi}{\partial r} = \frac{(1-f)}{2}\frac{\rho}{\epsilon_0} r$$

$$f = f(\alpha) = \begin{cases} \frac{\alpha^2}{1-\alpha^2}\left[\frac{1}{\sqrt{1-\alpha^2}}\tanh^{-1}\sqrt{1-\alpha^2} - 1\right] & \alpha < 1 \\[2mm] \frac{1}{3} & \alpha = 1 \\[2mm] \frac{\alpha^2}{\alpha^2 - 1}\left[1 - \frac{1}{\sqrt{\alpha^2-1}}\tan^{-1}\sqrt{\alpha^2 - 1}\right] & \alpha > 1 \end{cases} \qquad \alpha \equiv \frac{r_\perp}{r_z}$$

Check: for a spherical bunch

$$\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_0} \quad \Rightarrow \quad 4\pi r^2 E_r = \frac{4\pi r^3}{3} \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow E_r = \frac{\rho}{3\varepsilon_0} r$$

THE FIELD FOR ALL RADII MAY BE WRITTEN:

$$E_r = \frac{\rho}{2\varepsilon_0}\left[\frac{\alpha^2}{(\alpha^2 + \chi)(1+\chi)^{1/2}} - F(\chi,\alpha)\right] r$$

$$E_z = \frac{\rho}{\varepsilon_0}\left[F(\chi,\alpha)\right] z$$

$$F(\chi,\alpha) = \begin{cases} \dfrac{\alpha^2}{1-\alpha^2}\left[\dfrac{1}{\sqrt{1-\alpha^2}}\tanh^{-1}\sqrt{\dfrac{1-\alpha^2}{1+\chi}} - \dfrac{1}{\sqrt{1+\chi}}\right] & \alpha < 1 \\[3mm] \dfrac{1}{3(1+\chi)^{3/2}} & \alpha = 1 \\[3mm] \dfrac{\alpha^2}{\alpha^2-1}\left[\dfrac{1}{\sqrt{1+\chi}} - \dfrac{1}{\sqrt{\alpha^2-1}}\tan^{-1}\left(\dfrac{\sqrt{\alpha^2-1}}{\sqrt{1+\chi}}\right)\right] & \alpha > 1 \end{cases}$$

χ satisfies:

$$\frac{\alpha^2 (r^2/r_\perp^2)}{\alpha^2 + \chi} + \frac{(z^2/r_z^2)}{1+\chi} = 1 \qquad \text{for exterior particle}$$

$$\chi = 0 \qquad \text{for interior particle}$$

$$\alpha \equiv \frac{r_\perp}{r_z}$$

FOR AN EXTERIOR PARTICLE AT $r \,\&\, z \Rightarrow \chi$ CAN BE SOLVED FOR (QUADRATIC EQUATION FOR χ).

$\Rightarrow E_r \,\&\, E_z$ ARE KNOWN ANALYTICALLY FOR ALL r, z.

EXAMPLE: $r = 0 \Rightarrow \chi = \dfrac{z^2}{r_z^2} - 1$

(and $z > r_z$)

RELATIVISTIC TRANSFORMATION FROM BEAM FRAME TO LAB FRAME:
(see e.g. JACKSON, CLASSICAL ELECTRODYNAMICS)

$$\underline{E} = \gamma\left(\underline{E}' - c\underline{\beta} \times \underline{B}'\right) - \frac{\gamma^2}{\gamma+1}\underline{\beta}\left(\underline{\beta}\cdot\underline{E}'\right)$$

$$\underline{B} = \gamma\left(\underline{B}' + \frac{\beta}{c}\times \underline{E}'\right) - \frac{\gamma^2}{\gamma+1}\underline{\beta}\left(\underline{\beta}\cdot\underline{B}'\right)$$

' = BEAM FRAME

NO PRIME = LAB FRAME

$$\underline{B}' = 0$$

$$\Rightarrow \quad E_z = \gamma E_z' - \frac{\gamma^2}{\gamma+1}\beta^2 E_z' = E_z'$$

$$E_r = \gamma E_r'$$

$$B_\theta = \frac{\gamma\beta}{c} E_r'$$

$$F_z = q E_z = q E_z'$$

$$F_r = q\left(E_r - v_z B_\theta\right) = \frac{1}{\gamma} q E_r'$$

BEAM   LAB

ΔZ'   Δz

$$\Delta z = \frac{1}{\gamma}\Delta z'$$

$$\underline{x}_\perp = \underline{x}_\perp'$$

$$\rho = \gamma \rho'$$
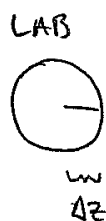
$$\underline{F}_\perp = \frac{d\underline{p}_\perp}{dt} = \gamma_s m v_{zs}^2 \frac{d^2 \underline{x}_\perp}{ds^2} \quad \left(\text{NEGLECTING } \frac{d\gamma_s}{dt}, \frac{dv_{zs}}{dt}\right)$$

$$\Delta F_z = \frac{dp_\parallel}{dt} - \frac{dp_{\parallel s}}{dt} = \gamma_s^3 m v_{zs}^2 \frac{d^2(z-z_s)}{ds^2} \quad \left(\text{USING } \frac{d}{ds}\gamma\beta = \gamma^3 \frac{d\beta}{ds}\right)$$

FOR RELATIVISTIC BEAM

(cf. & BARNARD & LUND 1997
Lund & BARNARD 1997.)
PAC 97 Conf Proceedings

$$\frac{d^2 x_\perp}{ds^2} = \frac{F_\perp}{\gamma_s \beta_s^2 mc^2}$$

$$F_{\perp s} = \frac{q\rho}{2\gamma_s^2 \varepsilon_0}\left[1 - f(\alpha)\right] x_\perp$$

$$\frac{d^2 \Delta z}{ds^2} = \frac{F_z}{\gamma_s^3 \beta_s^2 mc^2}$$

$$F_{zs} = \frac{q\rho}{\varepsilon_0} f(\alpha) \Delta z$$

$$\alpha = \frac{r_\perp}{\gamma r_z} \qquad \left[\alpha = \frac{r_\perp}{(r_z \text{ in comoving frame})}\right]$$

COMBINING FOCUSING + SELF FIELDS

$$\frac{d^2}{ds^2} \Delta z = -k_{s0}^2 \Delta z + \frac{q\rho \, f(\alpha)}{\gamma_s^3 \beta_s^2 mc^2 \varepsilon_0} \Delta z \qquad \text{(LINEAR rf)}$$

Total field seen by particle is sum of rf and spacecharge
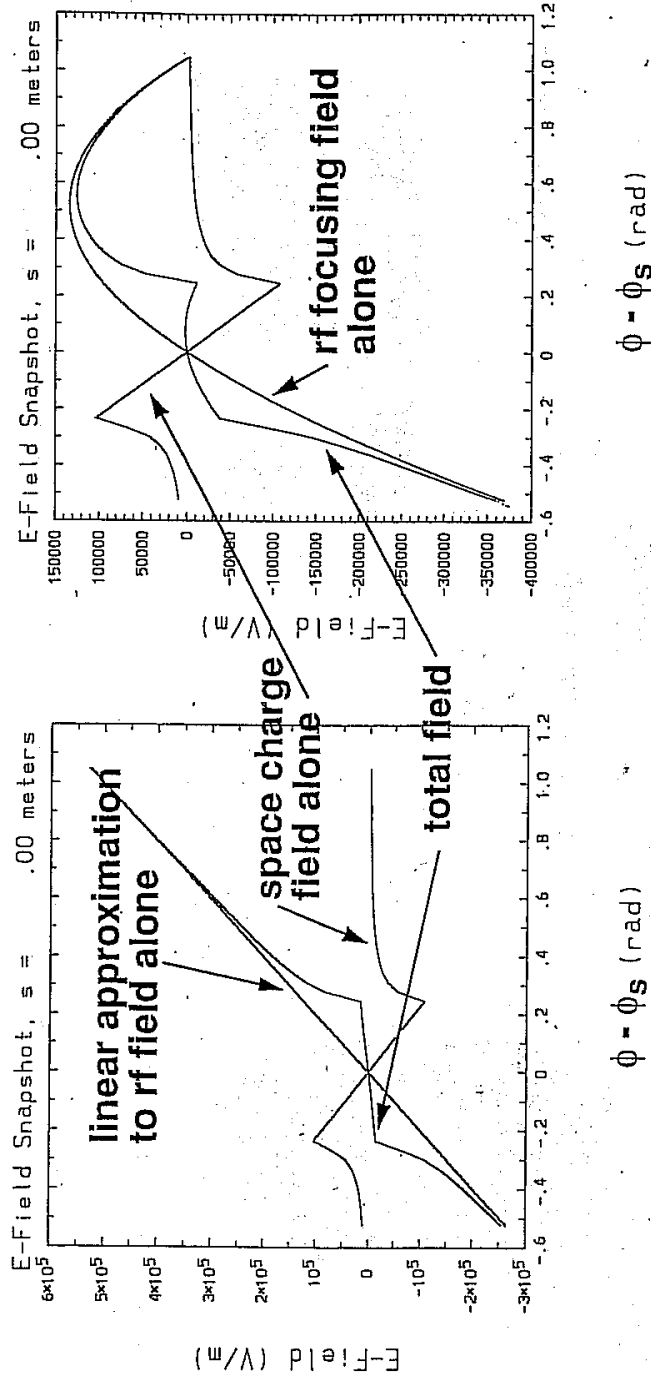
here $\phi - \phi_s = - (2\pi/\beta_s\lambda)\,\Delta z$, where $\beta_s c$ is the longitudinal velocity of the synchronous particle and $\lambda = c/\nu$ is the rf vacuum wavelength

Induction acceleration: Volt-second limits

Applied Voltage V

Switch

Energy Storage

Faraday's law, $\nabla \times E = -\dfrac{\partial B}{\partial t}$

$\Rightarrow V \Delta t = A \Delta B$

E

B

Ferromagnetic Material

Cross-sectional area A
$A = (R_o - R_i) \, l$

R_o

R_i

l
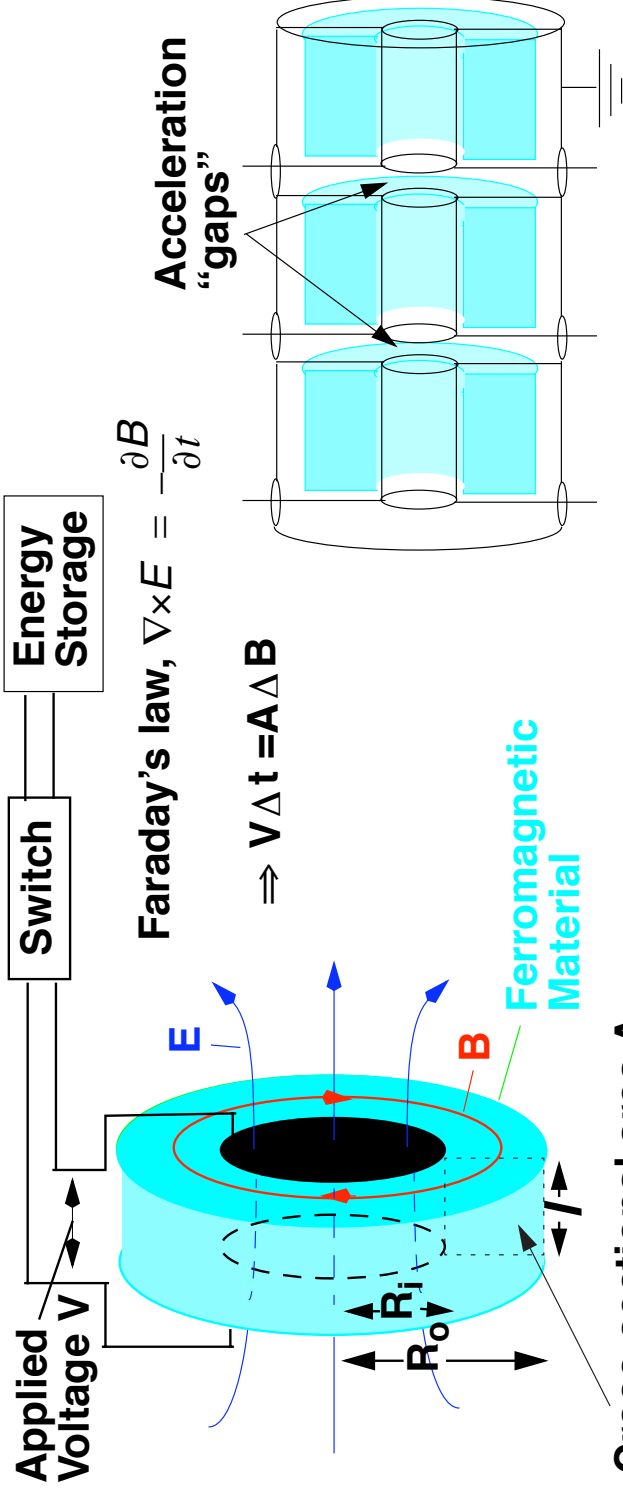
Volt-seconds per m: $(dV/dz) \, \Delta t = (R_o - R_i) \, \Delta B$

	$R_o - R_i$	ΔB	f_{radial}	$f_{longit.}$
	~1 m	~2.5 T	~0.8	~0.8

$(dV/dz) \, \Delta t < \sim 1.6 \text{ V-s/m}$

Acceleration "gaps"

Several types of waveform are needed to accelerate, compress, and confine the beam

Voltage

Current

Ear Voltage

ΔV_{tilt}

ΔV_{acc}

ΔV_{ears}

ΔV_{ears}

time

time
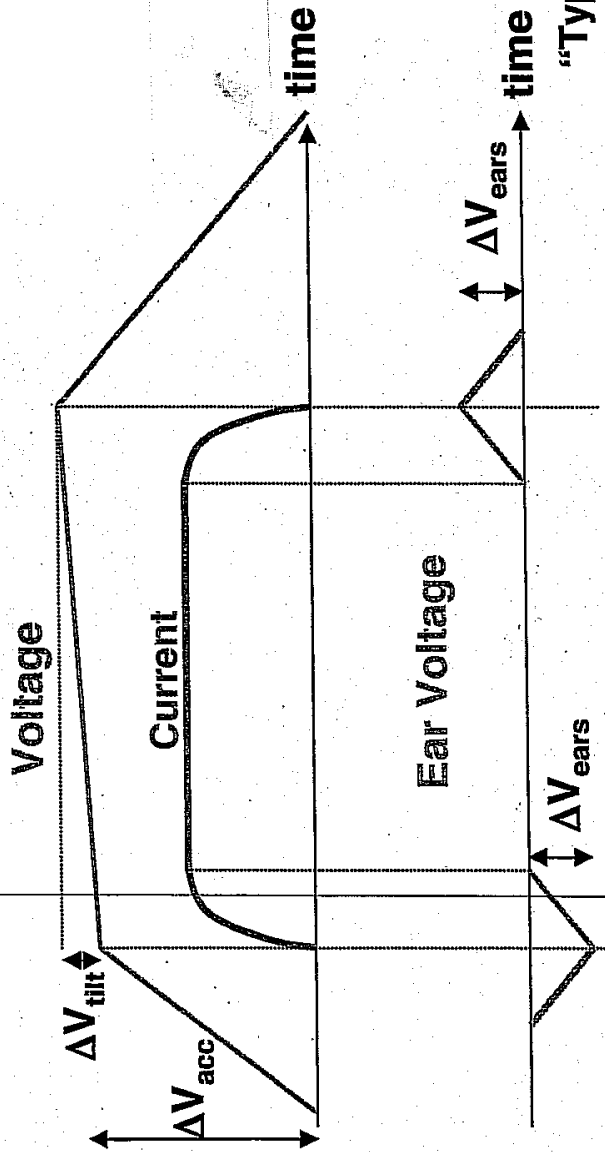
"Typical" numbers:

$\Delta V_{tilt} \sim \quad 1\ kV$
$\Delta V_{ears} \sim \quad 14\ kV$
$\Delta V_{acc} \sim \quad 100\ kV$

The Heavy Ion Fusion Virtual National Laboratory

⑫

COORDINATE SYSTEM
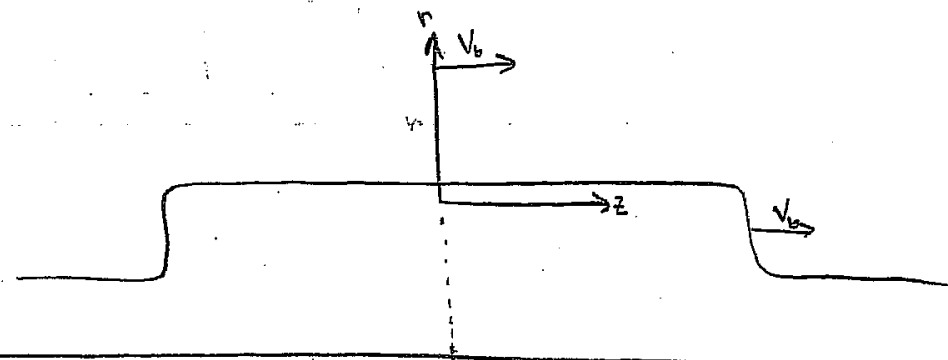
$s = \beta_b c t$ for drifting beam

= position of beam center in lab frame

$s \longleftrightarrow t$ are related by $\beta_b c$ for drifting beam

z = longitudinal coordinate in beam frame ($z=0$ = beam center)

r = radial coordinate in beam frame (or lab frame).

(This class will assume non-relativistic dynamics)
These are ions with $\beta < 0.2$).

LONGITUDINAL PHYSICS OF LONG PULSES (BUNCH LENGTH $\gg r_{pipe}$)

"g-factor" model

s = location of bunch center
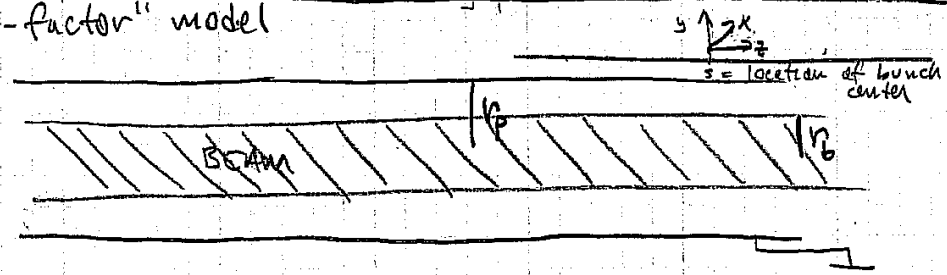
If $\quad \dfrac{\partial^2 \phi}{\partial z^2} \ll \dfrac{1}{r}\left(\dfrac{\partial}{\partial r} r \dfrac{\partial \phi}{\partial r}\right) \quad \Rightarrow \quad \dfrac{\partial \phi}{\partial r} = -\dfrac{\lambda(r)}{2\pi \varepsilon_0 r}$

Let $\rho = \begin{cases} \rho_0 & 0 < r < r_b \\ 0 & r_b < r < r_p \end{cases} \quad \Rightarrow \quad \lambda = \lambda_0 \left(\dfrac{r}{r_b}\right)^2$

$\phi = \displaystyle\int_r^{r_p} \dfrac{\partial \phi}{\partial r}\, dr = \begin{cases} \dfrac{\lambda}{2\pi \varepsilon_0}\left[\dfrac{1}{2}\left(1 - \dfrac{r^2}{r_b^2}\right) + \ln \dfrac{r_p}{r_b}\right] & 0 < r < r_b \\[2mm] \dfrac{\lambda}{2\pi \varepsilon_0} \ln\left(\dfrac{r_p}{r}\right) & r_b < r < r_p \end{cases}$

$\dfrac{\partial \phi}{\partial z} = \dfrac{1}{2\pi \varepsilon_0}\left[\dfrac{1}{2}\left(1 - \dfrac{r^2}{r_b^2}\right) + \ln \dfrac{r_p}{r_b}\right]\dfrac{\partial \lambda}{\partial z} - \dfrac{1}{2\pi \varepsilon_0}\left[1 - \dfrac{r^2}{r_b^2}\right]\dfrac{\lambda}{r_b}\dfrac{\partial r_b}{\partial z}$

If $\rho = const \Rightarrow \dfrac{\lambda}{r_b^2} = const \qquad \dfrac{\partial \lambda}{\partial z} = \dfrac{-2\lambda}{r_b}\dfrac{\partial r_b}{\partial z}$

$\Rightarrow \quad \dfrac{\partial \phi}{\partial z} = \dfrac{1}{2\pi \varepsilon_0} \ln\left(\dfrac{r_p}{r_b}\right)\dfrac{\partial \lambda}{\partial z}$

$E_z = \dfrac{-g}{4\pi \varepsilon_0}\dfrac{\partial \lambda}{\partial z}$

where $\quad g = 2 \ln\left(\dfrac{r_p}{r_b}\right)$

[SPACE-CHARGE DOMINATED BEAM]

[Example of $\rho = const$.
Magnetic Quad focusing
$\dfrac{\lambda}{4\pi \varepsilon_0 \sqrt{a}} \approx k_{\beta 0}^2 a$
$\Rightarrow \rho \sim V k_{\beta 0}^2 \approx const$]

(for space-charge dominated beam)

FOR EMITTANCE DOMINATED BEAMS:

(with constant emittance and radius)

RADIUS NOT DETERMINED BY λ

so $\quad \frac{\partial r_b}{\partial z} \simeq 0$

$$\left\langle \frac{\partial \phi}{\partial z} \right\rangle = \frac{1}{2\pi\varepsilon_0} \left[\frac{1}{2} \left(1 - \left\langle \frac{r^2}{r_b^2} \right\rangle \right) + \ln \frac{r_p}{r_b} \right] \frac{\partial \lambda}{\partial z}$$

$$\Rightarrow \quad g = 2 \ln \left(\frac{r_p}{r_b} \right) + \frac{1}{2} \qquad \text{(EMITTANCE DOMINATED BEAMS)}$$

(SEE REISER, SECTION 6.3 FOR DISCUSSION ON g-factor).

<u>Vlasov - equation for a drifting beam:</u>

$$\frac{\partial f}{\partial s} + x'\frac{\partial f}{\partial x} + x''\frac{\partial f}{\partial x'} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'} + z'\frac{\partial f}{\partial z} + z''\frac{\partial f}{\partial z'} = 0$$

Let $\tilde{f}(z,z',s) \equiv \iiiint f \, dx \, dx' \, dy \, dy'$

INTEGRATING VLASOV EQUATION: If $z'' \neq f(x,x',y,y')$:

$$\Rightarrow \frac{\partial \tilde{f}}{\partial s} + \iiint x'\underbrace{\int \frac{\partial f}{\partial x} \, dx}_{=f|_{-\infty}^{\infty} \atop \downarrow 0} dx' \, dy \, dy' + \cdots \qquad z'\frac{\partial \tilde{f}}{\partial z} + z''\frac{\partial \tilde{f}}{\partial z'} = 0$$

$$\Rightarrow \boxed{\frac{\partial \tilde{f}}{\partial s} + z'\frac{\partial \tilde{f}}{\partial z} + z''\frac{\partial \tilde{f}}{\partial z'} = 0} \qquad \text{1 D Vlasov}$$

Now let $\lambda \equiv q\int \tilde{f} \, dz'$; $\lambda \overline{z'} \equiv \int \tilde{f} z' \, dz'$; $\lambda \overline{z'^2} \equiv \int \tilde{f} z'^2 \, dz'$

Also, let $\Delta z'^2 \equiv \overline{z'^2} - (\overline{z'})^2$

<u>FLUID EQUATIONS</u>

INTEGRATING 1D VLASOV OVER z':

$$\boxed{\frac{\partial \lambda}{\partial s} + \frac{\partial}{\partial z}(\lambda \overline{z'}) = 0} \qquad (\text{CONTINUITY EQUATION})$$

MULTIPLYING BY z' & INTEGRATING VLASOV OR z':

$$\frac{\partial}{\partial s}\lambda \overline{z'} + \frac{\partial}{\partial z}\lambda \overline{z'^2} - \lambda \overline{z''} = 0$$

DIVIDING BY λ, USING CONTINUITY EQUATION & DEFINITION OF $\Delta z'^2$:

$$\boxed{\underbrace{\frac{\partial \overline{z'}}{\partial s} + \overline{z'}\frac{\partial}{\partial z}\overline{z'}}_{\text{INERTIAL}} + \underbrace{\frac{1}{\lambda}\frac{\partial}{\partial z}(\lambda \Delta z'^2)}_{\substack{\text{PRESSURE} \\ \text{TERM}}} = \underbrace{\overline{z''}}_{\text{FORCE}}} \qquad \begin{pmatrix} \text{MOMENTUM} \\ \text{EQUATION} \end{pmatrix}$$

$$= \frac{1}{v_0^2}\frac{qE_z}{m}$$

Combining g-factor model with fluid equations

$$\frac{\partial \lambda}{\partial s} + \frac{\partial}{\partial z}(\lambda \bar{z}') = 0$$

$$\frac{\partial \bar{z}'}{\partial s} + \bar{z}' \frac{\partial \bar{z}'}{\partial z} + \underbrace{\frac{1}{\lambda}\frac{\partial}{\partial z}(\lambda \Delta z'^2)}_{\text{PRESSURE TERM}} = \underbrace{\frac{-qg}{4\pi\varepsilon_0 m V_0^2}\frac{\partial \lambda}{\partial z}}_{\text{SPACE CHARGE TERM}}$$

WHEN PRESSURE TERM $\ll$ SPACE CHARGE TERM,

$\left(\text{\& LET } C_s^2 \equiv \dfrac{qg\lambda_0}{4\pi\varepsilon_0 m} = \text{``(SPACE CHARGE WAVE SPEED)}^2\text{''}\right.$

$$\Rightarrow \boxed{\begin{array}{l} \dfrac{\partial \lambda}{\partial s} + \lambda \dfrac{\partial \bar{z}'}{\partial z} + \bar{z}' \dfrac{\partial \lambda}{\partial z} = 0 \\[3mm] \dfrac{\partial \bar{z}'}{\partial s} + \bar{z}' \dfrac{\partial \bar{z}'}{\partial z} + \dfrac{C_s^2}{\lambda_0 V_0^2}\dfrac{\partial \lambda}{\partial z} = 0 \end{array}} \qquad (g1)$$

LINEARIZING g1

LET $\quad \lambda = \lambda_0 + \lambda_1 \qquad \bar{z}' = \bar{z}_0' + \bar{z}_1'$

EQUILIBRIUM $\quad \lambda_0 = $ CONSTANT

$$\bar{z}_0' = 0$$

LINEARIZING

$$\frac{\partial \lambda_1}{\partial s} + \lambda_0 \frac{\partial \bar{z}_1'}{\partial z} = 0 \qquad (g2a)$$

$$\frac{\partial \bar{z}_1'}{\partial s} + \frac{C_s^2}{\lambda_0 V_0^2}\frac{\partial \lambda_1}{\partial z} = 0 \qquad (g2b)$$

TAKING $\frac{\partial}{\partial s}$ of (g2a) \& $\frac{\partial}{\partial z}$ of g2b and combining:

$$\Rightarrow \boxed{\frac{\partial^2 \lambda_1}{\partial s^2} - \frac{C_s^2}{V_0^2}\frac{\partial^2 \lambda_1}{\partial z^2} = 0} \qquad \Rightarrow \text{WAVE EQUATION}$$

SOLVING WAVE EQUATION

$$\frac{\partial^2 \lambda_1}{\partial s^2} - \frac{c_s^2}{V_0^2}\frac{\partial^2 \lambda_1}{\partial z^2} = 0$$

Let $\quad \lambda_1 = \tilde{\lambda}_1 \exp\left[\frac{i\omega}{V_0}s \pm ikz\right]$

$$-\frac{\omega^2}{V_0^2} + \frac{k^2 c_s^2}{V_0^2} = 0 \quad \Rightarrow \quad \omega = c_s k$$

$\Rightarrow$ PHASE & GROUP VELOCITY OF WAVES $= c_s$

(in beam frame)

GENERAL SOLUTION

$$\lambda_1 = \lambda_0 f_+[u_+] + \lambda_0 f_-[u_-]$$

where $\quad u_+ = z + \frac{c_s s}{V_0} + C_0 \quad \& \quad u_- = z - \frac{c_s s}{V_0} + C_0$

& $f_+[u]$ & $f_-[u]$ are any functions of the argument
& C_0 is an arbitrary constant

$$\overline{z_1'} = \frac{c_s}{V_0}\left[-f_+[u_+] + f_-[u_-]\right]$$

$s = 0:$

$\lambda_1(z)$

$\overline{z_1'}(z)$

$s = s_0:$

$\lambda_1(z)$

BACKWARD WAVE   FORWARD WAVE
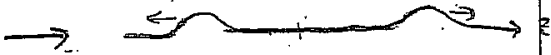
$\overline{z_1'}(z)$
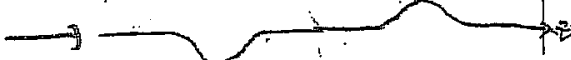
BEAM ENDS & RAREFACTION WAVES

SUPPOSE YOU START WITH A PULSE THAT ENDS WITH A STEP FUNCTION IN λ. WHAT HAPPENS TO THE END?

EXPECT $g\frac{\partial \lambda}{\partial z}$ TO DRIVE PARTICLE AHEAD OF BEAM

TO ANALYZE: RETURN TO __NON-LINEAR FLUID__ EQUATIONS (SINCE $\delta\lambda \sim \lambda_0$) (g1):

$$\frac{\partial \lambda}{\partial s} + \lambda \frac{\partial \bar{z}'}{\partial z} + \bar{z}' \frac{\partial \lambda}{\partial z} = 0 \qquad \text{(CONTINUITY)}$$

$$\frac{\partial \bar{z}'}{\partial s} + \bar{z}' \frac{\partial \bar{z}'}{\partial z} + \frac{c_s^2}{\lambda_0 v_0^2} \frac{\partial \lambda}{\partial z} = 0 \qquad \text{(momentum)}$$

1st IT IS CONVENIENT TO DEFINE: $\Lambda \equiv \lambda/\lambda_0$

$$\left(c_s^2 = \frac{g}{m}\frac{q\lambda_0}{4\pi\epsilon_0}\right)$$

$$V \equiv \frac{v_0}{c_s}\bar{z}'$$
$$\zeta \equiv \frac{v_0}{c_s}z$$

$$\Rightarrow \frac{\partial \Lambda}{\partial s} + \Lambda \frac{\partial V}{\partial \zeta} + V \frac{\partial \Lambda}{\partial \zeta} = 0 \qquad \text{(CONTINUITY)}$$

$$\frac{\partial V}{\partial s} + V \frac{\partial V}{\partial \zeta} + \frac{\partial \Lambda}{\partial \zeta} = 0 \qquad \text{(momentum)}$$

(r1)

TRY A SIMILARITY SOLUTION: $\Lambda = \Lambda(x)$ & $V = V(x)$

WHERE $x = \dfrac{z}{s} = \left(\dfrac{V_0 z}{c_s s}\right)$

$$\frac{\partial x}{\partial s} = -\frac{x}{s} \qquad \frac{\partial x}{\partial z} = \frac{x}{z}$$

$$\frac{\partial \Lambda}{\partial s} = -\frac{d\Lambda}{dx}\frac{x}{s} \qquad \frac{\partial \Lambda}{\partial z} = \frac{d\Lambda}{dx}\frac{x}{z} \qquad \frac{\partial V}{\partial s} = -\frac{dV}{dx}\frac{x}{s} \qquad \frac{\partial V}{\partial z} = \frac{dV}{dx}\frac{x}{z}$$

$$\left[-\frac{d\Lambda}{dx}\frac{x}{s} + \Lambda\frac{dV}{dx}\frac{x}{z} + V\frac{d\Lambda}{dx}\frac{x}{z}\right] = 0 \qquad \text{(CONTINUITY)}$$

$$\left[-\frac{dV}{dx}\frac{x}{s} + V\frac{dV}{dx}\frac{x}{z} + \frac{d\Lambda}{dx}\frac{x}{z}\right] = 0 \qquad \text{(momentum)}$$

MULTIPLY BY z/s & gather terms):

$$\Rightarrow \begin{bmatrix} V-x & \Lambda \\ 1 & V-x \end{bmatrix}\begin{bmatrix} d\Lambda/dx \\ dV/dx \end{bmatrix} = 0$$

FOR NON-TRIVIAL SOLUTION DETERMINANT MUST VANISH:

$$\boxed{\Lambda = [V-x]^2}$$

$$\Rightarrow \frac{d\Lambda}{dx} = 2[V-x]\left[\frac{dV}{dx}-1\right] \quad \& \quad \frac{d\Lambda}{dx} = -[V-x]\frac{dV}{dx}$$

$$\Rightarrow -\frac{dV}{dx} = 2\frac{dV}{dx} - 2 \quad \Rightarrow \quad \frac{dV}{dx} = \frac{2}{3}$$

$$\Rightarrow \boxed{\begin{array}{l} V = \dfrac{2}{3}x + C \\[2mm] \Lambda = \left[-\dfrac{1}{3}x + C\right]^2 \end{array}}$$

$$V = \frac{2}{3}x + C$$
$$\Lambda = \left[-\frac{1}{3}x + C\right]^2$$

ORIGINAL BEAM

x_- x_+ → X

At x_+: $\Lambda = 0$ ⟹ $C = \frac{1}{3}X$ $x_+ = 3C$

At x_-: $\Lambda = 1$ ⟹ $C = \frac{1}{3}X + 1$ ⟹ $x_- = 3C - 3$

MASS CONSERVATION ⟹ $(x_-) = -(3C-3) = \int_{3C-3}^{3C} \left[C - \frac{1}{3}x\right]^2 dx = -3\frac{\left[C - \frac{1}{3}x\right]^3}{3}\Big|_{3C-3}^{3C}$

$-(3C-3) = 1$

$$\boxed{C = \frac{2}{3}}$$ ⟹ $\begin{array}{l} x_- = -1 \\ x_+ = 2 \end{array}$
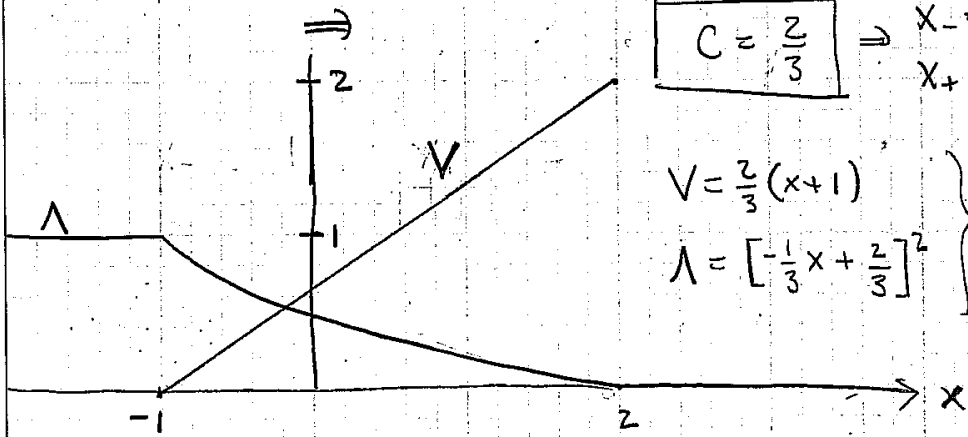
$V = \frac{2}{3}(x+1)$ ⎱ for
$\Lambda = \left[-\frac{1}{3}x + \frac{2}{3}\right]^2$ ⎰ $-1 < x < 2$

RECALL $X = \frac{V_0 z}{C_s S}$ so $X = 2$ ⟹ $z = 2C_s\left(\frac{S}{V_0}\right)$

$X = -1 \Rightarrow z = -C_s\left(\frac{S}{V_0}\right)$

SO BEAM END EXPANDS AT TWICE SPACE-CHARGE WAVE SPEED & RAREFACTION WAVE PROPAGATES INWARD AT THE SPACE-CHARGE WAVE SPEED.

SNAPSHOTS OF λ/λ_0 vs z AT VARIOUS s

$s = v_0/c_s$

$s = 2v_0/c_s$

$s = 3v_0/c_s$

HOW DOES ONE PREVENT "END EROSION"?

APPLY EAR PULSES AT END OF BEAM:

$$V \sim E_z = \frac{\mp g}{4\pi\epsilon_0} \frac{\partial\lambda}{\partial z}$$

Current

Head   time   Tail
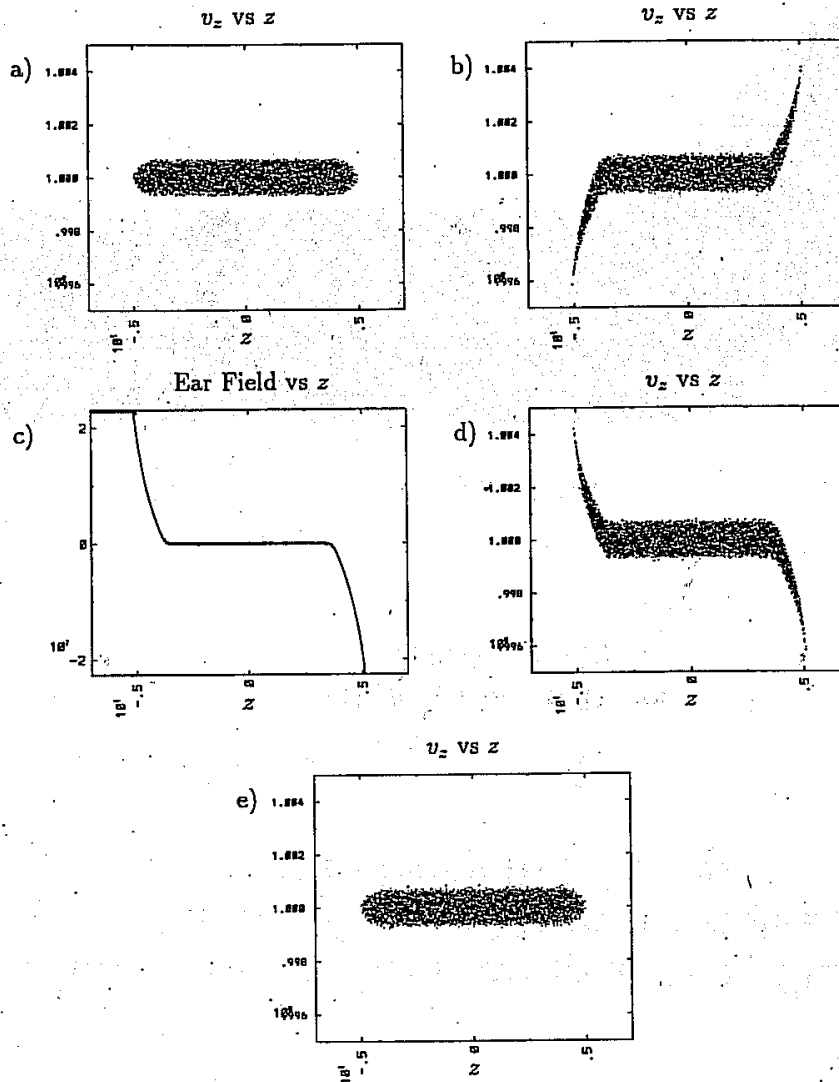
Voltage

Head   time   Tail
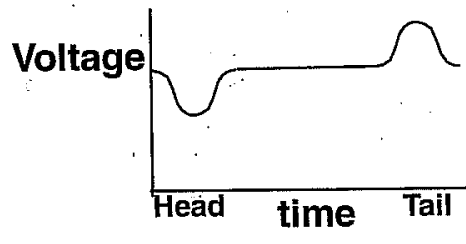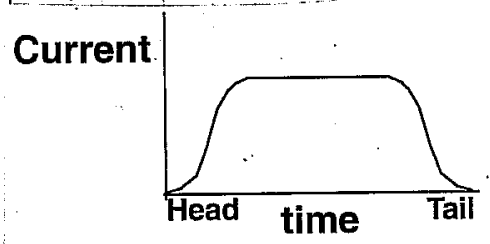
Figure 6.4: One cycle of intermittently-applied ears. (a) Initial phase space (b) Beam expands (c) Ear Field is applied (d) Beam is compressed (e) Beam expands back to its initial state

from D. Callahan Miller
PhD thesis, U. C. Davis, 1994.