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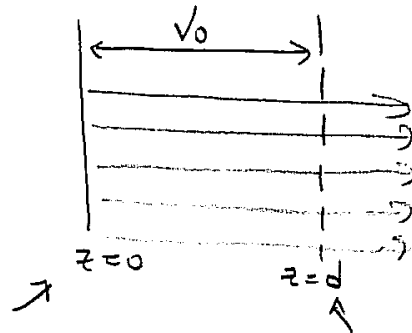
Injectors and longitudinal physics -- III

1. Longitudinal cooling from acceleration
2. Longitudinal instability
3. Bunch compression
4. Neuffer distribution

LONGITUDINAL COOLING

1. DURING INJECTION BEAM UNDERGOES LARGE LONGITUDINAL EXPANSION
2. $T_{\perp 0} = T_{\perp 1}$ AT SOURCE, BUT $T_{\perp} \neq T_{\parallel}$ AFTER ACCELERATION
3. IMPLICATIONS FOR BEAM STABILITY AND EMITTANCE EVOLUTION

CONSIDER 1D DIODE:



AT SOURCE

$$\Delta E \sim \frac{\langle p_{z0}^2 \rangle}{2m} = \frac{kT_{\parallel 0}}{2}$$

$$\Delta p_{z0} \sim \sqrt{mkT_{\parallel 0}}$$

AT END OF DIODE

$$E_f = qV_0 + \frac{p_{z0}^2}{2m}$$

$$\Delta E_{\parallel f} = \Delta E_{\parallel 0} \neq \frac{1}{2} kT_{\parallel f}$$

SINCE $E_{\parallel} = \frac{p_{\parallel}^2}{2m} \Rightarrow \Delta E_{\parallel} = \frac{2p_{\parallel} \Delta p_{\parallel}}{2m}$

$$\frac{\Delta E_{\parallel}}{E} = \frac{2 \Delta p_{\parallel}}{p_{\parallel}}$$

$$\frac{1}{2} kT_{\parallel f} \approx \frac{\Delta p_{\parallel}^2}{2m} = \left(\frac{p_{\parallel} \Delta E_{\parallel f}}{2E_f} \right) \frac{1}{2m} = \frac{\Delta E_{\parallel f}^2}{4E_f} = \frac{1}{2} kT_0 \left(\frac{\frac{1}{2} kT_0}{qV_0} \right)$$

$$\Rightarrow kT_{\parallel f} \sim \frac{1}{2} kT_0 \left(\frac{kT_0}{qV_0} \right)$$

$$kT_{\parallel f} \sim \frac{1}{2} kT_0 \left(\frac{kT_0}{qV_0} \right)$$

$\ll 1$

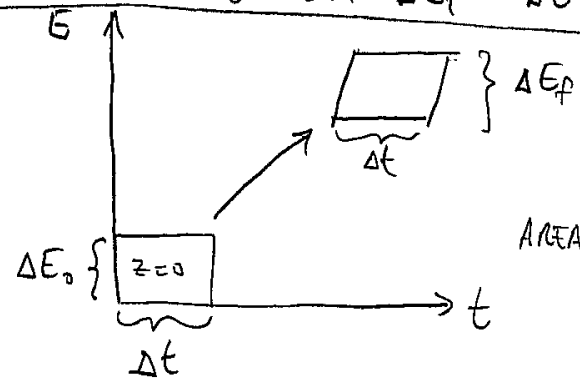
EXAMPLE $1000^\circ\text{C} \Leftrightarrow 0.1 \text{ eV}$

FOR $V_0 = 1 \text{ MeV}$

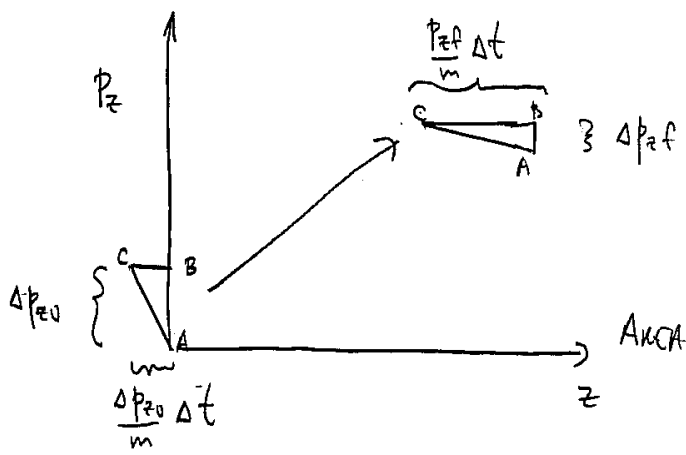
$kT_0 = 0.1 \text{ eV}$

$kT_f = 5 \cdot 10^{-9} \text{ eV}$

HOW CAN $kT_f \ll kT_0$ BUT $\Delta E_f = \Delta E_0$?



AREA IS CONSERVED
(PULSE DURATION STAYS THE SAME.)



(BUNCH LENGTH GROWS)
AREA IS CONSERVED

$$\frac{1}{2} \frac{\Delta p_{z0}^2}{m} \Delta t = \frac{1}{2} \Delta p_{zf} \left(\frac{p_{zf}}{m} \right) \Delta t$$

$$\Rightarrow \Delta p_{zf} = \frac{\Delta p_{z0}^2}{p_{zf}}$$

$$\Rightarrow kT_f = \frac{1}{2} kT_0 \left(\frac{kT_0}{qV_0} \right)$$

CHANGE IN NOTATION

④

NOTE: $\bar{z}' \equiv \left\langle \frac{dz}{ds} \right\rangle$; $s \equiv v_0 t$

Let $u = \left\langle \frac{dz}{dt} \right\rangle$; then $u = v_0 \bar{z}'$
 = fluid velocity in comoving frame

So

$$\frac{\partial \lambda}{\partial s} + \frac{\partial}{\partial z} (\lambda \bar{z}') = 0 \Rightarrow \boxed{\frac{\partial \lambda}{\partial t} + \frac{\partial}{\partial z} (\lambda u) = 0}$$

$$\frac{\partial \bar{z}'}{\partial s} + \bar{z}' \frac{\partial}{\partial z} \bar{z}' + \frac{1}{\lambda} \frac{\partial}{\partial z} (\lambda A \bar{z}'^2) = \bar{z}''$$

$$\Rightarrow \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} + \frac{1}{\lambda} \frac{\partial}{\partial z} (\lambda [\langle v_z^2 \rangle - u^2]) = \bar{z}''$$

Since $p_z = m \int_{-\infty}^{\infty} n [v_z^2 - u^2] dv_z =$ where $n = \frac{\lambda}{\pi v_0^2}$

$$\Rightarrow \boxed{\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} + \frac{\pi v_0^2}{m \lambda} \frac{\partial}{\partial z} p_z = \bar{z}''}$$

where $\bar{z}'' = \frac{d^2 z}{dt^2}$
 $= \frac{u}{s}$

National Brand 42-102 100 SHEETS Made in U.S.A.

"LONGITUDINAL" or "RESISTIVE WALL" INSTABILITY

Let us return to the 1-D FLUID EQUATIONS

$$\frac{\partial \lambda}{\partial t} + \frac{\partial}{\partial z} \lambda u = 0$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} + \frac{1}{\rho} \frac{\partial p_e}{\partial z} = \frac{-g g}{4\pi \epsilon_0 m v_0^2} \frac{\partial \lambda}{\partial z} + \frac{g E_z}{m}$$

↑
IGNORE
AGAIN

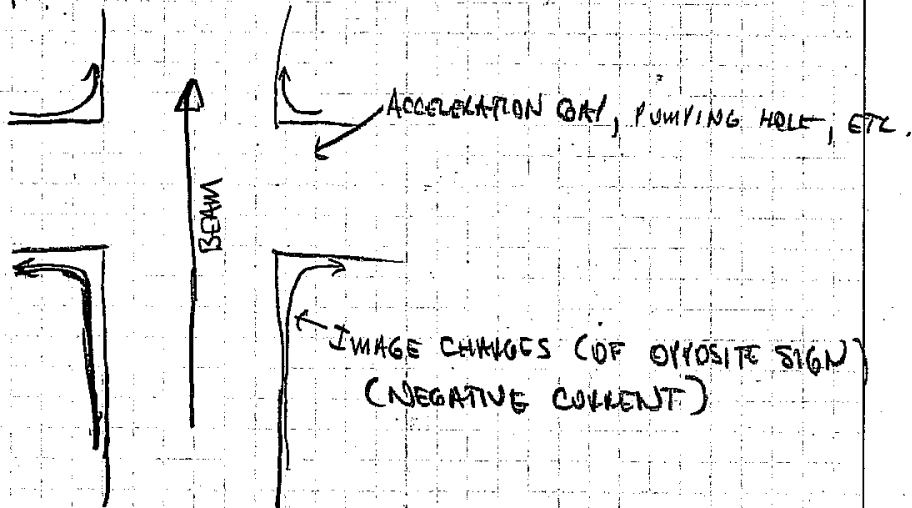
↑
EXTERNALLY
GENERATED

SEE
REISER 6.3.2.

CALKHAN-MILLER, PH
Ph.D. DISSERTATION,
U.C. DAVIS, 1994

42-102 100 SHEETS
National Brand
Made in U.S.A.

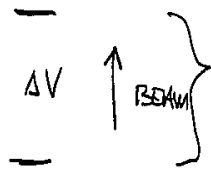
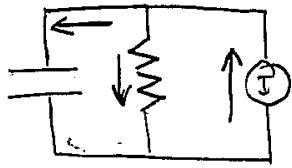
AS BEAM PASSES CONDUCTING SURFACE IMAGE CHARGE AND
CURRENT INTERACTS WITH BEAM. HIGHLY GEOMETRY
DEPENDENT.



CAN BE CALCULATED APPROXIMATELY USING CIRCUIT MODEL.

RESISTIVITY IN WALL, AND COMPLICATED ELECTRON FLOW
PATTERNS CREATE A RETARDING ELECTRIC FIELD ON
BEAM.

MODEL OF IMPEDANCE (IN LONG WAVELENGTH REGIME) (6)



ONE MODULE OF MANY, EACH SEPARATED BY DISTANCE L

$$I = C \frac{d\Delta V}{dt} + \frac{\Delta V}{R}$$

$$I = [CL] \frac{d\Delta V/L}{dt} + \frac{\Delta V/L}{R/L}$$

$$E = -\frac{\Delta V}{L}$$

$$C^* = CL$$

$$R^* = \frac{R}{L}$$

$$\text{LET } I = I_0 + I_1 e^{-i\omega t}$$

$$E = E_0 + E_1 e^{-i\omega t}$$

$$I_1 = i\omega C^* E_1 - \frac{E_1}{R^*}$$

$$\Rightarrow E_1 = \frac{-R^*}{1 - i\omega C^* R^*} I_1$$

$$Z^* = \frac{-E_1}{I_1} = \frac{R^*}{1 - i\omega C^* R^*}$$

RETURNING TO THE 1D FLUID EQUATIONS

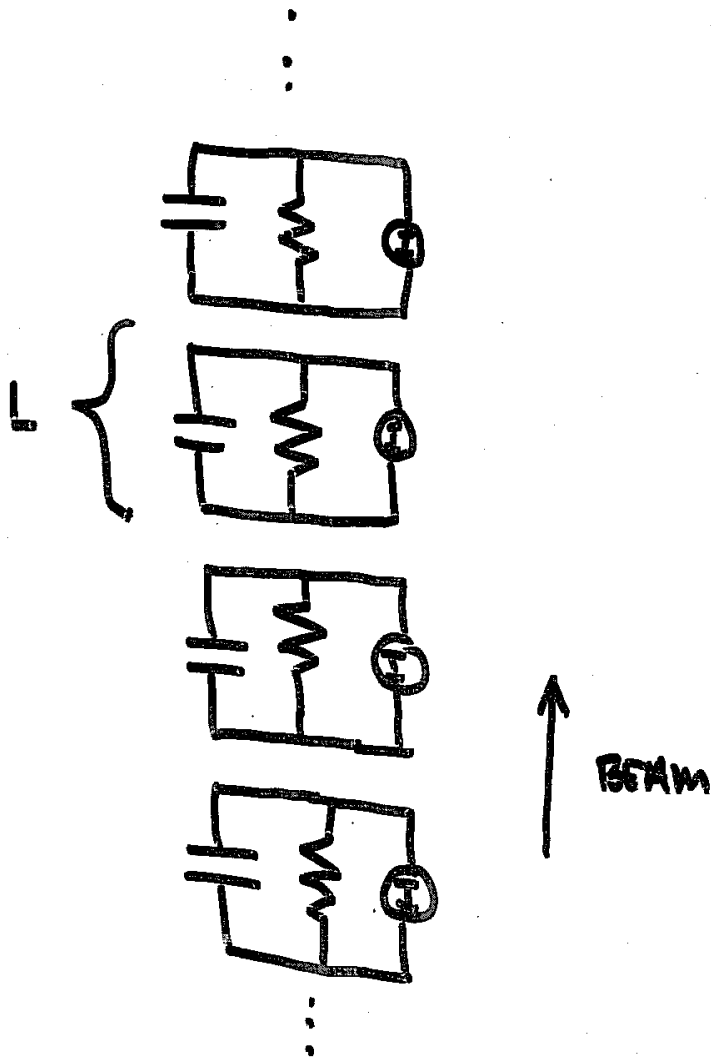
$$\frac{\partial \lambda}{\partial t} + \frac{\partial}{\partial z} \lambda v = 0$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} = \frac{-\rho g}{4\pi\epsilon_0 m} \frac{\partial \lambda}{\partial z} + \frac{\rho E_0}{m}$$

$$\text{Let } \lambda = \lambda_0 + \lambda_1 \exp[i(kz - \omega t)]$$

$$u = v_0 + u_1 \exp[i(kz - \omega t)]$$

$$(I = \lambda u \Rightarrow I_0 = \lambda_0 v_0 \quad \text{and} \quad I_1 = \lambda_0 u_1 + \lambda_1 v_0)$$



CONTINUOUS LIMIT :

$$R^* = R/L$$

$$C^* = CL$$

$$E = \frac{\Delta V}{L}$$

Resistance per unit length
 C^* per unit length

AVERAGE ELECTRIC
 FIELD

$$\frac{\partial \lambda}{\partial t} + \frac{\partial}{\partial z} \lambda v = 0$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial z} = \frac{-qg}{4\pi\epsilon_0 m} \frac{\partial \lambda}{\partial z} + \frac{qE}{m}$$

(7)

$$-i\omega \lambda_1 + ik \lambda_0 u_1 + ik v_0 \lambda_1 = 0$$

$$-i\omega u_1 + ik v_0 u_1 + \underbrace{\frac{ikgq}{4\pi\epsilon_0 m} \lambda_1}_{= \frac{ikc_s^2}{\lambda_0} \lambda_1} + \frac{q}{m} z^* (\lambda_0 v_1 + v_0 \lambda_1) = 0$$

$= I_1$

$$\begin{bmatrix} \omega - kv_0 & -k\lambda_0 \\ -\frac{c_s^2 k}{\lambda_0} + \frac{iq}{m} z^* v_0 & \omega - kv_0 + \frac{iq}{m} z^* \lambda_0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ u_1 \end{bmatrix} = 0$$

THE DETERMINANT OF THE ABOVE MATRIX MUST VANISH:

$$(\omega - kv_0)^2 + \frac{iq}{m} z^* \lambda_0 (\omega - kv_0) - c_s^2 k^2 + \frac{iq}{m} z^* \lambda_0 v_0 k = 0$$

$$\boxed{(\omega - kv_0)^2 - c_s^2 k^2 + \frac{iq z^* \lambda_0 \omega}{m} = 0} \quad (\text{LAB FRAME})$$

Using a Galilean transformation, in the beam frame:

$$\omega' = \omega - kv_0$$

$$k' = k$$

' denotes beam frame

$$\boxed{\omega'^2 - c_s^2 k'^2 + \frac{iq z^*(\omega') \lambda_0 (\omega' + kv_0)}{m} = 0} \quad (\text{BEAM FRAME})$$

Note $z^*(\omega') = z^*(\omega = \omega' + kv_0)$

CASE I PURE RESISTIVE IMPEDANCE $Z^* = R^*$ (REAL)

$$\omega'^2 = \pm c_s k' \sqrt{1 + i R^* \frac{g \lambda_0 (\omega' + k' V_0)}{m c_s^2 k'^2}}$$

Using $c_s^2 = \frac{g g \lambda_0}{4 \pi \epsilon_0 m}$ and $\frac{\omega'}{k'} \sim c_s \ll V_0$

$$\omega' = \pm c_s k' \sqrt{1 - i R^* \left(\frac{4 \pi \epsilon_0}{g}\right) \frac{V_0}{k'}}$$

$$\approx \pm \left[c_s k' - i \frac{c_s V_0}{2} \left(\frac{4 \pi \epsilon_0}{g}\right) R^* \right]$$

Since $\lambda_1, E_1 \sim \exp [i(k'x' - \omega't')]$

CHOOSING "+" ($\text{Re } \omega' > 0$) $\Rightarrow z' = c_s t'$ line of constant phase \Rightarrow Forward propagation

($\text{Im } \omega' < 0$) $\Rightarrow \lambda_1 \sim \exp \left[-\frac{c_s V_0}{2} \left(\frac{4 \pi \epsilon_0}{g}\right) R^* t' \right] \Rightarrow$ DECAYING PERTURBATION

CHOOSING "-"

($\text{Re } \omega' < 0$) $\Rightarrow z' = -c_s t'$ is line of constant phase \Rightarrow BACKWARD PROPAGATING

$$\Rightarrow \lambda_1 \sim \exp \left[\underbrace{+\frac{c_s V_0}{2} \left(\frac{4 \pi \epsilon_0}{g}\right) R^* t'}_G \right]$$

INSTABILITY!

$$\lambda_1 \approx \lambda_{10} \exp[G]$$

(9)

$$G \equiv \Gamma_{Rt} = \frac{4\pi\epsilon_0 c_s v_0 R^* t}{2g}$$

LOGARITHMIC
GAIN
OF
INSTABILITY = $\ln\left(\frac{\lambda_{\text{final}}}{\lambda_{\text{initial}}}\right)$

Now $t_{\text{max}} = \begin{cases} \min \left\{ \begin{array}{l} \lambda_b / c_s \\ t_{\text{residence}} \end{array} \right. \end{cases}$

TRANSIT TIME FOR
PERTURBATION TO TRAVEL
FROM HEAD TO TAIL
RESIDENCE TIME
WITHIN
ACCELERATOR

IF upper condition holds

$$G \sim \frac{v_0^2}{2} \left(\frac{4\pi\epsilon_0}{g} \right) R^* t$$

IF lower condition holds

$$G \sim \sqrt{\lambda}$$

$$\epsilon = QV$$

(10)

$$I \sim \frac{6 \text{ MJ}}{4 \text{ GeV} \cdot 200 \text{ ns}} \sim \frac{QV}{V \Delta t} \\ \sim 7.5 \text{ kA}$$

EXAMPLE:

FOR MATCHED BEAM IMPEDANCE

$$R^* = \frac{dV/ds}{I} \sim \frac{10^6 \text{ V/m}}{10 \text{ kA}} \sim 100 \Omega/\text{m}$$

$$v_0 \sim 0.2c$$

$$\Delta t \sim 200 \text{ ns}$$

$$G \sim \frac{v_0^2}{2} \left(\frac{4\pi\epsilon_0}{g} \right) R^* \Delta t$$

$$\sim 3.6$$

(AN EARLY CONCERN FOR HEAVY ION FUSION)

$$R^* = 100 \Omega/m$$

(1)

FOR ALL
SIMULATIONS
(p 11-15)

$$V_0 = C/3$$

$$I = 3 \text{ kA}$$

$$l_b = 10 \text{ m}$$

$$\frac{V_b}{V_p} = 0.4$$

$$kT_{\perp} = kT_{\parallel} = 10 \text{ keV}$$

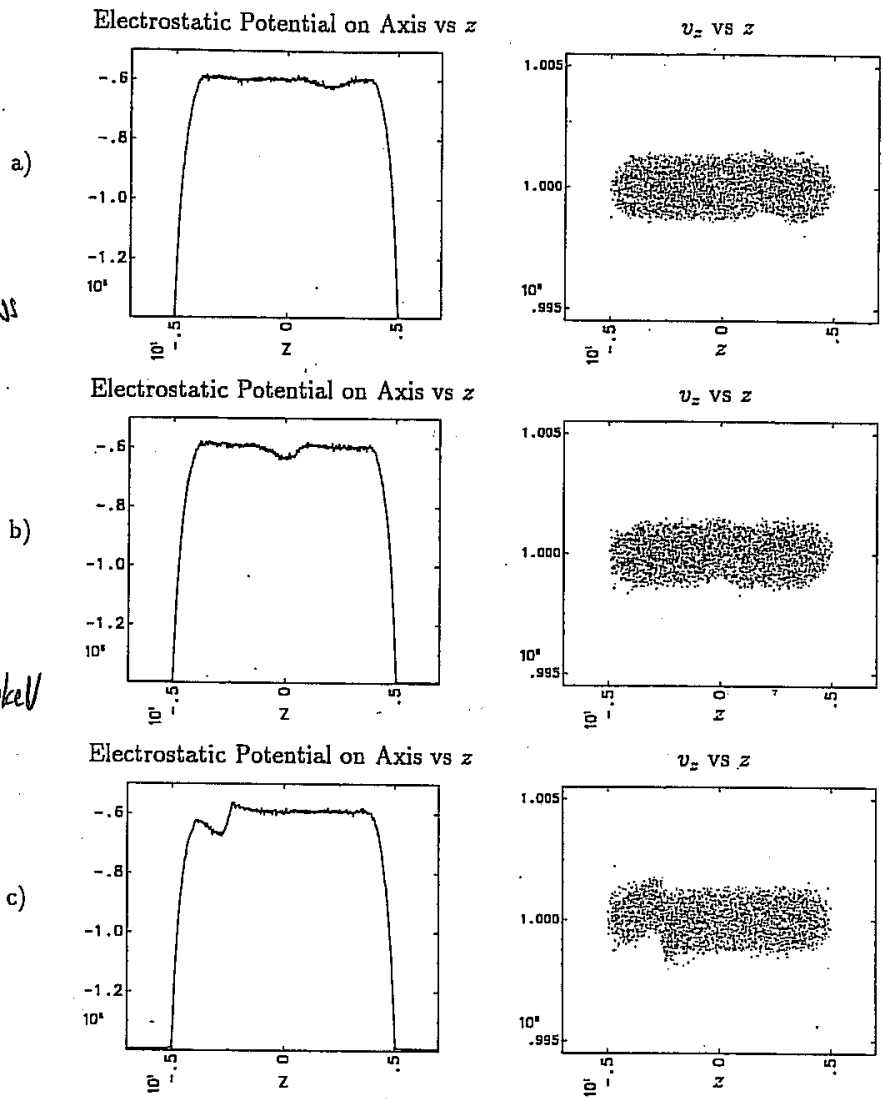


Figure 4.2: A simulation with $100 \Omega/m$ resistance shows moderate growth. (a) $6.6 \mu\text{s}$, (b) $10.9 \mu\text{s}$, (c) $17.5 \mu\text{s}$

from D.A. Callahan Miller, Ph.D. Thesis
U.C. Davis, 1994

$$R^* = 100 \Omega/m$$

(12)

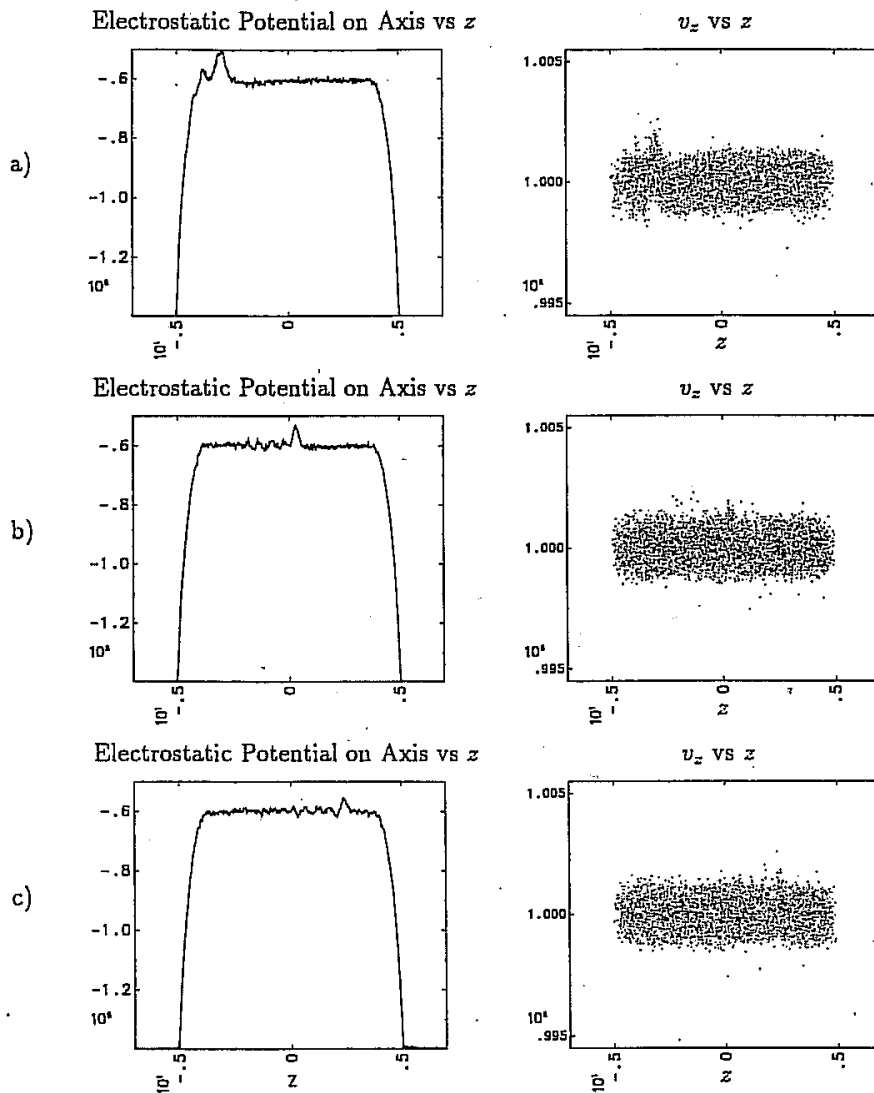


Figure 4.3: The perturbation reflects off the beam end and decays as it travels forward.
(a) 28.4 μs , (b) 35.0 μs , (c) 39.4 μs

from D.A. Callahan Miller, Ph.D. Thesis
U.C. Davis, 1994
(FORWARD WAVE)

$$R^* = 200 \Omega/m$$

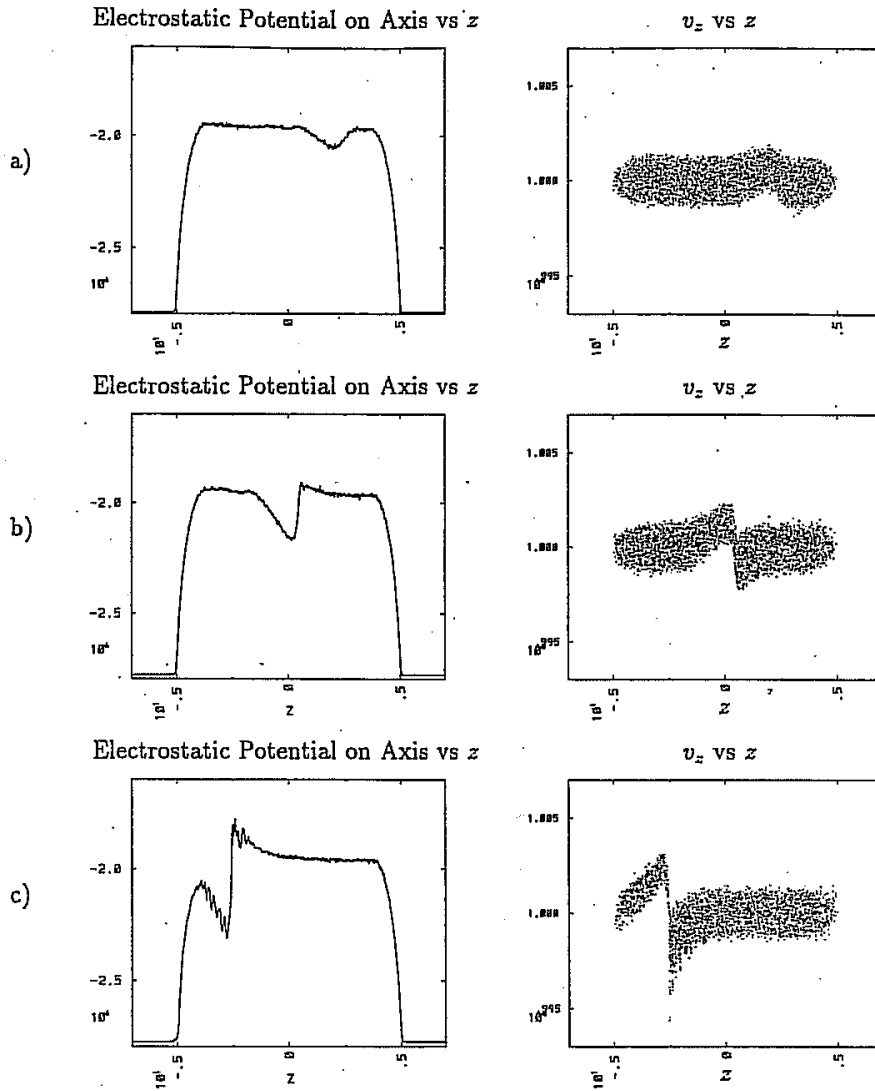


Figure 4.1: A simulation with 200 Ω/m resistance shows large amounts of growth. (a) 6.6 μs , (b) 10.9 μs , (c) 17.5 μs

from D.A. Callahan Miller, Ph.D. Thesis
U.C. Davis, 1994

$$(\omega - kv_0)^2 - c_s^2 k^2 + \frac{i q z^* \lambda_0 \omega}{m} = 0$$

(10)

CASE II RESISTIVE + CAPACITIVE IMPEDANCE

$$z^* = \frac{R^*}{1 - i\omega C^+ R^*} = \frac{R^* + i\omega C^+ R^{*2}}{1 + \omega^2 C^{+2} R^{*2}}$$

GOING BACK TO (10):

IN LAB FRAME:

$$(\omega - kv_0)^2 - c_s^2 k^2 + \frac{i q R^* \lambda_0 \omega}{m(1 + \omega^2 C^{+2} R^{*2})} - \frac{q \omega^2 C^+ R^{*2} \lambda_0}{m(1 + \omega^2 C^{+2} R^{*2})} = 0$$

$$(\omega - kv_0)^2 - c_s^2 k^2 - \frac{4\pi\epsilon_0 \omega^2 C^+ R^{*2} c_s^2}{g(1 + \omega^2 C^{+2} R^{*2})} + \frac{i(4\pi\epsilon_0 c_s^2 R^* \omega)}{g(1 + \omega^2 C^{+2} R^{*2})}$$

IN BEAM FRAME:

$$\omega'^2 - c_s^2 k'^2 - \frac{2\Gamma_R(c_s/v_0)(\omega' + k'v_0)^2 C^+ R^*}{(1 + (\omega' + k'v_0)^2 C^{+2} R^{*2})} + \frac{i 2\Gamma_R(c_s/v_0)(\omega' + k'v_0)}{(1 + (\omega' + k'v_0)^2 C^{+2} R^{*2})} = 0$$

So if one takes limit $C \rightarrow \infty$ the final two terms tend to zero. Thus capacitance has reduced the instability growth rate.

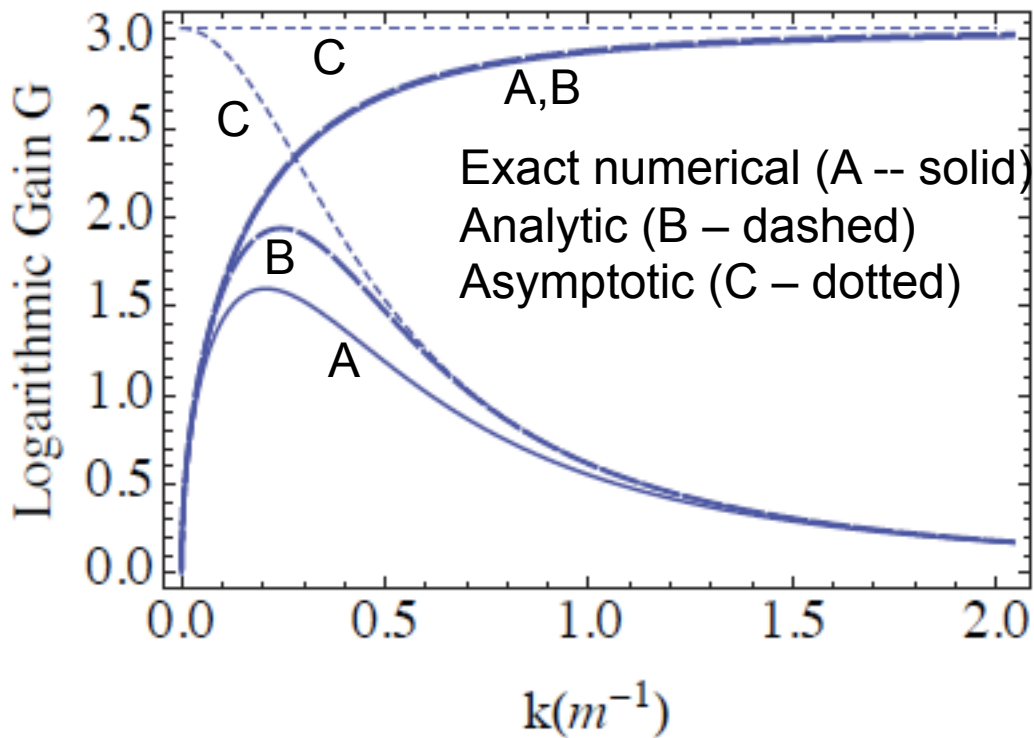
Here $\Gamma_R \equiv \frac{4\pi\epsilon_0 c_s v_0 R^*}{2g}$

$$\omega'^2 - c_s^2 k'^2 - \frac{2\Gamma_R(c_s/v_0)(\omega' + k'v_0)^2 C^\dagger R^*}{(1 + (\omega' + k'v_0)^2 C^{\dagger 2} R^{*2})} + \frac{i 2\Gamma_R(c_s/v_0)(\omega' + k'v_0)}{(1 + (\omega' + k'v_0)^2 C^{\dagger 2} R^{*2})} = 0$$

For $2(c_s/v_0)\Gamma_R R^* C^\dagger \ll 1$: (A)

$$\begin{aligned} \omega' &= \pm c_s k' \sqrt{1 - i \frac{2\Gamma_R}{c_s k' (1 + (k'v_0 R^* C^\dagger)^2)}} \\ &= \pm c_s k' \left(1 + \left(\frac{2\Gamma_R}{c_s k' (1 + (k'v_0 R^* C^\dagger)^2)} \right)^2 \right)^{1/4} \times \\ &\quad \times \left(\cos \left[\frac{1}{2} \tan^{-1} \left(\frac{2\Gamma_R}{c_s k' (1 + (k'v_0 R^* C^\dagger)^2)} \right) \right] \right. \\ &\quad \left. - i \sin \left[\frac{1}{2} \tan^{-1} \left(\frac{2\Gamma_R}{c_s k' (1 + (k'v_0 R^* C^\dagger)^2)} \right) \right] \right) \end{aligned} \quad (B)$$

$$\omega' \simeq \pm c_s k' \mp i \frac{\Gamma_R}{1 + (k'v_0 R^* C^\dagger)^2} \quad \text{for} \quad \frac{2\Gamma_R}{c_s k' (1 + (k'v_0 R^* C^\dagger)^2)} \ll 1 \quad (C)$$



Logarithmic Gain G of the longitudinal instability as a function of perturbation wavenumber k , for $R = 100 \text{ } \Omega/\text{m}$, $C^\dagger = 0$ (upper curves) and $C^\dagger = 2 \times 10^{-10} \text{ F}\cdot\text{m}$ (lower curves), after a growth time corresponding to l_b/c_s , where $l_b = 10 \text{ m}$ and $c_s = 4.9 \times 10^5 \text{ m/s}$

$$RC^* = 2 \times 10^{-8} \text{ s}$$

$$R^* = 100 \Omega/\text{m}$$

15

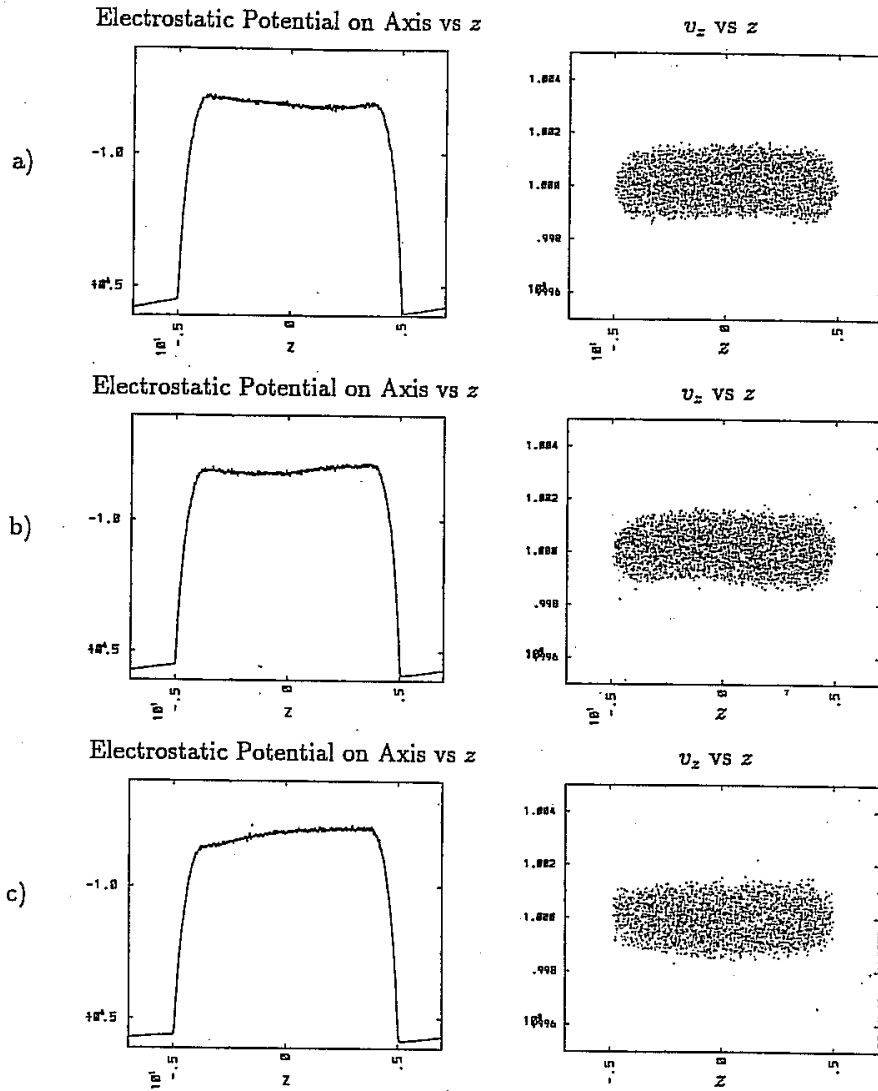


Figure 4.6: When capacitance is added to the system, a larger perturbation is launched, but little growth occurs (a) 6.6 μs , (b) 10.9 μs , (c) 17.5 μs

from D.A. Callahan Miller, Ph. D. Thesis,
U.C. Davis, 1994

Summary of LONGITUDINAL INSTABILITY

"RESISTIVE WALL" OR "LONGITUDINAL" INSTABILITY HAS POTENTIAL TO DEGRADE LONGITUDINAL EMITTANCE IN HIGH CURRENT ACCELERATORS.

HOWEVER, CAPACITANCE (e.g. FROM ACCELERATING GUNS) DECREASES GROWTH CAN MITIGATE THE INSTABILITY,

NOT DISCUSSED:

1. LONGITUDINAL TEMPERATURE DRIFTS INSTABILITY (e.g. REISER 6.3.3)
2. FEED BACK HAS BEEN PROPOSED TO CONTROL INSTABILITY IF NEEDED

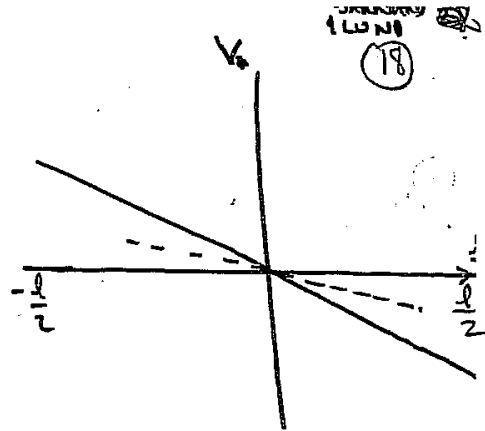
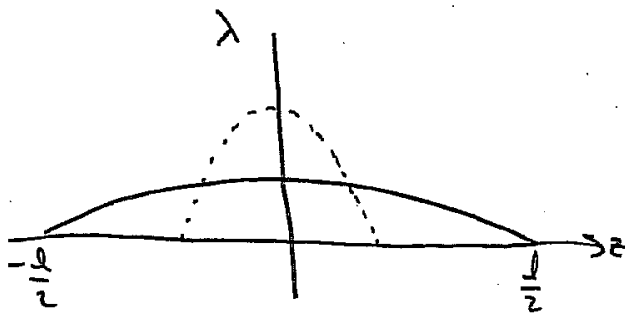
DRIFT COMPRESSION

OBJECTS :

APPLY A HEAD-TO-TAIL VELOCITY TILT TO
INCREASE CURRENT BY DECREASING PULSE DURATION

DURING COMPRESSION "TAILS" ARE NOT REQUIRED

AT END OF DRIFT COMPRESSION, VELOCITY "TILT"
SHOULD BE MINIMIZED, SO THAT CHROMATIC
ABERRATIONS IN FINAL FOCUS ARE MINIMIZED.



$$\frac{\partial \lambda}{\partial t} + \frac{\partial}{\partial z} \lambda v = 0$$

CONTINUITY EQUATION

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial z} = \frac{-qg}{4\pi\epsilon_0} \frac{\partial \lambda}{\partial z}$$

MOMENTUM EQUATION

$$\text{LET } \lambda = \lambda_0(t) \left(1 - \frac{4z^2}{l^2(t)} \right)$$

← PARABOLIC LINE
CHARGE PROFILE

$$v = -\Delta V(t) \frac{z}{l(t)}$$

← LINEAR VELOCITY
PROFILE

① MASS conservation:

$$Q_c = \int_{-l/2}^{l/2} \lambda dz = \lambda_0 \int_{-l/2}^{l/2} \left(1 - \frac{4z^2}{l^2} \right) dz = \frac{2}{3} \lambda_0 l = \text{constant}$$

(but
 $\lambda_0 = \lambda_0(t)$
& $l = l(t)$)

$$(1) \quad \frac{z}{3} \lambda_0 l = \text{constant}$$

CALCULATING PARTIAL DERIVATIVES:

$$\frac{\partial \lambda}{\partial t} = \dot{\lambda}_0 \left(1 - \frac{4z^2}{l^2}\right) + 2\lambda_0 \left(\frac{4z}{l^3}\right) \dot{l}$$

$$\frac{\partial \lambda}{\partial z} = -\frac{8z}{l^2} \lambda_0$$

$$\frac{\partial V}{\partial t} = -\dot{\Delta V} \left(\frac{z}{l}\right) + \frac{\Delta V}{l^2} \dot{l}$$

$$\frac{\partial V}{\partial z} = -\frac{\Delta V}{l}$$

$$\frac{\partial \lambda}{\partial t} + \frac{\partial}{\partial z} \lambda v = 0$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial z} = \frac{-9g}{m 4\pi\epsilon_0} \frac{\partial \lambda}{\partial z}$$

$$\lambda = \lambda_0(t) \left(1 - \frac{4z^2}{l^2(t)}\right)$$

$$V = -\Delta V(t) \frac{z}{l(t)}$$

$$(2) \text{ CONTINUITY EQUATION } \Rightarrow \begin{cases} \frac{8z^2 \lambda_0}{l^3} (\Delta v + \dot{l}) = 0 \Rightarrow \dot{l} = -\Delta v \\ \left(1 - \frac{4z^2}{l^2}\right) \left(\dot{\lambda}_0 - \frac{\Delta V \lambda_0}{l}\right) = 0 \end{cases}$$

$$(3) \text{ MOMENTUM EQUATION } \Rightarrow \left(\frac{z}{l}\right) \left[-\dot{\Delta v} + \frac{\dot{l} \Delta V}{l} + \frac{\Delta V^2}{l} + \frac{9g}{m 4\pi\epsilon_0} \frac{\lambda_0}{l}\right] = 0$$

$$(1) \& (2) \Rightarrow \frac{\dot{\lambda}_0}{\lambda_0} = \frac{\Delta V}{l} = -\frac{\dot{l}}{l} \quad (4)$$

$$(3) \& (4) \Rightarrow \ddot{l} - \frac{12g}{4\pi\epsilon_0 m} \frac{Q_c}{l^2} = 0$$

where $Q_c = \frac{z}{3} \lambda_0 l = \text{const.}$

|||
CHARGE
IN
WAVELENGTH (NOT
PERIOD)

LONGITUDINAL "ENVELOPE" EQUATION
(WITHOUT EMITTANCE)

MULTIPLY BY \dot{l} & INTEGRATE:

ORANGE & RED (20)

$$\frac{\dot{l}^2}{2} + \frac{1299}{4\pi\epsilon_0 m} \frac{Q_c}{l} = \frac{\dot{l}_f^2}{2} + \frac{1299}{4\pi\epsilon_0 m} \frac{Q_c}{l_f}$$

HERE SUBSCRIPT "f"
= "final"

& SUBSCRIPT "o"
= original or initial

$$\Rightarrow \dot{l}_o = \sqrt{\frac{1699}{4\pi\epsilon_0 m} \lambda_f \left[1 - \frac{l_f}{l_o} \right]}$$

Now $Q_f = \frac{\lambda_f}{4\pi\epsilon_0 V_f}$ = FINAL PULVANCE AT CENTER OF HYPERBOLIC TULSE

(NOTE $Q_c = \frac{2}{3} \lambda_o l$
= CHARGE

WHEREAS $Q_f =$ PULVANCE (DIMENSIONLESS)

$C =$ COMPRESSION RATIO = $\frac{l_o}{l_f}$

$\frac{\Delta V}{V_o} =$ velocity tilt = $\frac{|\dot{l}|}{V_o}$

$$\rightarrow \frac{\Delta V}{V} = \sqrt{89 Q_f \left[1 - \frac{1}{C} \right]}$$

for $Q_f = 10^{-4}$
 $g = 1.1$
 $C = 20$

$$\Rightarrow \frac{\Delta V}{V} = 0.029$$

$$\text{DRIFT LENGTH} \cong \frac{l}{\Delta V} V_o = \frac{l}{\Delta V/V} = 345 \text{ m for } l = 10 \text{ m}$$

Vlasov - equation for a drifting beam:

$$\frac{\partial f}{\partial s} + x' \frac{\partial f}{\partial x} + x'' \frac{\partial f}{\partial x'} + y' \frac{\partial f}{\partial y} + y'' \frac{\partial f}{\partial y'} + z' \frac{\partial f}{\partial z} + z'' \frac{\partial f}{\partial z'} = 0$$

$$\text{Let } \tilde{f}(z, z', s) \equiv \iiint f \, dx \, dx' \, dy \, dy'$$

INTEGRATING VLASOV EQUATION:

if $z'' \neq f(x, x', y, y')$:

$$\Rightarrow \frac{\partial \tilde{f}}{\partial s} + \underbrace{\iiint x' \frac{\partial f}{\partial x} \, dx \, dx' \, dy \, dy'}_{= \tilde{f} z'} + \dots + z' \frac{\partial \tilde{f}}{\partial z} + z'' \frac{\partial \tilde{f}}{\partial z'} = 0$$

$$\Rightarrow \boxed{\frac{\partial \tilde{f}}{\partial s} + z' \frac{\partial \tilde{f}}{\partial z} + z'' \frac{\partial \tilde{f}}{\partial z'} = 0} \quad \text{1D Vlasov}$$

Now let $\lambda \equiv q \int \tilde{f} \, dz'$; $\lambda \bar{z}' = q \int \tilde{f} z' \, dz'$; $q \lambda \bar{z}'^2 \equiv \int \tilde{f} z'^2 \, dz'$

Also, let $\Delta z'^2 \equiv \bar{z}'^2 - (\bar{z}')^2$

FLUID EQUATIONS

INTEGRATING 1D VLASOV OVER z' :

$$\boxed{\frac{\partial \lambda}{\partial s} + \frac{\partial (\lambda \bar{z}')}{\partial z} = 0} \quad \text{(CONTINUITY EQUATION)}$$

MULTIPLYING BY z' & INTEGRATING VLASOV OVER z' :

$$\frac{\partial}{\partial s} \lambda \bar{z}' + \frac{\partial}{\partial z} \lambda \bar{z}'^2 - \lambda z'' = 0$$

DIVIDING BY λ , USING CONTINUITY EQUATION & DEFINITION OF $\Delta z'^2$:

$$\boxed{\underbrace{\frac{\partial \bar{z}'}{\partial s} + \bar{z}' \frac{\partial}{\partial z} \bar{z}'}_{\text{INERTIAL}} + \underbrace{\frac{1}{\lambda} \frac{\partial (\lambda \Delta z'^2)}{\partial z}}_{\text{PRESSURE TERM}} = \underbrace{z''}_{\text{FORCE}}} \quad \text{(MOMENTUM EQUATION)}$$



LONGITUDINAL ENVELOPE EQUATION

$$\frac{\partial f^2}{\partial s} + z' \frac{\partial f^2}{\partial z} + z'' \frac{\partial f^2}{\partial z'} = 0$$

$Q_c = \text{total charge in bunch}$

$$\text{If } z'' = -K(s)z + \frac{gg}{4\pi\epsilon_0 m v^2} \left(\frac{12Q_c}{L^3} \right) z$$

$$\Rightarrow \frac{\partial}{\partial s} \langle z^2 \rangle = 2 \langle z z' \rangle$$

$$\frac{\partial}{\partial s} \langle z z' \rangle = \langle z'^2 \rangle + \frac{gg}{4\pi\epsilon_0 m v^2} \left(\frac{12Q_c}{L^3} \right) \langle z^2 \rangle - K(s) \langle z^2 \rangle$$

$$\frac{\partial}{\partial s} \langle z'^2 \rangle = 2 \left(\frac{gg}{4\pi\epsilon_0 m v^2} \right) \left(\frac{12Q_c}{L^3} \right) \langle z z' \rangle - 2K(s) \langle z z' \rangle$$

NOTE $\langle z^2 \rangle = \frac{1}{Q_c} \int_{-L}^L \int_{-L/2}^{L/2} z^2 f(z, z') dz dz' = \frac{1}{20} L^2$

$$\epsilon_z^2 = 25 \left[\langle z^2 \rangle \langle z'^2 \rangle - \langle z z' \rangle^2 \right]$$

$$\Rightarrow \frac{d^2 L}{ds^2} = \frac{16 \epsilon_z^2}{L^3} + \frac{12 gg Q_c}{4\pi\epsilon_0 m v^2 L^2} - K(s) L$$

Let $v_z = L/z$

$$\Rightarrow \frac{d^2 v_z}{ds^2} = \frac{\epsilon_z^2}{v_z^3} + \frac{3}{2} \frac{gg Q_c}{4\pi\epsilon_0 m v^2} \frac{1}{v_z^2} - K(s) v_z$$

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If we regard the envelope radii r_x , r_y as specified functions of s , then these equations of motion are [Hill's equations](#) familiar from elementary accelerator physics:

$$x''(s) + \kappa_x^{\text{eff}}(s)x(s) = 0$$

$$y''(s) + \kappa_y^{\text{eff}}(s)y(s) = 0$$

$$\kappa_x^{\text{eff}}(s) = \kappa_x(s) - \frac{2Q}{[r_x(s) + r_y(s)]r_x(s)}$$

$$\kappa_y^{\text{eff}}(s) = \kappa_y(s) - \frac{2Q}{[r_x(s) + r_y(s)]r_y(s)}$$

Suggests Procedure:

- ♦ Calculate Courant-Snyder invariants under assumptions made
- ♦ Construct a distribution function of Courant-Snyder invariants that generates the uniform density elliptical beam projection assumed
 - **Nontrivial step:** guess and show that it works

Resulting distribution will be an [equilibrium](#) that does not evolve in s in 4D phase-space, but lower-dimensional phase-space projections can evolve in s

Self – consistent longitudinal distribution

Recall Hill's equation: (From Steve Lund's notes on "Transverse equilibrium distributions," p. 20 -26.)

$$z'' + K(s)z = 0$$

The Courant-Snyder invariant C_z for this equation can be written:

$$C_z = \left(\frac{z}{r_z} \right)^2 + \left(\frac{r_z z' - r_z' z}{\epsilon_z} \right)^2 = \text{constant along a particle trajectory}$$

At each s , particle lies on ellipse of constant area $\pi\epsilon_z$.

Along each trajectory: $\frac{dC_z}{ds} = 0$

$$\frac{df}{ds} = \frac{df}{dC_z} \frac{dC_z}{ds} = 0 \quad \text{so } f(C_z) \text{ is a solution of the Vlasov equation.}$$

BUT $\lambda = \int f(z, z', s) dz'$ must be of the form

$$\lambda = (a_0 + b_0 z^2) \Rightarrow K(s)z \sim E_z = -g \frac{\partial \lambda}{\partial z} \sim z$$

So what $f(C_z)$ yields $\lambda = (a_0 + b_0 z^2)$?

Answer: $f(C_z) = \frac{3N}{2\pi\epsilon_z} \sqrt{1 - C_z}$

SELF-CONSISTENT LONGITUDINAL DISTRIBUTION:

NEUFFER DISTRIBUTION

D. NEUFFER, PARTICLE ACCELERATORS, Vol 11, p 23 (1980)

RETURNING TO THE 1D VLASOV EQUATION:

$$\frac{\partial f}{\partial s} + z' \frac{\partial f}{\partial z} + z'' \frac{\partial f}{\partial z'} = 0$$

If $z'' = -A \frac{\partial \lambda}{\partial z} - K(s) z$

THEN,

$$f(z, z', s) = \frac{3N}{2\omega E_z} \sqrt{1 - \frac{z^2}{v_z^2} - \frac{v_z^2 (z' - v_z' z / v_z)^2}{E_z^2}}$$

for $-v_z < z < v_z$

$$\frac{v_z' z}{v_z} - \frac{E_z}{v_z} \sqrt{1 - \frac{z^2}{v_z^2}} \leq z' \leq \frac{v_z' z}{v_z} + \frac{E_z}{v_z} \sqrt{1 - \frac{z^2}{v_z^2}}$$

is a solution to the 1D Vlasov equation.

HERE $E_z^2 = 25 [\langle z^2 \rangle \langle z'^2 \rangle - \langle z z' \rangle^2] = \text{CONSTANT}$

N = total number of particles in bunch

v_z = half edge of bunch

NOTE THAT $\lambda(z) = \frac{3}{4} \frac{N}{v_z} \left(1 - \frac{z^2}{v_z^2}\right) = \int_{-v_z}^{v_z} f(z, z', s) dz'$

$\Rightarrow \frac{\partial \lambda}{\partial z} \propto z \Rightarrow \text{LINEAR SPACE CHARGE FIELD}$

Note that $f(z, z', s) = 0$ outside the above ellipse in z, z' phase space.

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 125 SHEETS QUARTER SIZE
 62 SHEETS EIGHTH SIZE
 31 SHEETS SIXTEENTH SIZE
 16 SHEETS THIRTYSECOND SIZE
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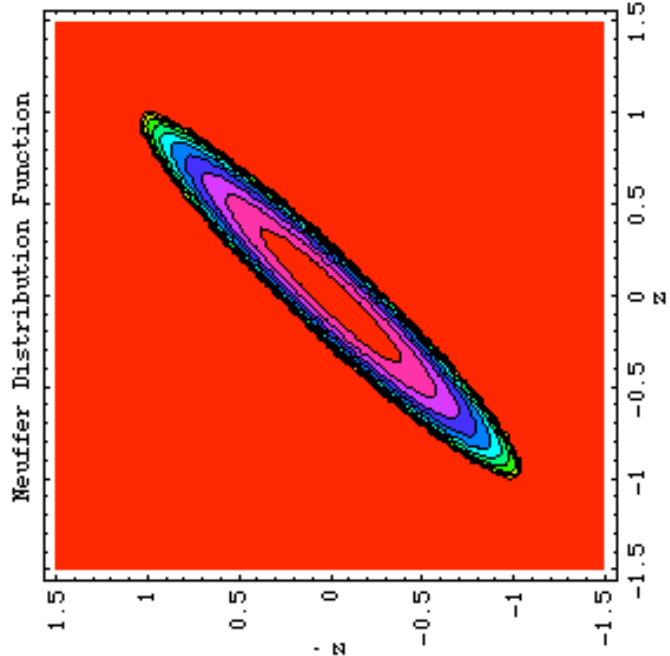
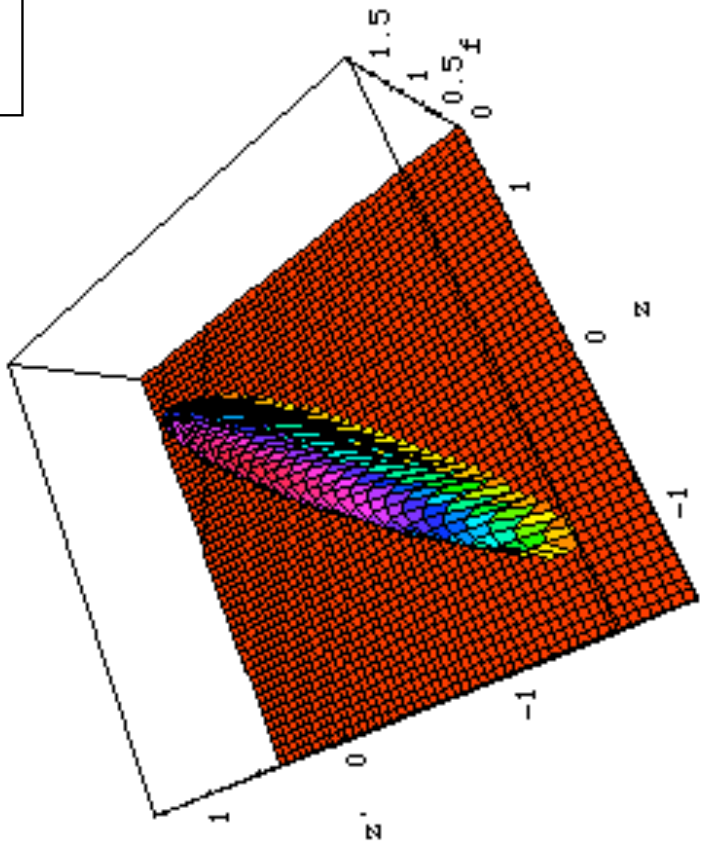
Neuffer Distribution Function

$$f[z, z'] = \frac{3N}{2\pi\epsilon_z} \sqrt{\frac{z^2}{r_z^2} - \frac{r_z^2(z' - r_z'z/r_z)^2}{\epsilon_z^2}}$$

for:

$$-r_z \leq z \leq r_z$$

$$r_z'z - \frac{\epsilon_z}{r_z} \sqrt{1 - \frac{z^2}{r_z^2}} \leq z' \leq \frac{r_z'z}{r_z} + \frac{\epsilon_z}{r_z} \sqrt{1 - \frac{z^2}{r_z^2}}$$



Here $N=r_z=r_z'=1$; $\epsilon_z=0.3$

Summary

1D VLAROV EQUATION

g-factor model

$$\frac{\partial f^2}{\partial s} + z' \frac{\partial f^2}{\partial z} + z'' \frac{\partial f^2}{\partial z'} = 0$$

$$z'' = -g \frac{\partial \lambda}{4\pi \epsilon_0 m v_z^2 \partial z}$$

Leads to fluid equations:

$$\frac{\partial \lambda}{\partial s} + \frac{\partial}{\partial z} (\lambda z') = 0$$

$$\frac{\partial z'}{\partial s} + z' \frac{\partial z'}{\partial z} + \frac{1}{\lambda} \frac{\partial}{\partial z} (\lambda z z') + \frac{c^2}{\lambda_0 v_0^2} \frac{\partial \lambda}{\partial z} = 0$$

⇒ SPACE CHARGE WAVES

↳ LONGITUDINAL
ON RESISTIVE WAVE INSTABILITY

⇒ SPACE CHARGE LATERALION WAVES

⇒ PARABOLIC BUNCH COMPRESSION $\frac{\partial \lambda}{\partial z} \propto z$

VLAROV EQUATION ALSO ⇒ ENVELOPE EQUATION

$$\frac{d^2 r_z}{ds^2} = \frac{E_z^2}{V_z^3} + \frac{3}{2} \frac{g Q_0}{4\pi \epsilon_0 m v_z^2} \frac{1}{V_z^2} - K(s) r_z$$

KINETIC SOLUTION TO VLAROV EQUATION SATISFYING RMS ENVELOPE EQUATION IS "MUFFET DISTRIBUTION" (ANALOGOUS TO KV).

$$f(z, z') = \frac{3N}{2\pi E_+} \sqrt{1 - \frac{z^2}{V_z^2} - \frac{V_z^2}{E_+^2} (z' - V_z' z / V_z)^2}$$

Summary

1D VLAROV EQUATION

g-factor model

$$\frac{\partial f^2}{\partial s} + z' \frac{\partial f^2}{\partial z} + z'' \frac{\partial f^2}{\partial z'} = 0$$

$$z'' = \frac{-g}{4\pi\epsilon_0 m v_z^2} \frac{\partial \lambda}{\partial z}$$

Leads to fluid equations:

$$\frac{\partial \lambda}{\partial s} + \frac{\lambda}{z} \left(\frac{\partial z'}{\partial z} \right) = 0$$

$$\frac{\partial z'}{\partial s} + z' \frac{\partial z'}{\partial z} + \frac{1}{\lambda} \frac{\partial (\lambda z z')}{\partial z} + \frac{c_s^2}{\lambda_0 v_0^2} \frac{\partial \lambda}{\partial z} = 0$$

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↳ LONGITUDINAL
OR RESISTIVE WAVE INSTABILITY

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KINETIC SOLUTION TO VLAROV EQUATION SATISFYING VMC ENVELOPE EQUATION IS "MUFFET DISTRIBUTION" (ANALOGOUS TO KV).

$$f(z, z') = \frac{3N}{2\pi E_z} \sqrt{1 - \frac{z^2}{V_z^2} - \frac{V_z^2}{E_z^2} \left(z' - \frac{1}{V_z} z \right)^2}$$