Intro. Lecture 02: Classes of Self-Consistent Simulations*

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US Particle Accelerator School (USPAS) Lectures On
"Self-Consistent Simulations of Beam and Plasma Systems"
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Outline
Introductory Lectures on Self-Consistent Simulations

Classes of Self-Consistent Beam Simulations

Classes of Intense Beam Simulations
A. Overview

There are three distinct classes of modeling of intense beams applicable to numerical simulation

0) Particle methods (see: Sec. B)
1) Distribution methods (see: Sec. C)
2) Moment methods (see: Sec. D)

All of these draw heavily on methods developed for the simulation of neutral plasmas. The main differences are:

- Lack of overall charge neutrality
  - Single species typical, though electron + ion simulations (Ecloud) and beam in plasma simulations are common also
- Directed motion of the beam along accelerator axis
- Applied field descriptions of the lattice
  - Optical focusing and bending elements
  - Accelerating structures

We will review and contrast these methods before discussing specific numerical implementations and concentrate mainly on particle in cell simulations

B. Particle Methods
B.1 Equations of Motion

Classical point particles are advanced with self-consistent interactions given by the Maxwell Equations

- Most general: If actual number of particles are used, this is approximately the physical beam under a classical (non-quantum) theory
- Often intractable using real number of beam particles due to numerical work and problem size
- Method also commonly called *Molecular Dynamics* simulations

Equations of motion (time domain, 3D, for generality)
ith particle moving in electric and magnetic fields

$$\frac{d\mathbf{p}_i}{dt} = \mathbf{F}_i = q_i\left[\mathbf{E}(\mathbf{x}_i,t) + \frac{d\mathbf{x}_i}{dt}\times\mathbf{B}(\mathbf{x}_i,t)\right]$$

$$m_i\gamma_i\frac{d\mathbf{x}_i}{dt} = \mathbf{p}_i \quad ; \quad \gamma_i = \left[1 + \frac{\mathbf{p}_i^2}{(m_i c)^2}\right]^{1/2}$$

Initial conditions

$$\mathbf{x}_i(t=0)$$
$$\mathbf{p}_i(t=0)$$

Particle phase-space orbits $\mathbf{x}_i(t),\ \mathbf{p}_i(t)$ are solved as a function of time in the self-consistent electric and magnetics fields $\mathbf{E}(\mathbf{x},t),\ \mathbf{B}(\mathbf{x},t)$

The *Lorentz force equation* of a charged particle is given by (MKS Units):

$$\boxed{\frac{d}{dt}\mathbf{p}_i(t) = q_i\left[\mathbf{E}(\mathbf{x}_i,t) + \mathbf{v}_i(t)\times\mathbf{B}(\mathbf{x}_i,t)\right]}$$

$m_i,\ q_i$ .... particle mass, charge

$\mathbf{x}_i(t)$ .... particle coordinate

$\mathbf{p}_i(t) = m_i\gamma_i(t)\mathbf{v}_i(t)$ .... particle momentum

$\mathbf{v}_i(t) = \frac{d}{dt}\mathbf{x}_i(t) = c\vec{\beta}_i(t)$ .... particle velocity

$\gamma_i(t) = \frac{1}{\sqrt{1-\beta_i^2(t)}}$ .... particle gamma factor

$i =$ particle index
$t =$ time

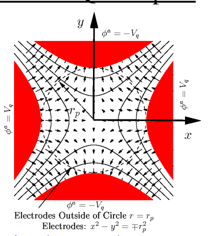
The electric and magnetic fields $\mathbf{E},\ \mathbf{B}$ are consistent with the Maxwell Equations and the linearity of the Maxwell equations can be exploited to resolve the fields into applied (lattice) and self (beam generated) components:

	Total		Applied		Self
Electric Field:	$\mathbf{E}(\mathbf{x},t)$	$=$	$\mathbf{E}^a(\mathbf{x},t)$	$+$	$\mathbf{E}^s(\mathbf{x},t)$
Magnetic Field:	$\mathbf{B}(\mathbf{x},t)$	$=$	$\mathbf{B}^a(\mathbf{x},t)$	$+$	$\mathbf{B}^s(\mathbf{x},t)$

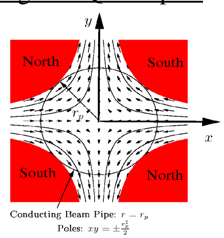
B.2 Applied Fields used to Focus, Bend, and Accelerate Beam

Transverse optics for focusing:

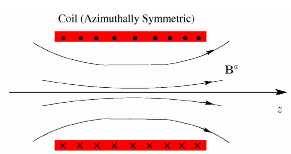
Electric Quadrupole



Magnetic Quadrupole

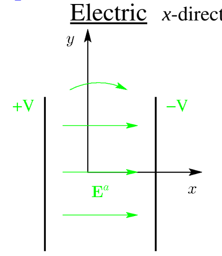


Solenoid

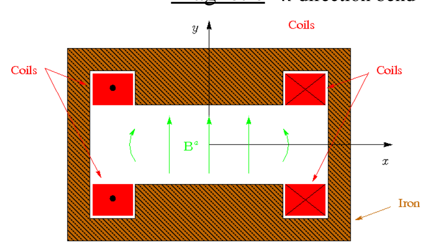


Dipole Bends:

Electric *x*-direction bend

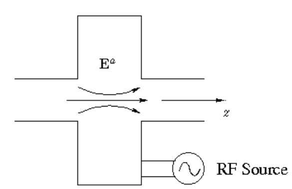


Magnetic *x*-direction bend

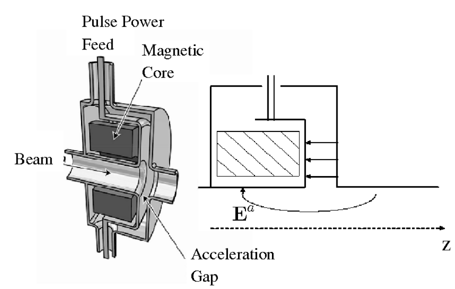


Longitudinal Acceleration:

RF Cavity



Induction Cell

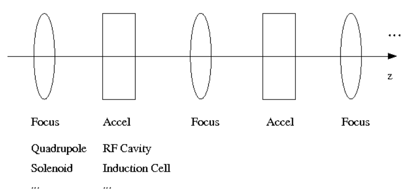


We will cover primarily simulations. More information on the basic physics can be found in USPAS lectures by Lund and Barnard on *Beam Physics with Intense Space Charge*:

https://people.nscl.msu.edu/~lund/uspas/bpisc_2015/

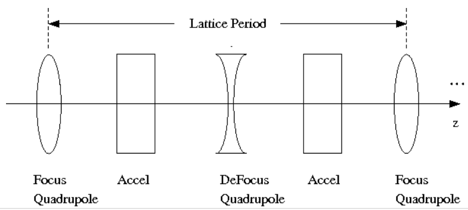
B.3 Machine Lattice

Applied field structures are often arraigned in a regular (periodic) lattice for beam transport/acceleration:

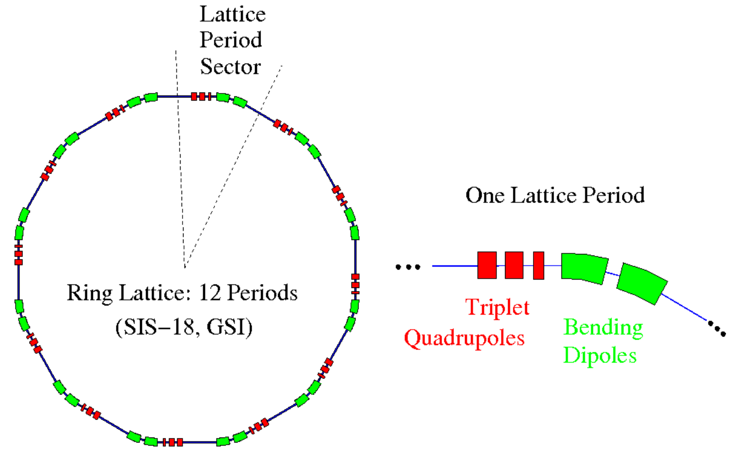


- Sometimes functions like bending/focusing are combined into a single element

Example – Linear FODO lattice (symmetric quadrupole doublet)



Lattices for rings and some beam insertion/extraction sections also incorporate bends and more complicated periodic structures:



- Elements to insert beam into and out of ring further complicate lattice
- Acceleration cells also (typically RF cavities at one or more locations)
- Errors (systematic, every element; random, once per lap) also complicate

B.4 Maxwell Equations

Fields (electromagnetic in most general form) $\mathbf{E},\ \mathbf{B}$ evolve consistently with the coupling to the particles according to the Maxwell Equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

+ boundary conditions on $\mathbf{E},\ \mathbf{B}$

Charge Density

$$\rho(\mathbf{x}, t) = \rho_{\text{ext}}(\mathbf{x}, t) + \sum_i q_i \delta[\mathbf{x} - \mathbf{x}_i(t)]$$

external (applied)

particle beam

Current Density

$$\mathbf{J}(\mathbf{x}, t) = \mathbf{J}_{\text{ext}}(\mathbf{x}, t) + \sum_i q_i \frac{d\mathbf{x}_i}{dt} \delta[\mathbf{x} - \mathbf{x}_i(t)]$$

$$\delta(\mathbf{x}) \equiv \delta(x)\delta(y)\delta(z)$$

$$\delta(x) \equiv \text{Dirac-delta function}$$

$$\mu_0 \epsilon_0 = \frac{1}{c^2}$$

$c =$ speed of light in *vacuuo*

Resolved into applied (lattice element) and self (beam generated) components

	Total		Applied		Self
Electric Field:	$\mathbf{E}(\mathbf{x}, t)$	$=$	$\mathbf{E}^a(\mathbf{x}, t)$	$+$	$\mathbf{E}^s(\mathbf{x}, t)$
Magnetic Field:	$\mathbf{B}(\mathbf{x}, t)$	$=$	$\mathbf{B}^a(\mathbf{x}, t)$	$+$	$\mathbf{B}^s(\mathbf{x}, t)$

Applied

$$\nabla \cdot \mathbf{E}^a = \frac{\rho_{\text{ext}}}{\epsilon_0}$$

$$\nabla \times \mathbf{E}^a = -\frac{\partial \mathbf{B}^a}{\partial t}$$

$$\nabla \cdot \mathbf{B}^a = 0$$

$$\nabla \times \mathbf{B}^a = \mu_0 \mathbf{J}_{\text{ext}} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}^a}{\partial t}$$

+ boundary conditions on $\mathbf{E}^a,\ \mathbf{B}^a$

Defer applied field discussion till later, but:

- Assume continuous
- Time varying (e.g. RF cavities) or static (typical DC optics) both possible
- Boundary conditions may not be fully separable from self-field

B.5 Self Fields

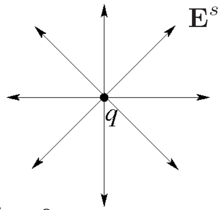
Self-fields are generated by the distribution of beam particles:

Charges

Currents

Particle at Rest

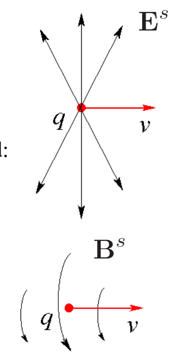
(pure electrostatic)



$\mathbf{B}^s = 0$

Particle in Motion

Obtain from Lorentz boost of rest-frame field: see Jackson, *Classical Electrodynamics*



- Superimpose for all particles in the beam distribution
- Accelerating particles also radiate
  - Often less important for heavy ions, but can be important in electron accelerators, light sources, and beam + plasma systems

/// Aside: Notation:

$$\nabla \equiv \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$ - Cartesian Representation

$$= \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$ - Cylindrical Representation

$x = r\cos\theta$ $\qquad \hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta$

$y = r\sin\theta$ $\qquad \hat{\theta} = -\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta$

$$= \frac{\partial}{\partial \mathbf{x}}$$ - Abbreviated Representation

$$= \frac{\partial}{\partial \mathbf{x}_\perp} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$ - Resolved Abbreviated Representation

Resolved into Perpendicular $(\perp)$ and Parallel (z) components

$$\mathbf{x} = \hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z$$
$$= \mathbf{x}_\perp + \hat{\mathbf{z}}z \qquad \mathbf{x}_\perp \equiv \hat{\mathbf{x}}x + \hat{\mathbf{y}}y$$

In integrals, we denote:

$$\int d^3x \; \cdots = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \; \cdots = \int d^2x_\perp \int_{-\infty}^{\infty} dz \; \cdots$$

$$\int d^2x_\perp \; \cdots = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; \cdots = \int_0^{\infty} dr \; r \int_{-\pi}^{\pi} d\theta \; \cdots$$

///

Self-Fields (dynamic, evolve with beam)

Generated by particle of the beam rather than (applied) sources outside beam

$$\nabla \cdot \mathbf{E}^s = \frac{\rho^s}{\epsilon_0} \qquad \nabla \times \mathbf{B}^s = \mu_0 \mathbf{J}^s + \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}^s$$

$$\nabla \times \mathbf{E}^s = -\frac{\partial}{\partial t}\mathbf{B}^s \qquad \nabla \cdot \mathbf{B}^s = 0$$

$\rho^s =$ beam charge density

$$= \sum_{i=1}^{N} q_i \delta[\mathbf{x} - \mathbf{x}_i(t)]$$

$\mathbf{J}^s =$ beam current density

$$= \sum_{i=1}^{N} q_i \mathbf{v}_i(t) \delta[\mathbf{x} - \mathbf{x}_i(t)]$$

$i =$ particle index (N particles)

$q_i =$ particle charge

$\mathbf{x}_i =$ particle coordinate

$\mathbf{v}_i =$ particle velocity

$\delta(\mathbf{x}) \equiv \delta(x)\delta(y)\delta(z)$

$\delta(x) \equiv$ Dirac-delta function

$\sum_{i=1}^{N} \cdots =$ sum over beam particles

+ Boundary Conditions on $\mathbf{E}^s$ and $\mathbf{B}^s$ from material structures, radiation conditions, etc.

In accelerators, typically there is ideally a single species of particle:

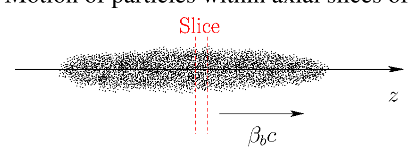
$q_i \to q$

$m_i \to m$

Large Simplification!

Multi-species results in more complex collective effects

Motion of particles within axial slices of the "bunch" are highly directed:



$$\beta_b(z)c \equiv \frac{1}{N'}\sum_{i=1}^{N'} \mathbf{v}_i \cdot \hat{\mathbf{z}}$$

$=$ Mean axial velocity of N' particles in beam slice

$$\frac{d}{dt}\mathbf{x}_i(t) = \mathbf{v}_i(t) = \hat{\mathbf{z}}\beta_b(z)c + \delta\mathbf{v}_i$$

$|\delta\mathbf{v}_i| \ll |\beta_b|c$ Paraxial Approximation

If there are typically many particles: (see Sec C, Vlasov Models for more details)

$$\rho^s = \sum_{i=1}^{N} q_i \delta[\mathbf{x} - \mathbf{x}_i(t)] \qquad \mathbf{J}^s = \sum_{i=1}^{N} q_i \mathbf{v}_i(t)\delta[\mathbf{x} - \mathbf{x}_i(t)]$$

$\simeq \rho(\mathbf{x}, t)$ continuous charge-density $\qquad \simeq \beta_b c \rho(\mathbf{x}, t)\hat{\mathbf{z}}$ continuous axial current-density

The beam evolution is typically sufficiently slow (for heavy ions) where we can neglect radiation and approximate the self-field Maxwell Equations as:

See: USPAS, *Beam Physics with Intense Space Charge* lecture notes

$$\boxed{\begin{aligned} &\mathbf{E}^s = -\nabla\phi \\ &\mathbf{B}^s = \nabla\times\mathbf{A} \qquad \mathbf{A} = \hat{\mathbf{z}}\frac{\beta_b}{c}\phi \\ &\nabla^2\phi = \frac{\partial}{\partial\mathbf{x}}\cdot\frac{\partial}{\partial\mathbf{x}}\phi = -\frac{\rho^s}{\epsilon_0} \\ &+ \text{Boundary Conditions on } \phi \end{aligned}}$$

Vast Reduction of self-field model:

Approximation equiv to electrostatic interactions in frame moving with beam

But still complicated!

Resolve the Lorentz force acting on beam particles into Applied and Self-Field terms:

$$\mathbf{F}_i(\mathbf{x}_i, t) = q\mathbf{E}(\mathbf{x}_i, t) + q\mathbf{v}_i(t)\times\mathbf{B}(\mathbf{x}_i, t)$$

$$\mathbf{F}_i = \mathbf{F}_i^a + \mathbf{F}_i^s$$
$$\mathbf{E} = \mathbf{E}^a + \mathbf{E}^s$$
$$\mathbf{B} = \mathbf{B}^a + \mathbf{B}^s$$

Applied:

$$\boxed{\mathbf{F}_i^a = q\mathbf{E}_i^a + q\mathbf{v}_i\times\mathbf{B}_i^a}$$

$$\mathbf{E}^a(\mathbf{x}_i, t) \equiv \mathbf{E}_i^a \quad \text{etc.}$$

Self-Field:

$$\boxed{\mathbf{F}_i^s = q\mathbf{E}_i^s + q\mathbf{v}_i\times\mathbf{B}_i^s}$$

The self-field force can be simplified:

Plug in self-field forms:

$$\left.\cdots\right|_i \equiv \left.\cdots\right|_{\mathbf{x}=\mathbf{x}_i}$$

~0 Neglect: Paraxial

$$\begin{aligned} \mathbf{F}_i^s &= q\mathbf{E}_i^s + q\mathbf{v}_i\times\mathbf{B}_i^s \\ &\simeq q\left[-\left.\frac{\partial\phi}{\partial\mathbf{x}}\right|_i + (\beta_b c\hat{\mathbf{z}} + \delta\mathbf{v}_i)\times\left.\left(\frac{\partial}{\partial\mathbf{x}}\times\hat{\mathbf{z}}\frac{\beta_b}{c}\phi\right)\right|_i\right] \end{aligned}$$

Resolve into transverse (*x* and *y*) and longitudinal (*z*) components and simplify:

$$\begin{aligned} \beta_b c\hat{\mathbf{z}}\times\left.\left(\frac{\partial}{\partial\mathbf{x}}\times\hat{\mathbf{z}}\frac{\beta_b}{c}\phi\right)\right|_i &= \beta_b^2\hat{\mathbf{z}}\times\left.\left(\frac{\partial}{\partial\mathbf{x}_\perp}\times\hat{\mathbf{z}}\phi\right)\right|_i \\ &= \beta_b^2\hat{\mathbf{z}}\times\left.\left(\frac{\partial\phi}{\partial y}\hat{\mathbf{x}} - \frac{\partial\phi}{\partial x}\hat{\mathbf{y}}\right)\right|_i \\ &= \beta_b^2\left.\left(\frac{\partial\phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\phi}{\partial y}\hat{\mathbf{y}}\right)\right|_i \\ &= \beta_b^2\left.\frac{\partial\phi}{\partial\mathbf{x}_\perp}\right|_i \end{aligned}$$

also

$$-\left.\frac{\partial\phi}{\partial\mathbf{x}}\right|_i = -\left.\frac{\partial\phi}{\partial\mathbf{x}_\perp}\right|_i - \left.\frac{\partial\phi}{\partial\mathbf{z}}\right|_i\hat{\mathbf{z}}$$

Together, these results give:

$$\boxed{\begin{aligned} &\mathbf{F}_i^s = \underbrace{-\frac{q}{\gamma_b^2}\left.\frac{\partial\phi}{\partial\mathbf{x}_\perp}\right|_i}_{\text{Transverse}} \underbrace{-\hat{\mathbf{z}}q\left.\frac{\partial\phi}{\partial z}\right|_i}_{\text{Longitudinal}} \\ &\gamma_b \equiv \frac{1}{\sqrt{1-\beta_b^2}} \qquad \text{Axial relativistic gamma of beam} \end{aligned}}$$

- Transverse and longitudinal forces have different axial gamma factors
- $1/\gamma_b^2$ factor in transverse force shows the space-charge forces become weaker as axial beam kinetic energy increases
  - Most important in low energy (nonrelativistic) beam transport
  - Strong in/near injectors before much acceleration

/// Aside: Singular Self Fields

In *free space*, the beam potential generated from the singular charge density:

$$\rho^s = \sum_{i=1}^{N} q_i\delta[\mathbf{x} - \mathbf{x}_i(t)]$$

is

$$\phi(\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{1}{|\mathbf{x} - \mathbf{x}_i|}$$

Thus, the force of a particle at $\mathbf{x} = \mathbf{x}_i$ is:

$$\mathbf{F}_i = -q\left.\frac{\partial\phi}{\partial\mathbf{x}}\right|_i = \frac{q^2}{4\pi\epsilon_0}\sum_{j=1}^{N}\frac{(\mathbf{x}_i - \mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^{3/2}}$$

Which diverges due to the *i* = *j* term. This divergence is essentially "erased" when the continuous charge density is applied:

$$\rho^s = \sum_{i=1}^{N} q_i\delta[\mathbf{x} - \mathbf{x}_i(t)] \quad \longrightarrow \quad \rho(\mathbf{x}, t)$$

- Effectively removes effect of collisions

  See: Secs. C.6 and C.7 for more details
  - Find collisionless Vlasov model of evolution is often adequate ///

The particle equations of motion in $\mathbf{x}_i - \mathbf{v}_i$ phase-space variables become:

- Separate parts of $q\mathbf{E}^a_i + q\mathbf{v}_i \times \mathbf{B}^a_i$ into transverse and longitudinal comp

Transverse

$$\frac{d}{dt}\mathbf{x}_{\perp i} = \mathbf{v}_{\perp i}$$

$$\frac{d}{dt}(m\gamma_i \mathbf{v}_{\perp i}) \simeq \underbrace{q\mathbf{E}^a_{\perp i} + q\beta_b c\hat{\mathbf{z}} \times \mathbf{B}^a_{\perp i} + qB^a_{zi}\mathbf{v}_{\perp i} \times \hat{\mathbf{z}}}_{\text{Applied}} \underbrace{- q\frac{1}{\gamma_b^2}\frac{\partial \phi}{\partial \mathbf{x}_\perp}\Big|_i}_{\text{Self}}$$

Longitudinal

$$\frac{d}{dt}z_i = v_{zi}$$

$$\frac{d}{dt}(m\gamma_i v_{zi}) \simeq \underbrace{qE^a_{zi} - q(v_{xi}B^a_{yi} - v_{yi}B^a_{xi})}_{\text{Applied}} \underbrace{- q\frac{\partial \phi}{\partial z}\Big|_i}_{\text{Self}}$$

- Except near injector, acceleration is typically slow
  - Fractional change in γ_b, β_b small over characteristic transverse dynamical scales such as lattice period and betatron oscillation periods
- Regard γ_b, β_b as specified functions given by the "acceleration schedule" in transverse dynamics and calculate from longitudinal dynamics
  - Idealized, but can be good approximation: real dynamics fully 3D

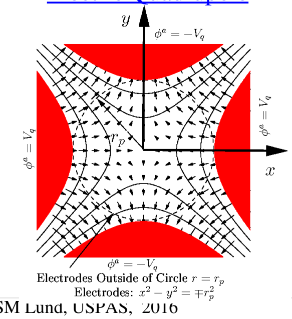
B.6 Applied Focusing Fields
Overview

Applied fields for focusing, bending, and acceleration enter the equations of motion via:

$$\mathbf{E}^a = \text{Applied Electric Field}$$
$$\mathbf{B}^a = \text{Applied Magnetic Field}$$

Generally, these fields are produced by sources (often static or slowly varying in time) located outside an aperture or so-called pipe radius $r = r_p$. For example, the electric and magnetic quadrupoles:

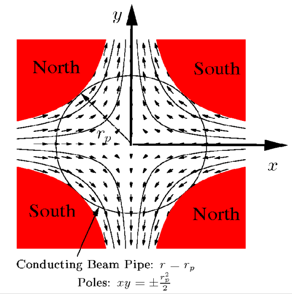
Electric Quadrupole



Hyperbolic material surfaces outside pipe radius

$r = r_p$

Magnetic Quadrupole



The fields of such classes of magnets obey the vacuum Maxwell Equations within the aperture:

$$\nabla \cdot \mathbf{E}^a = 0 \qquad \nabla \cdot \mathbf{B}^a = 0$$
$$\nabla \times \mathbf{E}^a = -\frac{\partial}{\partial t}\mathbf{B}^a \qquad \nabla \times \mathbf{B}^a = \frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}^a$$

If the fields are static or sufficiently slowly varying (quasistatic) where the time derivative terms can be neglected, then the fields in the aperture will obey the static vacuum Maxwell equations:

$$\nabla \cdot \mathbf{E}^a = 0 \qquad \nabla \cdot \mathbf{B}^a = 0$$
$$\nabla \times \mathbf{E}^a = 0 \qquad \nabla \times \mathbf{B}^a = 0$$

In general, optical elements are tuned to limit the strength of nonlinear field terms so the beam experiences primarily linear applied fields.

- Linear fields allow better preservation of beam quality

Removal of *all* nonlinear fields cannot be accomplished

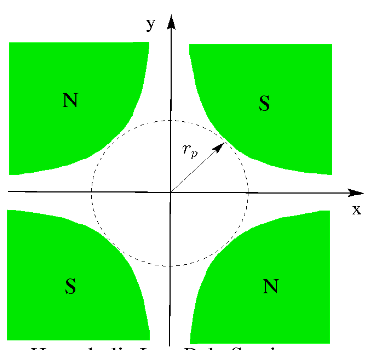
- 3D structure of the Maxwell equations precludes for finite geometry optics
- Even in finite geometries deviations from optimal structures and symmetry will result in nonlinear fields

As an example of this, when an ideal 2D iron magnet with infinite hyperbolic poles is truncated radially for finite 2D geometry, this leads to nonlinear focusing fields even in 2D:

- Truncation necessary along with confinement of return flux in yoke

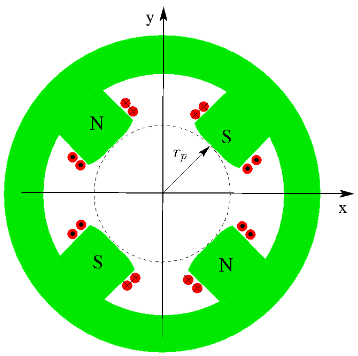
Cross-Sections of Iron Quadrupole Magnets

Ideal (infinite geometry)



Hyperbolic Iron Pole Sections (infinite)

Practical (finite geometry)



Shaped Iron Pole Sections (finite)

The design of optimized electric and magnetic optics for accelerators is a specialized topic with a vast literature. It is not be possible to cover this topic in this brief survey. In the remaining part of this section we will overview a limited subset of material on static magnetic optics including:

- Magnetic field expansions for focusing and bending
- Hard edge equivalent models
- 2D multipole models and nonlinear field scalings
- Good field radius

Much of the material presented can be immediately applied to static Electric Optics since the vacuum Maxwell equations are the same for static Electric $\mathbf{E}^a$ and Magnetic $\mathbf{B}^a$ fields in vacuum.

Magnetic Field Expansions for Focusing and Bending

Transverse forces from transverse $(B_z^a = 0)$ magnetic fields enter the transverse equations of motion via:

Force: $\mathbf{F}_\perp^a \simeq q\beta_b c\hat{\mathbf{z}} \times \mathbf{B}_\perp^a$

Field: $\mathbf{B}_\perp^a = \hat{\mathbf{x}} B_x^a + \hat{\mathbf{y}} B_y^a$

Combined these give:

$$\boxed{\begin{aligned} F_x^a &\simeq -q\beta_b c B_y^a \\ F_y^a &\simeq q\beta_b c B_x^a \end{aligned}}$$

Field components entering these expressions can be expanded about $\mathbf{x}_\perp = 0$

- Element center and design orbit taken to be at $\mathbf{x}_\perp = 0$

$$B_x^a = \overset{1}{B_x^a(0)} + \overset{2}{\frac{\partial B_x^a}{\partial y}(0)y} + \overset{3}{\frac{\partial B_x^a}{\partial x}(0)x}$$

Nonlinear Focus

$$+ \frac{1}{2}\frac{\partial^2 B_x^a}{\partial x^2}(0)x^2 + \frac{\partial^2 B_x^a}{\partial x \partial y}(0)xy + \frac{1}{2}\frac{\partial B_x^a}{\partial y^2}(0)y^2 + \cdots$$

$$B_y^a = \overset{1}{B_y^a(0)} + \overset{2}{\frac{\partial B_y^a}{\partial x}(0)x} + \overset{3}{\frac{\partial B_y^a}{\partial y}(0)y}$$

Nonlinear Focus

$$+ \frac{1}{2}\frac{\partial^2 B_y^a}{\partial x^2}(0)x^2 + \frac{\partial^2 B_y^a}{\partial x \partial y}(0)xy + \frac{1}{2}\frac{\partial B_y^a}{\partial y^2}(0)y^2 + \cdots$$

Terms:
1: Dipole Bend
2: Normal Linear Quad Focus
3: Skew Linear Quad Focus

Sources of undesired nonlinear applied field components include:

- Intrinsic finite 3D geometry and the structure of the Maxwell equations
- Systematic errors or sub-optimal geometry associated with practical trade-offs in fabricating the optic
- Random construction errors in individual optical elements
- Alignment errors of magnets in the lattice giving field projections in unwanted directions
- Excitation errors effecting the field strength
  - Currents in coils not correct and/or unbalanced

More advanced treatments exploit less simple power-series expansions to express symmetries more clearly:

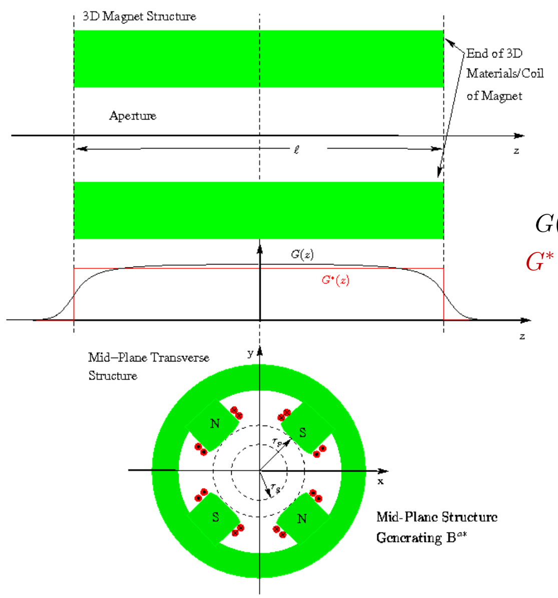
- Maxwell equations constrain structure of solutions
  - Expansion coefficients are NOT all independent
- Only fields consistent with the Maxwell equations make physical sense
  - Must be careful not to let model errors introduce nonphysical forces
- Forms appropriate for bent coordinate systems in dipole bends can become complicated

Hard Edge Equivalent Models

Real 3D magnets can often be modeled with sufficient accuracy by 2D hard-edge "equivalent" magnets that give the same approximate focusing impulse to the particle as the full 3D magnet

- Objective is to provide same approximate applied focusing "kick" to particles with different focusing gradient functions $G(s)$

See Figure Next Slide



$G(z) =$ 3D Field Gradient

$G^*(z) =$ Hard-Edge Equivalent Field Gradient

Many prescriptions exist for calculating the effective axial length and strength of hard-edge equivalent models

- See Review: Lund and Bukh, PRSTAB 7 204801 (2004), Appendix C

Here we overview a simple equivalence method that has been shown to work well:

For a relatively long, but finite axial length magnet with 3D gradient function:

$$G(z) \equiv \left.\frac{\partial B_x^a}{\partial y}\right|_{x=y=0}$$

Take hard-edge equivalent parameters:

- Take $z = 0$ at the axial magnet mid-plane

Gradient: $G^* \equiv G(z=0)$

Axial Length: $\ell \equiv \frac{1}{G(z=0)} \int_{-\infty}^{\infty} dz\, G(z)$

- More advanced equivalences can be made based more on particle optics
  - Disadvantage of such methods is "equivalence" changes with particle energy and must be revisited as optics are tuned

2D Transverse Multipole Magnetic Fields

In many cases, it is sufficient to characterize the field errors in 2D hard-edge equivalent as:

$$\overline{B}_x(x,y) = \frac{1}{\ell}\int_{-\infty}^{\infty} dz\; B_x^a(x,y,z)$$

$$\overline{B}_y(x,y) = \frac{1}{\ell}\int_{-\infty}^{\infty} dz\; B_y^a(x,y,z)$$

2D Effective Fields 3D Fields

Operating on the vacuum Maxwell equations with: $\int_{-\infty}^{\infty} \frac{dz}{\ell} \cdots$

yields the (exact) 2D Transverse Maxwell equations :

$$\frac{\partial \overline{B}_x(x,y)}{\partial y} = \frac{\partial \overline{B}_y(x,y)}{\partial x} \quad \Leftarrow \text{ From } \nabla \times \mathbf{B} = 0$$

$$\frac{\partial \overline{B}_x(x,y)}{\partial x} = -\frac{\partial \overline{B}_y(x,y)}{\partial y} \quad \Leftarrow \text{ From } \nabla \cdot \mathbf{B} = 0$$

These equations are recognized as the Cauchy-Riemann conditions for a complex field variable:

$$\underline{B}^* \equiv \overline{B}_x - i\overline{B}_y \qquad i \equiv \sqrt{-1}$$

to be an analytical function of the complex variable:

$$\underline{z} \equiv x + iy \qquad i \equiv \sqrt{-1}$$

Notation: Underlines denote complex variables where confusion may arise

Cauchy-Riemann Conditions

$\underline{F} = u(x,y) + iv(x,y)$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \longrightarrow \frac{\partial \overline{B}_x(x,y)}{\partial x} = -\frac{\partial \overline{B}_y(x,y)}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \longrightarrow \frac{\partial \overline{B}_x(x,y)}{\partial y} = \frac{\partial \overline{B}_y(x,y)}{\partial x}$$

$\underline{F} = u + iv$ analytic func of $\underline{z} = x + iy$

2D Magnetic Field

$u = \overline{B}_x \quad v = -\overline{B}_y$

$\underline{F} = \overline{B}_x - i\overline{B}_y$ analytic func of $\underline{z} = x + iy$

Note the complex field which is an analytic function of $\underline{z} = x + iy$ is $\underline{B}^* = \overline{B}_x - i\overline{B}_y$ NOT $\underline{B} = \overline{B}_x + i\overline{B}_y$. This is *not* a typo and is necessary for $\underline{B}^*$ to satisfy the Cauchy-Riemann conditions.

- See problem sets for illustration

It follows that $\underline{B}^*(\underline{z})$ can be analyzed using the full power of the highly developed theory of analytical functions of a complex variable.

Expand $\underline{B}^*(\underline{z})$ as a Laurent Series within the vacuum aperture as:

$$\boxed{\begin{aligned} \underline{B}^*(\underline{z}) &= \overline{B_x}(x,y) - i\overline{B_y}(x,y) = \sum_{n=1}^{\infty} \underline{b}_n \underline{z}^{n-1} \\ \underline{b}_n &= \text{const (complex)} \\ n &= \text{Multipole Index} \end{aligned}}$$

The $\underline{b}_n$ are called "multipole coefficients" and give the structure of the field. The multipole coefficients can be resolved into real and imaginary parts as:

$$\boxed{\begin{aligned} \underline{b}_n &= \mathcal{A}_n - i\mathcal{B}_n \\ \mathcal{B}_n &\Longrightarrow \text{"Normal" Multipoles} \\ \mathcal{A}_n &\Longrightarrow \text{"Skew" Multipoles} \end{aligned}}$$

Some algebra identifies the polynomial symmetries of low-order terms as:

Cartesian projections: $\overline{B_x} - i\overline{B_y} = (\mathcal{A}_n - i\mathcal{B}_n)(x+iy)^{n-1}$

Index	Name	Normal ($\mathcal{A}_n = 0$)		Skew ($\mathcal{B}_n = 0$)	
n		$\overline{B_x}/\mathcal{B}_n$	$\overline{B_y}/\mathcal{B}_n$	$\overline{B_x}/\mathcal{A}_n$	$\overline{B_y}/\mathcal{A}_n$
1	Dipole	0	1	1	
2	Quadrupole	y	x	x	$-y$
3	Sextupole	$2xy$	$x^2 - y^2$	$x^2 - y^2$	$-2xy$
4	Octupole	$3x^2y - y^3$	$x^3 - 3xy^2$	$x^3 - 3xy^2$	$-3x^2y + y^3$
5	Decapole	$4x^3y - 4xy^3$	$x^4 - 6x^2y^2 + y^4$	$x^4 - 6x^2y^2 + y^4$	$-4x^3y + 4xy^3$

Comments:

- Reason for pole names most apparent from polar representation (see following pages) and sketches of the magnetic pole structure
- Caution: In so-called "US notation", poles are labeled with index *n* -> *n* -1
  - Arbitrary in 2D but US choice *not* good notation in 3D generalizations

Comments continued:

- Normal and Skew symmetries can be taken as a symmetry *definition*. But this choice makes sense for *n* = 2 quadrupole focusing terms:

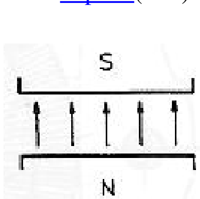
$$\begin{aligned} \overline{F^a_x} &= -q\beta_b c \overline{B_y} &&= -q\beta_b c(\mathcal{B}_2 x - \mathcal{A}_2 y) \\ \overline{F^a_y} &= \ \ q\beta_b c \overline{B_x} &&= \ \ q\beta_b c(\mathcal{B}_2 y + \mathcal{A}_2 x) \end{aligned}$$

In equations of motion:

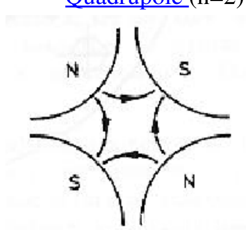
Normal	$\Rightarrow$	$\mathcal{B}_2$:	x-eqn, x-focus	y-eqn, y-defocus
Skew	$\Rightarrow$	$\mathcal{A}_2$:	x-eqn, y-defocus	y-eqn, x-defocus

Magnetic Pole Symmetries (normal orientation):

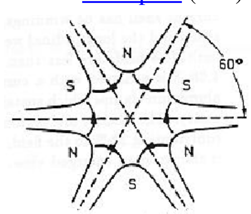
Dipole (n=1)



Quadrupole (n=2)



Sextupole (n=3)



- Actively rotate normal field structures clockwise through an angle of $\pi/(2n)$ for skew field component symmetries

Multipole scale/units

Frequently, in the multipole expansion:

$$\boxed{\underline{B}^*(\underline{z}) = \overline{B_x}(x,y) - i\overline{B_y}(x,y) = \sum_{n=1}^{\infty} \underline{b}_n \underline{z}^{n-1}}$$

the multipole coefficients $\underline{b}_n$ are rescaled as

$$\underline{b}_n \longrightarrow \underline{b}_n r_p^{n-1} \qquad r_p = \text{Aperture "Pipe" Radius}$$

Closest radius of approach of magnetic sources and/or aperture materials

so that the expansions becomes

$$\boxed{\underline{B}^*(\underline{z}) = \overline{B_x}(x,y) - i\overline{B_y}(x,y) = \sum_{n=1}^{\infty} \underline{b}_n \left(\frac{\underline{z}}{r_p}\right)^{n-1}}$$

Advantages of alternative notaiton:

- Multipoles $\underline{b}_n$ given directly in field units regardless of index *n*
- Scaling of field amplitudes with radius within the magnet bore becomes clear

Scaling of Fields produced by multipole term:
Higher order multipole coefficients (larger *n* values) leading to nonlinear focusing forces decrease rapidly within the aperture. To see this use a polar representation for $\underline{z}$, $\underline{b}_n$

$$\boxed{\begin{aligned} \underline{z} &= x + iy = re^{i\theta} & r &= \sqrt{x^2+y^2} \\ & & \theta &= \arctan[y,x] \\ \underline{b}_n &= |\underline{b}_n| e^{i\psi_n} & \psi_n &= \text{Real Const} \end{aligned}}$$

Thus, the nth order multipole terms scale as

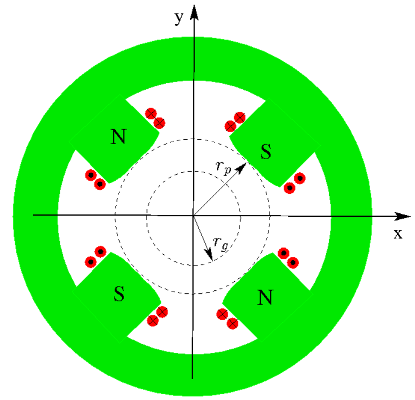
$$\boxed{\underline{b}_n \left(\frac{\underline{z}}{r_p}\right)^{n-1} = |\underline{b}_n| \left(\frac{r}{r_p}\right)^{n-1} e^{i[(n-1)\theta + \psi_n]}}$$

- Unless the coefficient $|\underline{b}_n|$ is very large, high order terms in *n* will become small rapidly as r_p decreases
- Better field quality can be obtained for a given magnet design by simply making the clear bore r_p larger, or alternatively using smaller bundles (more tight focus) of particles
  - Larger bore machines/magnets cost more. So designs become trade-off between cost and performance.
  - Stronger focusing to keep beam from aperture can be unstable

Good Field Radius

Often a magnet design will have a so-called "good-field" radius $r = r_g$ that the maximum field errors are specified on.

- In superior designs the good field radius can be around ~70% or more of the clear bore aperture to the beginning of material structures of the magnet.
- Beam particles should evolve with radial excursions with $r < r_g$



r_p = Clear Bore Radius
$\sim$ Pole Radius Typical

r_g = Good Field Radius
$\sim 70\%\ r_p$ Typical

Comments:

- Particle orbits are designed to remain within radius r_g
- Field error statements are readily generalized to 3D since:

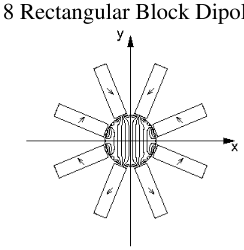
$$\begin{aligned} \nabla \cdot \mathbf{B}^a &= 0 \\ \nabla \times \mathbf{B}^a &= 0 \end{aligned} \quad \Longrightarrow \quad \nabla^2 \mathbf{B}^a = 0$$

and therefore each component of $\mathbf{B}^a$ satisfies a Laplace equation within the vacuum aperture. Therefore, field errors decrease when moving more deeply within a source-free region.

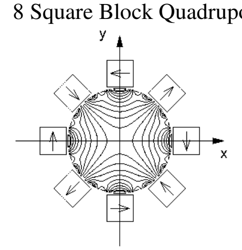
Example Permanent Magnet Assemblies

A few examples of practical permanent magnet assemblies with field contours are provided to illustrate error field structures in practical devices

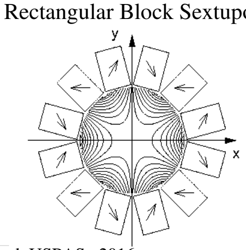
8 Rectangular Block Dipole



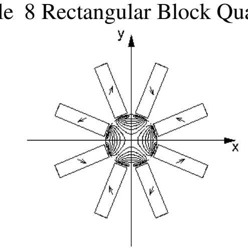
8 Square Block Quadrupole



12 Rectangular Block Sextupole



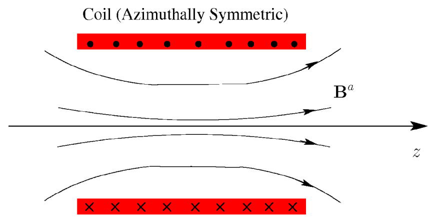
8 Rectangular Block Quadrupole



For more info on permanent magnet design see: Lund and Halbach, Fusion Engineering Design, **32-33**, 401-415 (1996)

Solenoidal Focusing

The field of an ideal magnetic solenoid is invariant under transverse rotations about it's axis of symmetry (z) can be expanded in terms of the on-axis field as as:



Vacuum Maxwell equations:

$$\nabla \cdot \mathbf{B}^a = 0$$
$$\nabla \times \mathbf{B}^a = 0$$

Imply $\mathbf{B}^a$ can be expressed in terms of on-axis field $\mathbf{B}^a_z(r=0,z)$

$$\mathbf{E}^a = 0$$

$$\mathbf{B}^a_\perp = \frac{1}{2}\sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu!(\nu-1)!} \frac{\partial^{2\nu-1} B_{z0}(z)}{\partial z^{2\nu-1}} \left(\frac{|\mathbf{x}_\perp|}{2}\right)^{2\nu-2} \mathbf{x}_\perp$$

$$B^a_z = B_{z0}(z) + \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{(\nu!)^2} \frac{\partial^{2\nu} B_{z0}(z)}{\partial z^{2\nu}} \left(\frac{|\mathbf{x}_\perp|}{2}\right)^{2\nu}$$

$$B_{z0}(z) \equiv B^a_z(\mathbf{x}_\perp = 0, z) = \text{On-Axis Field}$$

See Appendix A or Reiser, *Theory and Design of Charged Particle Beams*, Sec. 3.3.1

Writing out explicitly the terms of this expansion:

$$\mathbf{B}^a(r,z) = \hat{\mathbf{r}} B^a_r(r,z) + \hat{\mathbf{z}} B^a_z(r,z) \qquad r = \sqrt{x^2+y^2}$$
$$= (-\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta) B^a_r(r,z) + \hat{\mathbf{z}} B^a_z(r,z)$$

where

$$B^a_r(r,z) = \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu!(\nu-1)!} B^{(2\nu-1)}_{z0}(z) \left(\frac{r}{2}\right)^{2\nu-1}$$
$$= \boxed{-\frac{B'_{z0}(z)}{2} r} + \frac{B^{(3)}_{z0}(z)}{16} r^3 - \frac{B^{(5)}_{z0}(z)}{384} r^5 + \frac{B^{(7)}_{z0}(z)}{18432} r^7 - \frac{B^{(9)}_{z0}(z)}{1474560} r^9 + \ldots$$

$$B^a_z(r,z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{(\nu!)^2} B^{(2\nu)}_{z0}(z) \left(\frac{r}{2}\right)^{2\nu}$$
$$= \boxed{B_{z0}(z)} - \frac{B''_{z0}(z)}{4} r^2 + \frac{B^{(4)}_{z0}(z)}{64} r^4 - \frac{B^{(6)}_{z0}(z)}{2304} r^6 + \frac{B^{(8)}_{z0}(z)}{147456} r^8 + \ldots$$

$$B_{z0}(z) \equiv B^a_z(r=0,z) = \text{On-axis Field}$$

$\boxed{\cdots}$ Linear Terms

$$B^{(n)}_{z0}(z) \equiv \frac{\partial^n B_{z0}(z)}{\partial z^n} \qquad B'_{z0}(z) \equiv \frac{\partial B_{z0}(z)}{\partial z} \qquad B''_{z0}(z) \equiv \frac{\partial^2 B_{z0}(z)}{\partial z^2}$$

For modeling, we truncate the expansion using only leading-order terms to obtain:

- Corresponds to linear dynamics in the equations of motion

$$B^a_x = -\frac{1}{2}\frac{\partial B_{z0}(z)}{\partial z} x$$
$$B^a_y = -\frac{1}{2}\frac{\partial B_{z0}(z)}{\partial z} y \qquad B_{z0}(z) \equiv B^a_z(\mathbf{x}_\perp = 0, z) = \text{On-Axis Field}$$
$$B^a_z = B_{z0}(z)$$

Note that this truncated expansion is divergence free:

$$\nabla \cdot \mathbf{B}^a = -\frac{1}{2}\frac{\partial B_{z0}}{\partial z}\frac{\partial}{\partial \mathbf{x}_\perp} \cdot \mathbf{x}_\perp + \frac{\partial}{\partial z} B_{z0} = 0$$

but not curl free within the vacuum aperture:

$$\nabla \times \mathbf{B}^a = \frac{1}{2}\frac{\partial^2 B_{z0}(z)}{\partial z^2}(-\hat{\mathbf{x}} y + \hat{\mathbf{y}} x)$$
$$= \frac{1}{2}\frac{\partial^2 B_{z0}(z)}{\partial z^2} r(-\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta) = \frac{1}{2}\frac{\partial^2 B_{z0}(z)}{\partial z^2} r\hat{\theta}$$

- Nonlinear terms needed to satisfy 3D Maxwell equations
- Impossible to have a pure linear force with smoothly varying $B_{z0}(z)$

Solenoid equations of motion:

- Insert field components into equations of motion and collect terms

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} x' - \frac{B'_{z0}(s)}{2[B\rho]} y - \frac{B_{z0}(s)}{[B\rho]} y' = -\frac{q}{m\gamma_b^3\beta_b^2 c^2}\frac{\partial \phi}{\partial x}$$
$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)} y' + \frac{B'_{z0}(s)}{2[B\rho]} x + \frac{B_{z0}(s)}{[B\rho]} x' = -\frac{q}{m\gamma_b^3\beta_b^2 c^2}\frac{\partial \phi}{\partial y}$$

$$[B\rho] \equiv \frac{\gamma_b\beta_b mc}{q} = \text{Rigidity} \qquad \frac{B_{z0}(s)}{[B\rho]} = \frac{\omega_c(s)}{\gamma_b\beta_b c}$$

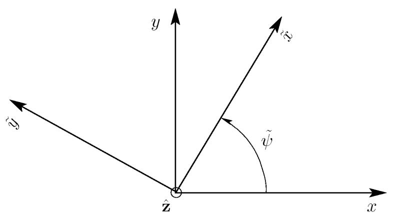
$$\omega_c(s) = \frac{qB_{z0}(s)}{m} = \text{Cyclotron Frequency (in applied axial magnetic field)}$$

- Equations are linearly cross-coupled in the applied field terms
  - x equation depends on y, y'
  - y equation depends on x, x'

If the beam space-charge is *axisymmetric*:

$$\frac{\partial \phi}{\partial \mathbf{x}_\perp} = \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial \mathbf{x}_\perp} = \frac{\partial \phi}{\partial r}\frac{\mathbf{x}_\perp}{r}$$

then the space-charge term also decouples under the Larmor transformation to a rotating frame and the equations of motion can be expressed in uncoupled form:



$$\tilde{x} = \quad x\cos\tilde{\psi}(s) + y\sin\tilde{\psi}(s)$$
$$\tilde{y} = -x\sin\tilde{\psi}(s) + y\cos\tilde{\psi}(s)$$

$$\tilde{\psi}(s) = -\int_{s_i}^{s} d\bar{s}\, k_L(\bar{s})$$

$$k_L(s) \equiv \frac{B_{z0}(s)}{2[B\rho]} = \frac{\omega_c(s)}{2\gamma_b\beta_b c}$$

= Larmor wave number

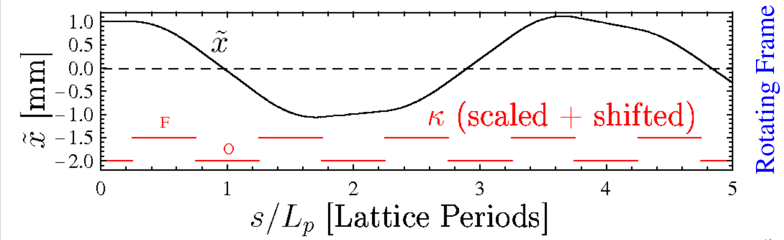
$s = s_i$ defines initial condition

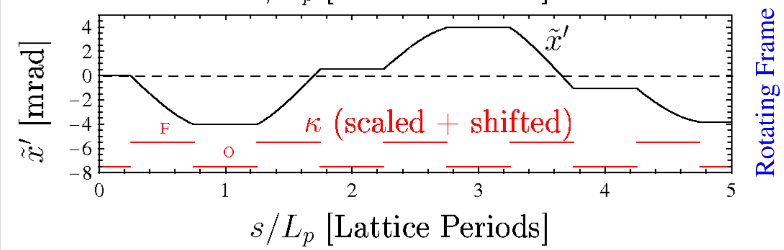
$$\boxed{\begin{aligned} &\tilde{x}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{x}' + \kappa(s)\tilde{x} = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{x}}{r} \\ &\tilde{y}'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\tilde{y}' + \kappa(s)\tilde{y} = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{y}}{r} \\ &\kappa(s) = k_L^2(s) \equiv \left[\frac{B_{z0}(s)}{2[B\rho]}\right]^2 = \left[\frac{\omega_c(s)}{2\gamma_b\beta_b c}\right]^2 \end{aligned}}$$

/// Example: Larmor Frame Particle Orbits in a Periodic Solenoidal Focusing Lattice: $\tilde{x}-\tilde{x}'$ phase-space for hard edge elements and applied fields

$L_p = 0.5$ m $\quad \kappa = 20$ rad/m^2 in Solenoids $\quad \tilde{x}(0) = 1$ mm $\quad \tilde{y}(0) = 0$

$\eta = 0.5 \quad \phi \simeq 0 \quad \gamma_b\beta_b = \text{const} \quad \tilde{x}'(0) = 0 \quad \tilde{y}'(0) = 0$

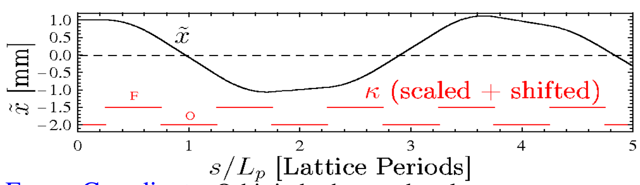




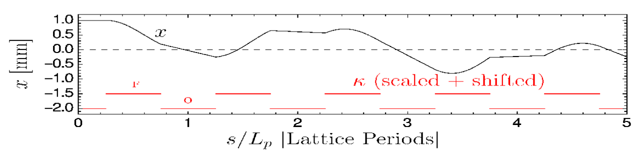
Contrast of Rotating Larmor-Frame and Lab-Frame Orbits

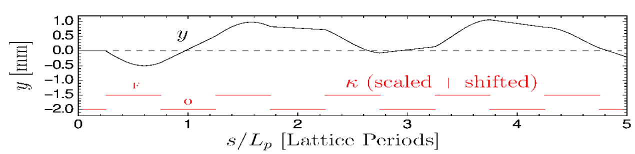
- Same initial condition

Larmor-Frame Coordinate Orbit in rotating frame *x*-plane only, zero in rotating y-plane



Lab-Frame Coordinate Orbit in both *x*- and *y*-planes

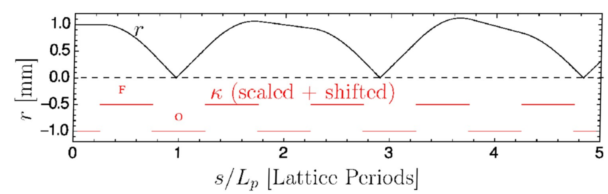




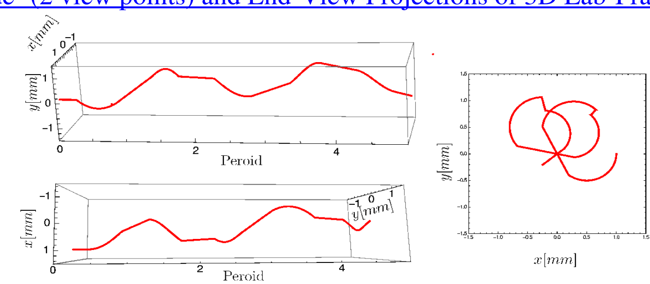
Additional perspectives of particle orbit in solenoid transport channel

- Same initial condition

Radius evolution (Lab or Rotating Frame: radius same)



Side- (2 view points) and End-View Projections of 3D Lab-Frame Orbit



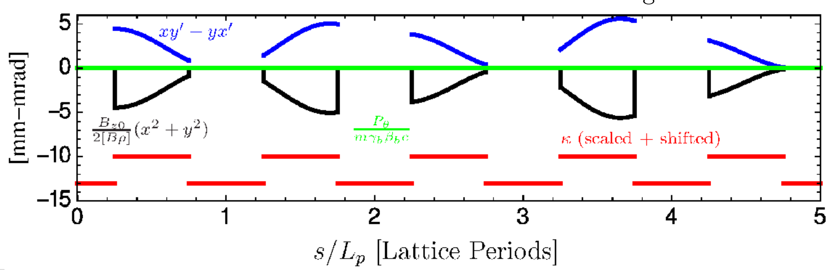
Invariant help gain confidence that the answer is correct!
Employ the solenoid focusing channel example and plot:

- Mechanical angular momentum $\propto xy' - yx'$
- Vector potential contribution to canonical angular momentum $\propto B_{z0}(x^2+y^2)$
- Canonical angular momentum (constant) P_θ

$$\frac{P_\theta}{m\gamma_b\beta_b c} = xy' - yx' + \frac{B_{z0}}{2[B\rho]}(x^2+y^2) = \text{const} = \text{Canonical Angular Momentum}$$

$$xy' - yx' = r^2\theta' = \text{Mechanical Angular Momentum}$$

$$\frac{B_{z0}}{2[B\rho]}(x^2+y^2) = \sqrt{\kappa}(x^2+y^2) = \text{Vector Potential Component Canonical Angular Momentum}$$



Appendix A: Axisymmetric Applied Magnetic or Electric Field Expansion

Static, rationally symmetric static applied fields $\mathbf{E}^a$, $\mathbf{B}^a$ satisfy the vacuum Maxwell equations in the beam aperture:

$$\nabla\cdot\mathbf{E}^a = 0 \qquad \nabla\times\mathbf{E}^a = 0 \qquad \nabla\cdot\mathbf{B}^a = 0 \qquad \nabla\times\mathbf{B}^a = 0$$

This implies we can take for some electric potential $\phi^e$and magnetic potential ϕ^m:

$$\mathbf{E}^a = -\nabla\phi^e \qquad \mathbf{B}^a = -\nabla\phi^m$$

which in the vacuum aperture satisfies the Laplace equations:

$$\nabla^2\phi^e = 0 \qquad \nabla^2\phi^m = 0$$

We will analyze the magnetic case and the electric case is analogous. In axisymmetric $(\partial/\partial\theta = 0)$ geometry we express Laplace's equation as:

$$\boxed{\nabla^2\phi^m(r,z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi^m}{\partial r}\right) + \frac{\partial^2\phi^m}{\partial z^2} = 0}$$

$\phi^m(r,z)$ can be expanded as (odd terms in *r* would imply nonzero $B_r = -\frac{\partial\phi_m}{\partial r}$ at *r = 0*):

$$\phi^m(r,z) = \sum_{\nu=0}^{\infty} f_{2\nu}(z)r^{2\nu} = f_0 + f_2r^2 + f_4r^4 + \ldots$$

where $f_0 = \phi^m(r=0,z)$ is the on-axis potential A1

Plugging ϕ^m into Laplace's equation yields the recursion relation for $f_{2\nu}$

$$(2\nu+2)^2 f_{2\nu+2} + f''_{2\nu} = 0$$

Iteration then shows that

$$\phi^m(r,z) = \sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{(\nu!)^2}\frac{\partial^{2\nu}f(0,z)}{\partial z^{2\nu}}\left(\frac{r}{2}\right)^{2\nu}$$

Using $B_z^a(r=0,z) \equiv B_{z0}(z) = -\frac{\partial\phi_m(0,z)}{\partial z}$ and diffrentiating yields:

$$\boxed{\begin{aligned} B_r^a(r,z) &= -\frac{\partial\phi_m}{\partial r} = \sum_{\nu=1}^{\infty}\frac{(-1)^\nu}{(\nu!)(\nu-1)!}\frac{\partial^{2\nu-1}B_{z0}(z)}{\partial z^{2\nu-1}}\left(\frac{r}{2}\right)^{2\nu-1} \\ B_z^a(r,z) &= -\frac{\partial\phi_m}{\partial z} = \sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{(\nu!)^2}\frac{\partial^{2\nu}B_{z0}(z)}{\partial z^{2\nu}}\left(\frac{r}{2}\right)^{2\nu} \end{aligned}}$$

- Electric case immediately analogous and can arise in electrostatic Einzel lens focusing systems often employed near injectors
- Electric case can also be applied to RF and induction gap structures in the quasistatic (long RF wavelength relative to gap) limit. A2

Applied Fields in Codes

Codes for beam and plasma simulation generally have extensive provisions for including applied fields of accelerator or confinement device.

- Idealized linear Field Elements
  - Hard edge
  - With axial fringe field variation
  - Possibly with provision for misalignments, etc.
- Multipole moments for electric or magnetic optics in 2D hard-edge or 3D form
- Gridded field elements for $\mathbf{E}$, $\mathbf{B}$ allowing general fields
  - Interpolate to position of particles or where needed (macro-element etc)
  - Possibly with symmetry options for efficiency
  - Possibly with provision for time modulation to allow application for resonant RF cavities, pulsed elements, etc.
- For electric elements may also effect by potential grid ϕ
  - Similar to field elements: interpolate, symmetry, and possibly time modulate
- Electric gridded field elements for $\mathbf{E}$ should be setup carefully:
  - Alternative: load biased conductors in self-field solver and calculate with self-field for correct image charges
  - Can still apply and add conductors with zero bias to obtain images. This can increase efficiency (detailed applied field and approx image).

B.7 Symplectic Formulation of Dynamics Expression of Hamiltonian Dynamics

Following Lectures by: Andy Wolski, U. Liverpool

Hamilton's form of the equations of motion for a coasting beam (no accel):

- For solenoid focusing, will need to employ appropriate canonical variables

$$\frac{d}{ds}\mathbf{x}_\perp = \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} \qquad \frac{d}{ds}\mathbf{x}'_\perp = -\frac{\partial H_\perp}{\partial \mathbf{x}_\perp}$$

can be equivalently expressed as:

$$\frac{d}{ds}\vec{x} = \mathbf{S}\cdot\nabla_{\vec{x}}H_\perp$$

where

$$\vec{x} \equiv \begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix} \qquad \nabla_{\vec{x}} \equiv \begin{bmatrix} \partial_x \\ \partial_{x'} \\ \partial_y \\ \partial_{y'} \end{bmatrix} \qquad \mathbf{S} \equiv \begin{bmatrix} \mathbf{S}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_2 \end{bmatrix} \qquad \mathbf{S}_2 \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- Immediately generalizable to 3D dynamics
- Formulation applies in general canonical variables

$s \to t \qquad x \to q \qquad x' \to p \qquad$ etc.

// Review Transverse Hamiltonian formulation: for coasting beam $(\gamma_b\beta_b = \text{const})$

1) Continuous or quadrupole (electric or magnetic) focusing:

Canonical variables:

$$x,\ x' \qquad y,\ y'$$

Hamiltonian:

"Kinetic" Applied Potential Self Potential

$$H_\perp = \frac{1}{2}x'^2 + \frac{1}{2}y'^2 + \frac{1}{2}\kappa_x x^2 + \frac{1}{2}\kappa_y y^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^3}$$

$$\frac{d}{ds}x = \frac{\partial H_\perp}{\partial x} \qquad \frac{d}{ds}x = \frac{\partial H_\perp}{\partial y}$$

$$\frac{d}{ds}x' = -\frac{\partial H_\perp}{\partial x'} \qquad \frac{d}{ds}y' = -\frac{\partial H_\perp}{\partial y'}$$

Giving the familiar equations of motion:

$$x'' + \kappa_x x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \kappa_y y = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

Solenoidal magnetic focusing:

Canonical variables:

$$\tilde{x} = x \qquad \tilde{y} = y$$

$$\tilde{x}' = x' - \frac{B_{z0}}{2[B\rho]}y \qquad \tilde{y}' = y' + \frac{B_{z0}}{2[B\rho]}x \qquad [B\rho] \equiv \frac{m\gamma_b\beta_b c}{q}$$

Hamiltonian:

$$\tilde{H}_\perp = \frac{1}{2}\left[\left(\tilde{x}' + \frac{B_{z0}}{2[B\rho]}\tilde{y}\right)^2 + \left(\tilde{y}' - \frac{B_{z0}}{2[B\rho]}\tilde{x}\right)^2\right] + \frac{q\phi}{m\gamma_b^3\beta_b^2c^3}$$

$$\frac{d}{ds}\tilde{x} = \frac{\partial\tilde{H}_\perp}{\partial\tilde{x}} \qquad \frac{d}{ds}\tilde{y} = \frac{\partial\tilde{H}_\perp}{\partial\tilde{y}}$$

$$\frac{d}{ds}\tilde{x}' = -\frac{\partial H_\perp}{\partial\tilde{x}'} \qquad \frac{d}{ds}\tilde{y}' = -\frac{\partial H_\perp}{\partial\tilde{y}'}$$

Caution:
Primes do not mean *d/ds* in tilde variables here: just notation to distinguish "momentum" variable!

Giving (after some algebra) the familiar equations of motion:

$$x'' - \frac{B'_{z0}(s)}{2[B\rho]}y - \frac{B_{z0}(s)}{[B\rho]}y' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \frac{B'_{z0}(s)}{2[B\rho]}x + \frac{B_{z0}(s)}{[B\rho]}x' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

Symplectic Matrix

Definition:

An *2n x 2n symplectic matrix* ***M*** satisfies

$$\mathbf{M}^T\cdot\mathbf{S}\cdot\mathbf{M} = \mathbf{S}$$

where ***S*** has the *n*-dimensional *block diagonal* structure:

$$\mathbf{S} = \begin{bmatrix} \mathbf{S}_2 & & \\ & \ddots & \mathbf{0} \\ & \mathbf{0} & \ddots \\ & & & \mathbf{S}_2 \end{bmatrix} \qquad \mathbf{S}_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The matrix ***S*** satisfies:

$$\mathbf{S}^T = -\mathbf{S}$$
$$\mathbf{S}^2 = -\mathbf{I}$$

- Follow direct from definition

Illustrative Example: Linear Dynamics with constant applied fields

For a general 2nd order Hamiltonian that generates linear dynamics we have:

$$\boxed{\nabla_{\vec{x}} H_\perp = \mathbf{J} \cdot \vec{x} \qquad \text{with} \qquad \mathbf{J}^T = \mathbf{J}}$$

- Transpose symmetry result of expected physical forces

Hamilton's equation of motion become:

$$\boxed{\frac{d}{ds}\vec{x} = \mathbf{S} \cdot \nabla_{\vec{x}} H_\perp = \mathbf{S} \cdot \mathbf{J} \cdot \vec{x}}$$

If ***J*** has *no explicit s variation*, then this equation can be solved as

- Applies to piecewise constant focusing systems within each element/drift
  - In context coupled motion ok
  - Transitions between elements need to be analyzed separately

$$\vec{x}(s) = \mathbf{M}(s - s_i) \cdot \vec{x}(s_i) = \exp[(s - s_i)\mathbf{S} \cdot \mathbf{J}] \cdot \vec{x}(s_i)$$
$$\mathbf{M}(s - s_i) = \exp\left[(s - s_i)\mathbf{S} \cdot \mathbf{J}\right] \qquad \mathbf{M} = \text{Transfer Matrix}$$

Because:

$$\mathbf{J}^T = \mathbf{J} \qquad \mathbf{S}^T = -\mathbf{S}$$

and for any matrices **A** and **B** the transpose property :

$$[\mathbf{A} \cdot \mathbf{B}]^T = \mathbf{B}^T \cdot \mathbf{A}^T$$

and (proof next page)

$$\mathbf{S} \cdot \exp(s\mathbf{S} \cdot \mathbf{J}) = \exp(s\mathbf{J} \cdot \mathbf{S}) \cdot \mathbf{S}$$

it follows that the transfer matrix ***M*** is *symplectic*:

$$\mathbf{M}(s - s_i) = \exp\left[(s - s_i)\mathbf{S} \cdot \mathbf{J}\right] \qquad s \equiv s - s_i$$

$$\begin{aligned}
\mathbf{M}^T(s) \cdot \mathbf{S} \cdot \mathbf{M}(s) &= \left[\exp(s\mathbf{S} \cdot \mathbf{J})\right]^T \cdot \mathbf{S} \cdot \exp(s\mathbf{S} \cdot \mathbf{J}) \\
&= \exp\left(s[\mathbf{S} \cdot \mathbf{J}]^T\right) \cdot \exp(s\mathbf{J} \cdot \mathbf{S}) \cdot \mathbf{S} \\
&= \exp\left(s\mathbf{J}^T \cdot \mathbf{S}^T\right) \cdot \exp(s\mathbf{J} \cdot \mathbf{S}) \cdot \mathbf{S} \\
&= \exp(-s\mathbf{J} \cdot \mathbf{S}) \cdot \exp(s\mathbf{J} \cdot \mathbf{S}) \cdot \mathbf{S} \\
&= \mathbf{S}
\end{aligned}$$

$$\boxed{\Longrightarrow \quad \mathbf{M}^T(s) \cdot \mathbf{S} \cdot \mathbf{M}(s) = \mathbf{S}}$$

Satisfies symplectic condition

// Aside Proof of formula applied:

Definition of the exponential of a matrix ***A***:

$$\exp(\mathbf{A}) \equiv \sum_{n=0}^{\infty} \frac{\mathbf{A}^n}{n!} \qquad \mathbf{S}^2 = -I$$

Then:

$$\begin{aligned}
\mathbf{S} \cdot \exp(s\mathbf{S} \cdot \mathbf{J}) &= \mathbf{S} \cdot \left(1 + s\mathbf{S} \cdot \mathbf{J} + \frac{1}{2}s^2\mathbf{S} \cdot \mathbf{J} \cdot \mathbf{S} \cdot \mathbf{J} + \cdots\right) \\
&= -\mathbf{S} \cdot \left(1 + s\mathbf{S} \cdot \mathbf{J} + \frac{1}{2}s^2\mathbf{S} \cdot \mathbf{J} \cdot \mathbf{S} \cdot \mathbf{J} + \cdots\right) \cdot \mathbf{S}^2 \\
&= \left([-\mathbf{S}^2] + s[-\mathbf{S}^2] \cdot \mathbf{J} \cdot \mathbf{S} + \frac{1}{2}s^2[-\mathbf{S}]^2 \cdot \mathbf{J} \cdot \mathbf{S} \cdot \mathbf{J} \cdot \mathbf{S} + \cdots\right) \\
&= \left(1 + s\mathbf{J} \cdot \mathbf{S} + \frac{1}{2}s^2\mathbf{J} \cdot \mathbf{S} \cdot \mathbf{J} \cdot \mathbf{S} + \cdots\right) \cdot \mathbf{S} \\
&= \exp(s\mathbf{J} \cdot \mathbf{S}) \cdot \mathbf{S}
\end{aligned}$$

//

Comments:

- Example presented illustrating symplectic structure of Hamiltonian dynamics only applies to piecewise constant linear forces
- Generalizations show that the symplectic structure of Hamiltonian dynamics persists into fully general cases with *s*-dependent Hamiltonians and nonlinear effects. Showing this requires development of more formalism beyond the scope of this course. See for more info:

  A. Dragt, *Lectures on Nonlinear Orbit Dynamics*, in "Physics of High Energy Accelerators," (AIP Conf. Proc. No. 87, New York, 1982), p. 147

  2300 page book distributed freely:
  Alex Dragt, *Lie Methods for Nonlinear Dynamics with Applications to Accelerator Physics* (2015)
  http://www.physics.umd.edu/dsat/dsatliemethods.html

- The symplectic structure of Hamiltonian dynamics is important in numerical codes for long-term tracking of particles in rings
  - Special map-based movers preserve symplectic structure
  - Insures no artificial numerical growth or damping in particle orbits over very long evolution

Symplectic Dynamics = Phase-Space Area Preservation

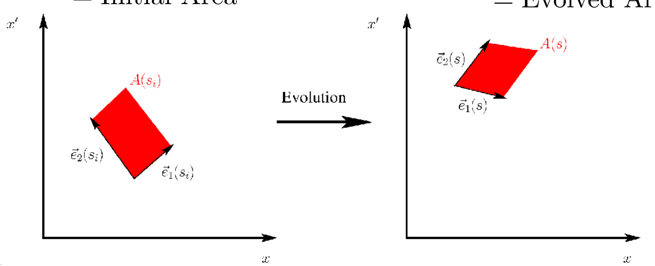
Why the emphasis on the symplectic structure of Hamiltonian dynamics?

Because symplectic dynamics implies that that phase-space area is preserved in the particle evolution.

Illustration: Consider the phase-space area A bounded by two vectors $\vec{e}_1, \vec{e}_2$ which evolve according to symplectic dynamics

- Area calcuated using cross-product: sketch 1D phase space but works 2D, 3D

$$\begin{aligned} A(s_i) &= |\vec{e}_1(s_i) \otimes \vec{e}_2(s_i)| \\ &= \vec{e}_1^T(s_i) \cdot \mathbf{S} \cdot \vec{e}_2(s_i) \\ &= \text{Initial Area} \end{aligned} \qquad \begin{aligned} A(s) &= |\vec{e}_1(s) \otimes \vec{e}_2(s)| \\ &= \vec{e}_1^T(s) \cdot \mathbf{S} \cdot \vec{e}_2(s) \\ &= \text{Evolved Area} \end{aligned}$$



With reference to the same constant field, linear dynamics formulation used to illustrate symplectic dynamics:

The vectors evolve according to Hamiltonian dynamics:

$$\vec{e}_1(s) = \mathbf{M}(s - s_i) \cdot \vec{e}_1(s_i)$$
$$\vec{e}_2(s) = \mathbf{M}(s - s_i) \cdot \vec{e}_2(s_i)$$

Thus, since the dynamics is symplectic with $\mathbf{M}^T \cdot \mathbf{S} \cdot \mathbf{M} = \mathbf{S}$

$$\begin{aligned} \text{Evolved Area} = A = \vec{e}_2^T(s) \cdot \mathbf{S} \cdot \vec{e}_1(s) & \\ &= [\mathbf{M}(s - s_i) \cdot \vec{e}_2(s_i)]^T \cdot \mathbf{S} \cdot [\mathbf{M}(s - s_i) \cdot \vec{e}_1(s_i)] \\ &= \vec{e}_2^T(s_i) \cdot [\mathbf{M}^T(s - s_i) \cdot \mathbf{S} \cdot \mathbf{M}(s - s_i)] \cdot \vec{e}_1(s_i) \\ &= \vec{e}_2^T(s_i) \cdot \mathbf{S} \cdot \cdot \vec{e}_1(s_i) \\ &= A(s_i) = \text{Initial Area} \end{aligned}$$

Giving the important results:

Symplectic dynamics implies conservation of phase-space area !

Comments:

- Illustration only applies to linear constant applied fields, but more advanced treatments (see Dragt refs in previous sub-section) show this property persists for s-varying Hamiltonians and nonlinear dynamics.
- IMPORTANT: conservation of phase-space area in nonlinear dynamics in the sense given does *NOT* imply that measures of *statistical* beam emittance calculated by averages over an ensemble of particles remain conserved in nonlinear dynamics
  - Statistical measures of phase space area impact beam focusability and can evolve in response to nonlinear effects with important implications
    - Effectively phase-space filaments with coarse grained measures of phase space area evolving
  - This is commonly seen in self-consistent simulations
- Acceleration can be dealt with by employing a 3D Hamiltonian formulation with a full set of proper canonical variables or using normalized variables (to effectively remove acceleration) in 4D transverse phase-space
- In numerical analysis of particle orbits in rings it is very important to advance particles that preserve the symplectic structure of the dynamics in the presence of numerical approximations/errors
  - Characteristics then faithful with those expected in real machine over many laps

C. Distribution Methods
C.1 Equations of Motion: Vlasov Equation

Distribution Methods

- Based on reduced (statistical) continuum models of the beam
- Two classes: (microscopic) kinetic models and (macroscopic) fluid models
- Here, distribution means a function of continuum variables
- Use a 3D collision-less Vlasov model to illustrate concept
  - Obtained from statistical averages of particle formulation (see Secs. C7, C8)

Example Kinetic Model: Vlasov Equation of Motion

$q_j = q\,;\ m_j = m\,;$ easy to generalize for multiple species (see later slide)

$$\left\{ \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} + q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial}{\partial \mathbf{p}} \right\} f(\mathbf{x}, \mathbf{p}, t) = 0$$

$$\mathbf{v} = \frac{\mathbf{p}}{\gamma m} = \frac{\mathbf{p}/m}{[1 + \mathbf{p}^2/(mc)^2]^{1/2}}$$

Initial condition $f(\mathbf{x}, \mathbf{p}, t = 0)$

$f(\mathbf{x}, \mathbf{p}, t)$ evolved from t = 0

$\mathbf{x},\ \mathbf{p},\ t$ independent variables

C.2 Distribution Methods: Fields

Fields: Same as in particle methods but with $\rho, \ \mathbf{J}$ expressed in proper form for coupling to the distribution f

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

+ boundary conditions on $\mathbf{E}, \ \mathbf{B}$

Charge Density

$$\rho(\mathbf{x},t) = \underbrace{\rho_{\text{ext}}(\mathbf{x},t)}_{\text{external (applied)}} + \underbrace{q \int d^3p f(\mathbf{x},\mathbf{p},t)}_{\text{beam}}$$

Current Density

$$\mathbf{J}(\mathbf{x},t) = \underbrace{\mathbf{J}_{\text{ext}}(\mathbf{x},t)}_{\text{external (applied)}} + \underbrace{q \int d^3p \ \mathbf{v} f(\mathbf{x},\mathbf{p},t)}_{\text{beam}}$$

$$n = \int d^3p \ f$$

establishes norm of f

- Applied field components continuous
- Self field components continuous for a smooth distribution function f

C.3 Vlasov Equation Incompressible Fluid in Phase-Space

The Vlasov Equation is essentially a continuity equation for an incompressible "fluid" in 6D phase-space. To see this, note that

$$\frac{\partial}{\partial \mathbf{p}} \cdot \mathbf{v} \times \mathbf{B} = 0$$

The Vlasov Equation can be expressed as

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot (\mathbf{v} f) + \frac{\partial}{\partial \mathbf{p}} \cdot (q[\mathbf{E} + \mathbf{v} \times \mathbf{B}] f) = 0$$
$$\Rightarrow \frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot \left(\left.\frac{d\mathbf{x}}{dt}\right|_{\text{orbit}} f \right) + \frac{\partial}{\partial \mathbf{p}} \cdot \left(\left.\frac{d\mathbf{p}}{dt}\right|_{\text{orbit}} f \right) = 0$$

which is manifestly the form of a continuity equation in 6D phase-space
– "probability" is not created or destroyed: flows somewhere in $\mathbf{x} - \mathbf{p}$ phase-space

C.4 Collision Corrections to Vlasov Equation

The effect of collisions can be included by adding a collision operator:

$$\boxed{\left\{ \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} + (q[\mathbf{E} + \mathbf{v} \times \mathbf{B}]) \cdot \frac{\partial}{\partial \mathbf{p}} \right\} f = \left.\frac{\partial f}{\partial t}\right|_{\text{coll}}}$$

For most applications in beam physics, $\left.\frac{\partial f}{\partial t}\right|_{\text{coll}}$ can be neglected.

- See: estimates in Sec. C.8

For cases where it is needed, specific forms of collisions terms can be found in Nicholson, *Intro to Plasma Theory*, Wiley 1983, and similar plasma physics texts

C.4 Distribution Methods: Comment on the PIC Method

The common Particle-in-Cell (PIC) method is *not* a particle method, but rather is a distribution method that uses a collection of smoothed "macro" particles to simulate Vlasov's Equation. This can understood roughly by noting that Vlasov's Equation can be interpreted as

$$\frac{d}{dt} f(\mathbf{x},\mathbf{p},t) = 0$$

Total derivative along a test particle's path

$\Longrightarrow$ Advance particles in a continuous field "fluid" to eliminate particle collisions

Important Point:

$\boxed{\text{PIC is a method to solve Vlasov's Equation, } \textit{not} \text{ a discrete particle method}}$

This will become clear later in the introductory lectures

C.5 Distribution Methods: Multispecies Generalizations

Subscript species with j. Then in the Vlasov equation replace:

$$f \longrightarrow f_j$$
$$m \longrightarrow m_j$$
$$q \longrightarrow q_j$$

and there is a separate Vlasov equation for each of the j species.

Replace the charge and current density couplings in the Maxwell Equations with and appropriate form to include charge and current contributions from all species:

$$\rho(\mathbf{x},t) = \rho_{\text{ext}}(\mathbf{x},t) + \sum_j q_j \int d^3p\, f_j(\mathbf{x},\mathbf{p},t)$$

$$\mathbf{J}(\mathbf{x},t) = \mathbf{J}_{\text{ext}}(\mathbf{x},t) + \sum_j q_j \int d^3p\, \mathbf{v} f_j(\mathbf{x},\mathbf{p},t)$$

Also, if collisions are included the collision operator should be generalized to include collisions between species as well as collisions of a species with itself

C.6 Klimontovich Equation for Self-Consistent Description of Beam/Plasma Evolution

Consider the evolution of N particles coupled to the Maxwell equations and describe the evolution in terms of a singular phase-space density function F evolving in 6D phase space:

$$\boxed{\begin{aligned} F(\mathbf{x},\mathbf{p},t) &= \sum_{i=1}^{N} \delta[\mathbf{x}-\mathbf{x}_i(t)]\delta[\mathbf{p}-\mathbf{p}_i(t)] \\ \mathbf{x}_i(t) &= \text{Position of } i\text{th particle} \\ \mathbf{p}_i(t) &= \text{Mechanical momentum of } i\text{th particle} \\ t &= \text{Time} \qquad N = \text{Number Particles} \end{aligned}}$$

Reminder:

$$\delta(\mathbf{x}) \equiv \delta(x)\delta(y)\delta(z)$$

$\delta(x) \equiv$ Dirac-delta function

- F is highly singular: infinite at location of classical point particles and zero otherwise.
- Here we implicitly assume a single species with charge q and mass m for simplicity. If there are more than one species, the formulation can be generalized by writing a separate density function for each species: $F \rightarrow F_s$
  - Most steps carry through with little modification outside of changes (sum over species) in coupling to the Maxwell equations. See discussion at end of section.

Note that:

$$\boxed{\int d^3x \int d^3p\; F(\mathbf{x},\mathbf{p},t) \;= N = \text{const}}$$

Particles evolve consistent with electromagnetic forces of "microscopic" classical point particles according to the Lorentz force equation:

$$\frac{d\mathbf{p}_i}{dt} = \mathbf{F}_i = q\left[\mathbf{E}^m(\mathbf{x}_i,t) + \frac{d\mathbf{x}_i}{dt}\times\mathbf{B}^m(\mathbf{x}_i,t)\right]$$

$$m\gamma_i\frac{d\mathbf{x}_i}{dt} = \mathbf{p}_i \quad ; \quad \gamma_i = \left[1 + \frac{\mathbf{p}_i^2}{(mc)^2}\right]^{1/2}$$

Initial conditions

$$\mathbf{x}_i(t=0)$$
$$\mathbf{p}_i(t=0)$$

Comments:

- Here we do not consider quantum mechanical effects in scattering but classical scattering and radiation is allowed consistent with electromagnetic forces
  - Ionizations, internal atom excitations, …. would require changes in F
- As written the system applies to one species (i.e., single q, m values) but easy to generalize by writing the same form of F for each species
- Denote superscript m on field components denote "microscopic" fields

Couple to the full set of microscopic fields (superscript m) via the Maxwell equations:

$$\nabla\cdot\mathbf{E}^m = \frac{\rho}{\epsilon_0}$$

$$\nabla\times\mathbf{E}^m = -\frac{\partial\mathbf{B}^m}{\partial t}$$

$$\nabla\cdot\mathbf{B}^m = 0$$

$$\nabla\times\mathbf{B}^m = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}^m}{\partial t}$$

Charge Density

$$\begin{aligned}\rho(\mathbf{x},t) &= \rho_{\text{ext}}(\mathbf{x},t) + q\sum_i \delta[\mathbf{x}-\mathbf{x}_i(t)] \\ &= \rho_{\text{ext}}(\mathbf{x},t) + q\int d^3p\; F(\mathbf{x},\mathbf{p},t)\end{aligned}$$

external (applied)

particle beam

Current Density

$$\begin{aligned}\mathbf{J}(\mathbf{x},t) &= \mathbf{J}_{\text{ext}}(\mathbf{x},t) + q\sum_i \frac{d\mathbf{x}_i}{dt}\delta[\mathbf{x}-\mathbf{x}_i(t)] \\ &= \mathbf{J}_{\text{ext}}(\mathbf{x},t) + q\int d^3p\; \mathbf{v}F(\mathbf{x},\mathbf{p},t)\end{aligned}$$

+ boundary conditions on $\mathbf{E}^m$, $\mathbf{B}^m$

$$\mathbf{v} \equiv \frac{\mathbf{p}}{\gamma m} = \frac{\mathbf{p}/m}{[1+\mathbf{p}^2/(mc)^2]^{1/2}}$$

Comments:

- Full form of Maxwell equations allow classical electromagnetic radiation
- Coupling to beam charge and current is shown for one species, but easy to generalize by summing contributions from all species

Derive an evolution equation for F:

- Note: explicit t dependence contained in $\mathbf{x}_i(t),\ \mathbf{p}_i(t)$

$$\frac{\partial F}{\partial t}(\mathbf{x},\mathbf{p},t) = -\sum_{i=1}^{N}\left[\frac{d\mathbf{x}_i(t)}{dt}\cdot\nabla_{\mathbf{x}} + \frac{d\mathbf{p}_i(t)}{dt}\cdot\nabla_{\mathbf{p}}\right]\delta[\mathbf{x}-\mathbf{x}_i(t)]\delta[\mathbf{p}-\mathbf{p}_i(t)]$$

$$= -\sum_{i=1}^{N}\left[\frac{\mathbf{x}_i(t)}{dt}\cdot\nabla_{\mathbf{x}} + q\left[\mathbf{E}^m(\mathbf{x}_i,\mathbf{p}_i,t) + \frac{d\mathbf{x}_i(t)}{dt}\times\mathbf{B}^m(\mathbf{x}_i,\mathbf{p}_i,t)\right]\cdot\nabla_{\mathbf{p}}\right]$$

$$\delta[\mathbf{x}-\mathbf{x}_i(t)]\delta[\mathbf{p}-\mathbf{p}_i(t)]$$

But:

$$\mathbf{x}\delta[\mathbf{x}-\mathbf{x}_i(t)]\delta[\mathbf{p}-\mathbf{p}_i(t)] = \mathbf{x}_i(t)\delta[\mathbf{x}-\mathbf{x}_i(t)]\delta[\mathbf{p}-\mathbf{p}_i(t)] \qquad \text{etc.}$$

$$\mathbf{v} = \frac{\mathbf{p}}{\gamma m} = \frac{\mathbf{p}/m}{[1+\mathbf{p}^2/(mc)^2]^{1/2}}$$

Giving, after some manipulation, the Klimontovich equation describing the classical collective evolution of the beam as:

$$\boxed{\begin{gathered}\left\{\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} + q\left[\mathbf{E}^m(\mathbf{x},\mathbf{p},t) + \mathbf{v}\times\mathbf{B}^m(\mathbf{x},\mathbf{p},t)\right]\cdot\nabla_{\mathbf{p}}\right] F(\mathbf{x},\mathbf{p},t) = 0 \\ \mathrm{F}(\mathbf{x},\mathbf{p},\ \mathrm{t}) \equiv \textstyle\sum_{i=1}^{N}\delta[\mathbf{x}-\mathbf{x}_i(t)]\delta[\mathbf{p}-\mathbf{p}_i(t)]\end{gathered}}$$

The derivative operator is recognized as acting along a particle orbit:

- Apply Lorentz force equation

$$\left\{\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_{\mathbf{x}} + q\left[\mathbf{E}^m(\mathbf{x},\mathbf{p},t) + \mathbf{v}\times\mathbf{B}^m(\mathbf{x},\mathbf{p},t)\right]\cdot\nabla_{\mathbf{p}}\right\} F(\mathbf{x},\mathbf{p},t)$$

$$= \left\{\frac{\partial}{\partial t} + \frac{d\mathbf{x}_i(t)}{dt}\cdot\nabla_{\mathbf{x}} + \frac{d\mathbf{p}_i(t)}{dt}\cdot\nabla_{\mathbf{p}}\right\} F(\mathbf{x},\mathbf{p},t)$$

$$= \left.\frac{d}{dt}\right|_{\text{orbit}} F(\mathbf{x},\mathbf{p},t)$$

Showing that the Klimontovich equation can be alternatively expressed as :

$$\boxed{\left.\tfrac{d}{dt}\right|_{\text{orbit}} F(\mathbf{x},\mathbf{p},t) = 0}$$

- Shows F is advected along characteristic particle orbits and is conserved
  - Orbits in this sense are trajectories of "marker" particles (same q/m as physical particles) evolving in the beam

Alternatively, some manipulations show that the Klimontovich equation can be expressed in the form of a higher dimensional relativistic continuity equation:

$$\mathbf{F} \equiv q\left[\mathbf{E}^m(\mathbf{x},t) + \mathbf{v}\times\mathbf{B}^m(\mathbf{x},t)\right]$$

$$\mathbf{v} = \frac{\mathbf{p}}{\gamma m} = \frac{\mathbf{p}/m}{[1+\mathbf{p}^2/(mc)^2]^{1/2}}$$

$$\Longrightarrow \nabla_{\mathbf{p}}\cdot\mathbf{F} = 0$$

Giving:

$$\boxed{\frac{\partial}{\partial t}F(\mathbf{x},\mathbf{p},t) + \nabla_{\mathbf{x}}\cdot[\mathbf{v}F(\mathbf{x},\mathbf{p},t)] + \nabla_{\mathbf{p}}\cdot[\mathbf{F}F(\mathbf{x},\mathbf{p},t)] = 0}$$

- Shows F is conserved in sense probability flows rather than created/destroyed
- Can apply analogy with familiar continuity equation in fluid mechanics (see Aside next page) to help interpret

// Aside: Analogy with the familiar continuity equation of a local fluid density n:

$$\frac{\partial}{\partial t}n(\mathbf{x},t) + \nabla_{\mathbf{x}}\cdot[n(\mathbf{x},t)\mathbf{V}(\mathbf{x},t)] = 0$$

$$n(\mathbf{x},t) = \text{Fluid number density}$$

$$\mathbf{V}(\mathbf{x},t) = \text{Fluid flow velocity}$$

- Shows fluid density n flows somewhere consistent with the fluid flow $\boldsymbol{V}$ and is not created/destroyed
- n is conserved in this sense and the continuity equation is the proper expression of local conversation
- Implies fluid weight not created or destroyed. Integrate over some volume V containing all fluid ($n = 0$ on surface)

$$\int_V d^3x\,\left\{\frac{\partial}{\partial t}n(\mathbf{x},t) + \nabla_{\mathbf{x}}\cdot[n(\mathbf{x},t)\mathbf{V}(\mathbf{x},t)]\right\} = 0$$

$$\frac{\partial}{\partial t}\int_V d^3x\ n(\mathbf{x},t)\ +\ +\int_{\partial V} d^2x\ \overset{0}{n(\mathbf{x},t)}\mathbf{V}(\mathbf{x},t)\cdot\hat{\mathbf{n}}(\mathbf{x})\ \ = 0$$

applied Divergence Theorem:

$$\partial V = \text{Surface bounding } V$$

$$\hat{\mathbf{n}}(\mathbf{x}) = \text{Local unit normal to } \partial V$$

$$\frac{\partial}{\partial t}\int_V d^3x\ n(\mathbf{x},t)\ = 0 \quad \Rightarrow \int_V d^3x\ n(\mathbf{x},t)\ = \text{const}$$

//

Comments on the Klimontovich formulation:

- Solved as an initial value problem: initial particle phase-space coordinates of particles specified and then solved with coupled Maxwell equations
  - Classically exact, but practically speaking intractable
  - Really just a restatement of classical point particles evolving consistently with the Maxwell equations: mostly just notation at this point
- Klimontovich equation is essentially a statement that local density in phase-space is conserved in the classical evolution of a charged particle system
  - Quantum theory of matter can change expression since collisions of particles can cause effects like ionization, internal atom excitations, ….
  - Klimontovich equation can be modified by including appropriate terms on the RHS of equation to model such classical and quantum scattering effects: would result in non-conservation of probabilities associated with quantum effects in "collisions"
- Sometimes called Liouville's theorem in micro-space
- Not particularly useful in this form other than conceptual grounding. Need an expression in terms of a *smoothed statistical measure* of the particle distribution. We construct this next section.

C.7 Motivation of Vlasov's Equation

Average of the singular (micro) Klimontovich distribution f to obtain a smooth *6D* phase-space distribution

$$\Delta x = \text{Position width}$$
$$\Delta p = \text{Momentum width}$$
$$f(\mathbf{x},\mathbf{p},t) \equiv \frac{1}{\Delta x^3 \Delta p^3} \int_{\Delta x^3} d^3x \int_{\Delta p^3} d^3p \, F(\mathbf{x},\mathbf{p},t) \quad = \langle F(\mathbf{x},\mathbf{p},t)\rangle$$

- Average taken about *local* phase-space coordinates $\mathbf{x},\ \mathbf{p}$ so after averaging, dependence of distribution remains in $\mathbf{x},\ \mathbf{p}$
- Average is essentially a coarse-graining to reduce detail

Arguments can be involved in taking appropriate coarse-grain measurements. Logic from plasma physics suggests:

$$n^{-1/3} \ll \Delta x \ll \lambda_D$$
$$0 \ll \Delta p \ll m v_t$$

$$n = \text{Characteristic number density}$$
$$\lambda_D = \left(\frac{\epsilon_0 T}{q^2 n}\right)^{1/2} = \text{Thermal debye length}$$
$$T = \text{Kinetic temp}$$
$$v_t = \left(\frac{T}{m}\right)^{1/2} = \text{Thermal velocity}$$

Better defining some of these measures:

- Take a nonrelativistic perspective here for simplicity

$$T = \text{Characteristic kinetic temperature (energy units)}$$
$$\frac{1}{2}T = \frac{1}{2}mv_t^2 \qquad \Longrightarrow \qquad v_t = \left(\frac{T}{m}\right)^{1/2} = \text{Thermal velocity}$$

- Kinetic temperature removing local flow measures (beam frame) of beam and measures strength of random spread of particle momentum
  - Relativity introduces some subtleties on how to best do this. See Reiser *Theory and Design of Charged Particle Beams* for a thorough discussion.

$$\lambda_D \equiv \frac{(T/m)^{1/2}}{\omega_p} = \frac{v_t}{\omega_p} = \left(\frac{\epsilon_0 T}{q^2 n}\right)^{1/2} = \text{Debye length}$$

- Characteristic screening/shielding distance within a plasma

$$\omega_p = \left(\frac{q^2 n}{\epsilon_0 m}\right)^{1/2} = \text{Plasma frequency}$$

- Characteristic collective oscillation frequency of electrostatic restoring forces in a plasma

Examine the local deviations of the distribution from the smoothed average:

$$F = f + \delta f$$
$$f \equiv \langle F \rangle \qquad \Longrightarrow \qquad \langle \delta f \rangle = 0$$

- Dependence of all distribution terms $(\mathbf{x},\mathbf{p},t)$

And make similar definitions for smoothed field components:

$$\begin{aligned} \mathbf{E}^m &= \mathbf{E} + \delta\mathbf{E} & \mathbf{E} &\equiv \langle \mathbf{E}^m \rangle & & \langle \delta\mathbf{E}\rangle = 0 \\ \mathbf{B}^m &= \mathbf{B} + \delta\mathbf{B} & \mathbf{B} &\equiv \langle \mathbf{B}^m \rangle & \Longrightarrow \quad & \langle \delta\mathbf{B}\rangle = 0 \end{aligned}$$

- Dependence of all field terms $(\mathbf{x},t)$

Averaging over the Kilmontovich equation in Sec. C.5 then obtains:

$$\left\{ \frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + q\left[\mathbf{E}(\mathbf{x},t) + \mathbf{v}\times\mathbf{B}(\mathbf{x},t)\right]\cdot\frac{\partial}{\partial \mathbf{p}} \right\} f(\mathbf{x},\mathbf{p},t) =$$
$$-q\langle [\delta\mathbf{E}(\mathbf{x},t) + \mathbf{v}\times\delta\mathbf{B}(\mathbf{x},t)]\,\delta f(\mathbf{x},\mathbf{p},t)\rangle$$

Update future version to give some of the simple steps to obtain terms in the coarse-grained averaged Kilmontovich equation

Similarly, averaging over the microscopic Maxwell equations gives the Maxwell equations for the smoothed fields:

- Equations are linear, so average is trivial. But coupling form to beam charges and currents changes to average form representing smoothed charge and current densities

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \rho(\mathbf{x},t) = \rho_{\text{ext}}(\mathbf{x},t) + q\int d^3p f(\mathbf{x},\mathbf{p},t)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \mathbf{J}(\mathbf{x},t) = \mathbf{J}_{\text{ext}}(\mathbf{x},t) + q\int d^3p\ \mathbf{v} f(\mathbf{x},\mathbf{p},t)$$

+ boundary conditions on $\mathbf{E},\ \mathbf{B}$

The fields can also be given, as usual, with potentials $\phi,\ \mathbf{A}$

- Specific form of potential equations and coupling to $\rho,\ \mathbf{J}$ depend on the gauge choices made: consult classical electrodynamics textbooks for details

$$\mathbf{E} = -\nabla\phi + \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Update future version to give some of the simple steps to obtain the coarse-grained Maxwell equations with Vlasov couplings in rho and J

In this averaged equation:

LHS: $$\left\{\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + q\left[\mathbf{E}(\mathbf{x},t) + \mathbf{v}\times\mathbf{B}(\mathbf{x},t)\right]\cdot\frac{\partial}{\partial \mathbf{p}}\right\} f(\mathbf{x},\mathbf{p},t)$$

- Represents the smoothly (continuum model theory) varying part
  - No scattering effects but retains self-consistent collective effects
  - Appropriate to model many (fluid like) particles interacting

RHS: $$-q\langle[\delta\mathbf{E}(\mathbf{x},t) + \mathbf{v}\times\delta\mathbf{B}(\mathbf{x},t)]\,\delta f(\mathbf{x},\mathbf{p},t)\rangle$$

- Represents an averaged interaction over rapidly varying quantities
- Retains information from classical discrete particle effects and collisions
  - In form given has no quantum mechanical effects such as ionizations, internal atom excitations, ... Model must be further augmented to analyze such effects
- Extensive treatments in plasma physics involve making approximations for this classical "collision operator" to statistically model scattering effects in plasmas
  - Beyond scope of this course: see treatments and references in the various plasma physics texts referenced at the end of these notes (recommend Nicholson)
- Will outline arguments that classical scattering effects associated with this term are negligible in many cases relevant to intense beam physics

When scattering effects on the RHS can be neglected, the evolution of the system is given by the Vlasov equation:

$$\boxed{\begin{aligned}&\left\{\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + q\left[\mathbf{E}(\mathbf{x},t) + \mathbf{v}\times\mathbf{B}(\mathbf{x},t)\right]\cdot\frac{\partial}{\partial \mathbf{p}}\right\} f(\mathbf{x},\mathbf{p},t) = 0 \\ &\text{Here,} \qquad \mathbf{v} = \frac{\mathbf{p}}{\gamma m} = \frac{\mathbf{p}/m}{[1+\mathbf{p}^2/(mc)^2]^{1/2}}\end{aligned}}$$

- Describes the evolution of the system in a classical continuum model sense
  - Includes collective effects
  - Does not include scattering and quantum mechanical effects
- Solved as an initial value problem with $f(\mathbf{x},\mathbf{p},t=0)$ specified and the fields given by the smoothed Maxwell equations

Simple estimate on when scattering can be neglected

- For more details consult plasma physics texts
- Take a nonrelativistic perspective for simplicity
- Use previous scales employed in coarse grain averages used to obtain f

Heuristically, on the RHS scattering term take:

$$\text{RHS} = -q\langle[\delta\mathbf{E} + \mathbf{v}\times\delta\mathbf{B}]\,\delta f\rangle \sim \nu_c f$$

$$\nu_c \sim \sigma n v_t = \text{Collision frequency}$$

$$\sigma \sim \pi r_c^2 = \text{Collision cross-section}$$

Estimate cross section by considering a large angle scatter where the thermal energy of the incident particle is of order the electrostatic potential energy at closest approach:

$$T \sim \frac{q^2}{4\pi\epsilon_0 r_c} \qquad \Longrightarrow r_c \sim \frac{q^2}{4\pi\epsilon_0 T} = \text{collision radius}$$

$$\Longrightarrow \nu_c \sim (\pi r_c^2) n v_t \sim \pi\left(\frac{q^2}{4\pi\epsilon_0 T}\right)^2 n \left(\frac{T}{m}\right)^{1/2} \sim \frac{1}{16\pi}\frac{v_t}{\lambda_D^4 n}$$

$$\boxed{\text{RHS} = -q\langle[\delta\mathbf{E} + \mathbf{v}\times\delta\mathbf{B}]\,\delta f\rangle \sim \nu_c f \sim \frac{1}{16\pi}\frac{v_t}{\lambda_D^4 n} f}$$

Reminder:

$$\lambda_D \equiv \left(\frac{\epsilon_0 T}{q^2 n}\right)^{1/2}$$

Heuristically, on the LHS collective term expect for electrostatic effects:

$$\boxed{\text{LHS} = \left\{\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + q\left[\mathbf{E} + \mathbf{v}\times\mathbf{B}\right]\cdot\frac{\partial}{\partial \mathbf{p}}\right\} f \sim \omega_p f}$$

$$\omega_p = \left(\frac{q^2 n}{\epsilon_0 m}\right)^{1/2} = \text{Angular frequency of plasma oscillations}$$

The relative order of the LHS (collective) and RHS (classical scattering) terms are:

$$\boxed{\frac{\text{RHS}}{\text{LHS}} = \frac{\text{Collisions}}{\text{Collective}} \sim \frac{1}{16\pi}\frac{v_t}{\lambda_D^4 n}\frac{1}{\omega_p} = \frac{1}{16\pi\lambda_D^3 n} \sim \frac{1}{\Lambda} \propto \frac{n^{1/2}}{T^{3/2}}} \qquad \lambda_D \propto \left(\frac{T}{n}\right)^{1/2}$$

$$\Lambda \equiv \frac{4\pi}{3}\lambda_D^3 n = \text{Particles per Debye sphere}$$

$$\gg 1 \text{ for intense beams}$$

We expect scattering effects to be weak relative to collective effects for typical intense beams

- Special situations can change this: very cold beams near source ….

Arguments here on ordering are somewhat circular but show consistency. To more rigorously motivate, usually careful comparisons with more complete models are necessary.

- Many studies in field motivate Vlasov model typically good for intense beams
- Often sense is try it and see if it works, analyze scattering when discrepancies appear that can plausibly be explained by scattering effects

Material presented follows formulations in:

- Nicholson, *Introduction to Plasma Physics,* Wiley, 1982
- USPAS Lectures on *Beam Physics with Intense Space-Charge*

Reiteration: Interpretation of Vlasov's Equation

The Vlasov Equation is essentially a continuity equation for an incompressible "fluid" in 6D phase-space. To see this, cast in standard continuity equation form using $\frac{\partial}{\partial \mathbf{p}} \cdot \mathbf{v} \times \mathbf{B} = 0$

To express the Vlasov equation equivalently as

$$\boxed{\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{x}} \cdot (\mathbf{v} f) + \frac{\partial}{\partial \mathbf{p}} \cdot (q[\mathbf{E} + \mathbf{v} \times \mathbf{B}] f) = 0}$$

- Manifestly the form of a continuity equation in 6D phase-space, i.e., "probability" f is not created or destroyed

Alternatively, we note that the total derivative along a single particle orbit in the continuum model is

$$\frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} + q[\mathbf{E} + \mathbf{v} \times \mathbf{B}] \cdot \frac{\partial}{\partial \mathbf{p}} = \left. \frac{d}{dt} \right|_{\text{orbit}}$$

So the Vlasov equation can be equivalently expressed as

$$\boxed{\left. \frac{d}{dt} \right|_{\text{orbit}} f(\mathbf{x}, \mathbf{p}, t) = 0}$$

- Expresses that f is advected along characteristic particle orbits in the continuum and is therefore manifestly conserved

C.8 Liouville's Theorem

These prove Liouville's theorem:

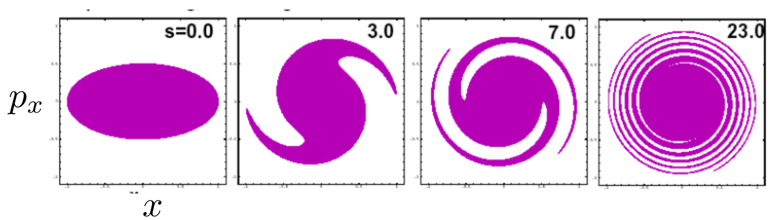
The density of particles in 6D phase-space is invariant when measured along the trajectories of characteristic particles

Comments:

- Although density in phase-space remains constant by Liouville's theorem, the shape in phase-space can vary in response to evolution and nonlinear effects
  - Distortions and Filamentation
- Consequently, coarse-grained or statistical measures of beam phase-space area such as rms emittances can evolve
  - Rms emittances provide important measures of statistical beam focusability
  - (see USPAS lectures on *Beam Physics with Intense Space-Charge*)
- Proved here using position-mechanical momentum phase space $(\mathbf{x}, \mathbf{p})$: later will show valid for all choices of canonical variables
- Valid in continuum mechanics approximation with average (mean field) self-consistent effects. Both classical scattering and quantum mechanical scattering processes result in violations of Liouville's theorem
  - Classical scattering tends to decrease density in phase-space
- Numerous versions in literature: often simplest case given for non-interacting particles evolving in response to prescribed forces

Phase-space area measures and Liouville's theorem

In spit of Liouville's theorem, nonlinear forces acting on a beam can filament phase space. Schematically:



Although phase-space area is conserved in such processes under Liouville's theorem, coarse-grained statistical projection measures of beam phase-space area such as rms beam emittances can evolve and tend to increase under the action of such effects.

$$\text{Projected Statistical Emittance} \sim \left[\langle x^2 \rangle_\perp \langle p_x^2 \rangle_\perp - \langle x p_x \rangle_\perp^2\right]^{1/2}$$

(rms measure)

- Much more on this topic in USPAS lecture notes on *Beam Physics with Intense Space Charge*. See: Transverse Equilibrium Distributions, Transverse Centroid and Envelope Descriptions, and Transverse Kinetic Stability

C.9 Canonical Variables, Vlasov's Equation, and a Generalized Expression of Liouville's Theorem

Louiville's theorem derived for a distribution expressed as a function of position-*mechanical* momentum variables $(\mathbf{x}, \mathbf{p})$ Here we will show that the same statement holds for any proper set of canonical variables to generalize the interpretation of the Liouville Theorem.

- Note that $(\mathbf{x}, \mathbf{p})$ variables are not necessarily a proper canonical pair in all relevant focusing systems considered: e.g., solenoid focusing

Consider a proper set of canonical variables from which a Hamiltonian H describes the continuum model trajectories consistent with the mean field model potentials $\phi,\ \mathbf{A}$

$$\boxed{\begin{aligned} q_i &= \text{Canonical Coordinate} & \frac{d}{dt} q_i &= \frac{\partial H}{\partial p_i} & H &= H(\{q_i\}, \{p_i\}, t) \\ p_i &= \text{Canonical Momentum} & & & & \\ i &= 1, 2, 3 & \frac{d}{dt} p_i &= -\frac{\partial H}{\partial q_i} & & \end{aligned}}$$

Next, we take the Vlasov model distribution f to be a function of the canonical variables

$$\boxed{f = f(\{q_i\}, \{p_i\}, t)}$$

// Aside: Canonical Variables in Magnetic Focusing

Illustrate the concept of canonical variables with transverse solenoid magnetic focusing. Velocity/angle dependent forces superficially are incompatible with a Hamiltonian formulation in x-x' variables.

For solenoidal magnetic focusing without acceleration, take (tilde) canonical variables:

- Tildes *do not* denote Larmor transform variables here !

$$\tilde{x} = x \qquad \tilde{y} = y$$
$$\tilde{x}' = x' - \frac{B_{z0}}{2[B\rho]}y \qquad \tilde{y}' = y' + \frac{B_{z0}}{2[B\rho]}x \qquad [B\rho] \equiv \frac{m\gamma_b\beta_b c}{q}$$

With Hamiltonian:

$$\tilde{H}_\perp = \frac{1}{2}\left[\left(\tilde{x}' + \frac{B_{z0}}{2[B\rho]}\tilde{y}\right)^2 + \left(\tilde{y}' - \frac{B_{z0}}{2[B\rho]}\tilde{x}\right)^2\right] + \frac{q\phi}{m\gamma_b^3\beta_b^2c^3}$$

$$\frac{d}{ds}\tilde{x} = \frac{\partial \tilde{H}_\perp}{\partial \tilde{x}'} \qquad \frac{d}{ds}\tilde{y} = \frac{\partial \tilde{H}_\perp}{\partial \tilde{y}'}$$
$$\frac{d}{ds}\tilde{x}' = -\frac{\partial H_\perp}{\partial \tilde{x}} \qquad \frac{d}{ds}\tilde{y}' = -\frac{\partial H_\perp}{\partial \tilde{y}}$$

Caution: Primes do not mean *d/ds* in tilde variables here: just notation to distinguish "momentum" variable!

Giving (after some algebra) the familiar cross-coupled equations of motion for the velocity dependant v x B force:

$$x'' - \frac{B'_{z0}(s)}{2[B\rho]}y - \frac{B_{z0}(s)}{[B\rho]}y' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$
$$y'' + \frac{B'_{z0}(s)}{2[B\rho]}x + \frac{B_{z0}(s)}{[B\rho]}x' = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

//

On general grounds, the distribution should evolve consistent with a continuity equation expressed in the canonical variables. We form this as:

$$\frac{\partial f}{\partial t} + \nabla_6\cdot(\vec{v}_6 f) = 0$$

where

$$\vec{v}_6 = \begin{bmatrix}\dot{q}_1\\ \dot{p}_1\\ \dot{q}_2\\ \dot{p}_2\\ \dot{q}_3\\ \dot{p}_3\end{bmatrix} \qquad \nabla_6\cdot\vec{A}_6 \equiv \sum_{i=1}^{3}\left[\frac{\partial A_i}{\partial q_i} + \frac{\partial A_i}{\partial p_i}\right]$$

$\vec{A}_6 \equiv$ 6-vector in obvious notation

$$\dot{} \equiv \frac{d}{dt}$$

Then using Hamilton's equations of motion of the characteristics:

$$\nabla_6\cdot\vec{v}_6 = \sum_{i=1}^{3}\left[\frac{\partial\dot{q}_i}{\partial q_i} + \frac{\partial\dot{p}_i}{\partial p_i}\right] = \sum_{i=1}^{3}\left[\frac{\partial^2 H}{\partial q_i\partial p_i} - \frac{\partial^2 H}{\partial p_i\partial q_i}\right] = 0$$

So:

$$\frac{\partial f}{\partial t} + \nabla_6\cdot(\vec{v}_6 f) = \frac{\partial f}{\partial t} + \cancelto{0}{\nabla_6\cdot\vec{v}_6} f + \vec{v}_6\cdot\nabla_6 f = 0$$

$$\Longrightarrow \quad \frac{\partial f}{\partial t} + \vec{v}_6\cdot\nabla_6 f = \left.\frac{d}{dt}\right|_{\text{orbit}} f = 0$$

Expected incompressible flow form

This shows that the distribution *f* evaluated along characteristic trajectories in any set of canonical variables remains invariant in the Vlasov model

- Liouville's theorem remains valid in any set of canonical variables

It is useful to also express the incompressible Vlasov equation in canonical variables:

$$\frac{\partial f}{\partial t} + \vec{v}_6\cdot\nabla_6 f = 0$$
$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left\{\dot{q}_i\frac{\partial f}{\partial q_i} + \dot{p}_i\frac{\partial f}{\partial p_i}\right\} = 0$$
$$\Longrightarrow \quad \frac{\partial f}{\partial t} + \sum_{i=1}^{3}\left\{\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i}\right\} = 0$$

Canonical form of Vlasov's equation

- See electrodynamics texts for form of *H* with (mean field) potentials $\phi, \mathbf{A}$ and various canonical variable choices

In the literature, sometimes

$$\{\, \mathrm{H},\ \mathrm{f}\,\} \equiv \left\{\frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i}\right\}$$

is called a *Poisson Bracket*

Further insight can be obtained on the canonical form of the Vlasov distribution by transforming from one set of canonical variables to another. Since $f d^3q d^3p$ is the physical number of particles (counting) and invariant with the choice of variables used to describe the problem, we have under canonical transform:

$$\boxed{f d^3q d^3p = \tilde{f} d^3\tilde{q} d^3\tilde{p}}$$

$$\tilde{q}_i = \text{Transformed canonical coordinate}$$
$$\tilde{p}_i = \text{Transformed canonical momentum}$$
$$\tilde{H} = \tilde{H}(\{\tilde{q}_i,\}, \{\tilde{p}_i\}, \tilde{t}) = \text{Transformed Hamiltonian}$$

- Canonical form is maintained in the transformed variables

In classical mechanics texts, canonical transform generating functions are used to prove that phase space area measures are invariant under canonical transform:

$$\boxed{d^3q d^3p = d^3\tilde{q} d^3\tilde{p}}$$

Therefore, the Vlasov distribution *f* is invariant under canonical transform:

$$\boxed{f\left(\{q_i\}, \{p_i\}, t\right) = \tilde{f}\left(\{\tilde{q}_i\}, \{\tilde{p}_i\}, \tilde{t}\right)}$$

Comments:

- Pure transverse theories of an accelerating beam cannot be cast in terms of a Hamiltonian theory. This is due to the acceleration induced damping terms like:

$$\frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x'$$

in transverse particle equations of motion.
  - For a coasting beam without acceleration or solenoid magnetic focusing *x,x' y,y'* form a convienient canonical set: these are used extensively in this context in following lecture sets
- Use of normalized variables can approximately bypass this limitation in transverse paraxial theories
- The canonical variables used in the 3D formulation here can include acceleration effects and can be thought of as "normalized" 3D variables
- Dimensionality and the scope of what is included (acceleration etc) in the Hamiltonian can cause confusion!
  - Clear concept of context, scope, and limitations can help clarify

C.10 Transverse Vlasov Formulation

Consider a coasting, single-species beam with electrostatic self-fields propagating in a linear focusing channel

$\mathbf{x}_\perp, \ \mathbf{x}'_\perp$ transverse particle coordinate, angle

$q, \ \ m$ charge, mass

$\gamma_b, \ \ \beta_b$ axial relativistic factors

$f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$ single particle distribution

$H_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s)$ single particle Hamiltonian

Vlasov Equation:

$$\frac{d}{ds}f_\perp = \frac{\partial f_\perp}{\partial s} + \frac{d\mathbf{x}_\perp}{ds}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} + \frac{d\mathbf{x}'_\perp}{ds}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}'_\perp} = 0$$

Particle Equations of Motion:

$$\frac{d}{ds}\mathbf{x}_\perp = \frac{\partial H_\perp}{\partial \mathbf{x}'_\perp} \qquad \frac{d}{ds}\mathbf{x}'_\perp = -\frac{\partial H_\perp}{\partial \mathbf{x}_\perp}$$

Hamiltonian:

$$H_\perp = \frac{1}{2}\mathbf{x}'^{\,2}_\perp + \frac{1}{2}\kappa_x(s)x^2 + \frac{1}{2}\kappa_y(s)y^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

Poisson Equation:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{q}{\epsilon_0}\int d^2\mathbf{x}'_\perp \ f_\perp$$

+ boundary conditions on

- Normalization of distribution chosen such that

$$\lambda = q\int d^x x \int d^2x'_\perp \ f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) \quad = \text{Line Charge} = \text{const}$$

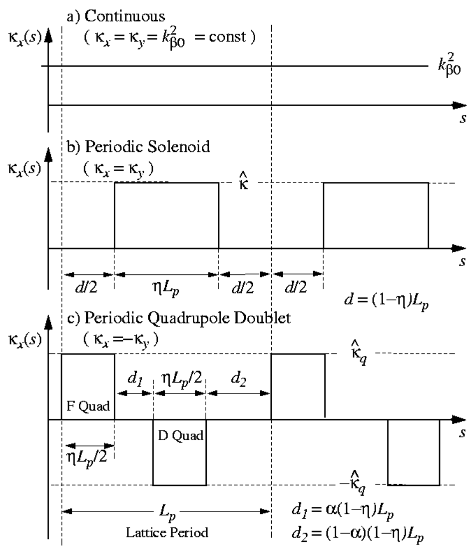
- The coupling to the self-field via the Poisson equation makes the Vlasov-Poisson model *highly* nonlinear

$$\rho = q\int d^2x'_\perp \ f_\perp \qquad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\phi = -\frac{\rho}{\epsilon_0}$$

+ aperture boundary condition on ϕ

- Vlasov-Poisson system is written without acceleration, but the transforms developed to identify the normalized emittance in the lectures on can be exploited to generalize to (weakly) accelerating beams
  - See USPAS, *Beam Physics with Intense Space-Charge* notes
- For solenoidal focusing the system can be interpreted in the rotating Larmor frame
  - See USPAS, *Beam Physics with Intense Space-Charge* notes
- System as expressed applies to 2D (unbunched) beam as expressed

Focusing lattices, continuous and periodic (simple piecewise constant):



Lattice Period L_p

Occupancy η

$\eta \in [0, 1]$

Solenoid description carried out implicitly in Larmor frame

Syncopation Factor α

$$\alpha \in [0, \frac{1}{2}]$$

$$\alpha = \frac{1}{2} \implies FODO$$

Continuous Focusing: $\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const}$

$$H_\perp = \frac{1}{2}\mathbf{x}_\perp'^{\,2} + \frac{1}{2}k_{\beta 0}^2 \mathbf{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

Solenoidal Focusing (in Larmor frame variables): $\kappa_x = \kappa_y = \kappa(s)$

$$H_\perp = \frac{1}{2}\mathbf{x}_\perp'^{\,2} + \frac{1}{2}\kappa \mathbf{x}_\perp^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

Quadrupole Focusing: $\kappa_x = -\kappa_y = \kappa_q(s)$

$$H_\perp = \frac{1}{2}\mathbf{x}_\perp'^{\,2} + \frac{1}{2}\kappa_q x^2 - \frac{1}{2}\kappa_q y^2 + \frac{q}{m\gamma_b^3\beta_b^2c^2}\phi$$

$$\boxed{\begin{aligned} &H_\perp = \frac{1}{2}x'^2 + \frac{1}{2}y'^2 + \frac{1}{2}\kappa_x x^2 + \frac{1}{2}\kappa_y y^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^3} \\ &\frac{d}{ds}x = \frac{\partial H_\perp}{\partial x'} \qquad \frac{d}{ds}x = \frac{\partial H_\perp}{\partial y'} \\ &\frac{d}{ds}x' = -\frac{\partial H_\perp}{\partial x} \qquad \frac{d}{ds}y' = -\frac{\partial H_\perp}{\partial y} \end{aligned}}$$

Giving the familiar equations of motion:

$$\text{x}'' + \kappa_x x = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x} \qquad y'' + \kappa_y y = -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

Although the equations have the same form, the couplings to the fields are different which leads to different regimes of applicability for the various focusing technologies with their associated technology limits:

Focusing:

Continuous:

$$\kappa_x(s) = \kappa_y(s) = k_{\beta 0}^2 = \text{const}$$

Good qualitative guide (see later material/lecture)

BUT not physically realizable (see S2B)

Quadrupole:

$$\kappa_x(s) = -\kappa_y(s) = \begin{cases} \frac{G(s)}{\beta_b c[B\rho]}, & \text{Electric} \\ \frac{G(s)}{c[B\rho]}, & \text{Magnetic} \end{cases} \qquad [B\rho] = \frac{m\gamma_b\beta_b c}{q}$$

G is the field gradient which for linear applied fields is:

$$G(s) = \begin{cases} -\frac{\partial E_x^a}{\partial x} = \frac{\partial E_y^a}{\partial y} = \frac{2V_q}{r_p^2}, & \text{Electric} \\ \frac{\partial B_x^a}{\partial y} = \frac{\partial B_y^a}{\partial x} = \frac{B_p}{r_p}, & \text{Magnetic} \end{cases}$$

Solenoid: (within a rotating frame)

$$\kappa_x(s) = \kappa_y(s) = k_L^2(s) = \left[\frac{B_{z0}(s)}{2[B\rho]}\right]^2 = \left[\frac{\omega_c(s)}{2\gamma_b\beta_b c}\right]^2 \qquad \omega_c(s) = \frac{qB_{z0}(s)}{m}$$

Expression of Vlasov Equation

Hamiltonian expression of the Vlasov equation:

$$\frac{d}{ds}f_\perp = \frac{\partial f_\perp}{\partial s} + \frac{d\mathbf{x}_\perp}{ds}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} + \frac{d\mathbf{x}_\perp'}{ds}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp'} = 0$$

$$= \frac{\partial f_\perp}{\partial s} + \frac{\partial H_\perp}{\partial \mathbf{x}_\perp'}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} - \frac{\partial H_\perp}{\partial \mathbf{x}_\perp}\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp'} = 0$$

Using the equations of motion:

$$\frac{d}{ds}\mathbf{x}_\perp = \frac{\partial H_\perp}{\partial \mathbf{x}_\perp'} = \mathbf{x}_\perp'$$

$$\frac{d}{ds}\mathbf{x}_\perp' = -\frac{\partial H_\perp}{\partial \mathbf{x}_\perp} = -\left(\kappa_x x\hat{\mathbf{x}} + \kappa_y y\hat{\mathbf{y}} + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial \mathbf{x}_\perp}\right)$$

Gives the explicit form of the Vlasov equation:

$$\boxed{\frac{\partial f_\perp}{\partial s} + \mathbf{x}_\perp'\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp} - \left(\kappa_x x\hat{\mathbf{x}} + \kappa_y y\hat{\mathbf{y}} + \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial \mathbf{x}_\perp}\right)\cdot\frac{\partial f_\perp}{\partial \mathbf{x}_\perp'} = 0}$$

C.11 Putting Additional Effects in Transverse Model Introduction

Transverse particle equations of motion in explicit component form in terms of applied field components $\mathbf{E}^a,\ \mathbf{B}^a$ can be applied to generalize model:

$$\boxed{\begin{aligned} x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' &= \frac{q}{m\gamma_b\beta_b^2c^2}E_x^a - \frac{q}{m\gamma_b\beta_b c}B_y^a + \frac{q}{m\gamma_b\beta_b c}B_z^a y' \\ &\quad - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x} \\ y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' &= \frac{q}{m\gamma_b\beta_b^2c^2}E_y^a + \frac{q}{m\gamma_b\beta_b c}B_x^a - \frac{q}{m\gamma_b\beta_b c}B_z^a x' \\ &\quad - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y} \end{aligned}}$$

Equations previously derived under assumptions:
- No bends (fixed *x*-*y*-*z* coordinate system with no local bends)
- Paraxial equations ($x'^2, y'^2 \ll 1$)
- No dispersive effects (β_b same all particles), acceleration allowed ($\beta_b \neq$ const)
- Electrostatic and leading-order (in β_b) self-magnetic interactions

In vector form these equations of motion can be expressed as:

$$\boxed{\begin{aligned} \mathbf{x}_\perp'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}\mathbf{x}_\perp' &= \frac{q}{m\gamma_b\beta_b^2c^2}\mathbf{E}_\perp^a + \frac{q}{m\gamma_b\beta_b c}\hat{\mathbf{z}}\times\mathbf{B}_\perp^a + \frac{qB_z^a}{m\gamma_b\beta_b c}\mathbf{x}_\perp'\times\hat{\mathbf{z}} \\ &\quad - \frac{q}{\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial\mathbf{x}_\perp}\phi \end{aligned}}$$

These equations can be reduced when the applied focusing fields are linear to:

$$\boxed{\begin{aligned} x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x(s)x &= -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial x}\phi \\ y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \kappa_y(s)y &= -\frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial}{\partial y}\phi \end{aligned}}$$

where

$$\kappa_x(s) = x\text{-focusing function of lattice}$$
$$\kappa_y(s) = y\text{-focusing function of lattice}$$

describe the linear applied focusing forces and the equations are implicitly analyzed in the rotating Larmor frame when $B_z^a \neq 0$.

Lattice designs attempt to minimize nonlinear applied fields. However, the 3D Maxwell equations show that there will *always* be some finite nonlinear applied fields for an applied focusing element with finite extent. Applied field nonlinearities also result from:
- Design idealizations
- Fabrication and material errors

The largest source of nonlinear terms will depend on the case analyzed. Beam self-fields, when strong, can also have large nonlinear components.

Nonlinear applied fields must be added back in the idealized model when it is appropriate to analyze their effects
- Common problem to address when carrying out large-scale numerical simulations to design/analyze systems

There are two basic approaches to carry this out:

Approach 1: Explicit 3D Formulation
Approach 2: Perturbations About Linear Applied Field Model

Discuss each of these in turn

Approach 1: Explicit 3D Formulation

This is the simplest. Just employ the full 3D equations of motion expressed in terms of the applied field components $\mathbf{E}^a,\ \mathbf{B}^a$ and avoid using the focusing functions $\kappa_x,\ \kappa_y$

Comments:
- Most easy to apply in computer simulations where many effects are simultaneously included
  - Simplifies comparison to experiments when many details matter for high level agreement
- Simplifies simultaneous inclusion of transverse and longitudinal effects
  - Accelerating field E_z^a can be included to calculate changes in $\beta_b,\ \gamma_b$
  - Transverse and longitudinal dynamics cannot be fully decoupled in high level modeling – especially try when acceleration is strong in systems like injectors
- Can be applied with time based equations of motion
  - Helps avoid unit confusion and continuously adjusting complicated equations of motion to identify the axial coordinate *s* appropriately

Approach 2: Perturbations About Linear Applied Field Model

Exploit the linearity of the Maxwell equations to take:

$$\boxed{\begin{aligned}\mathbf{E}^a_\perp &= \mathbf{E}^a_\perp|_L \;+\; \delta\mathbf{E}^a_\perp \\ \mathbf{B}^a &= \mathbf{B}^a|_L \;+\; \delta\mathbf{B}^a\end{aligned}}$$

where

$\mathbf{E}^a_\perp|_L,\ \mathbf{B}^a|_L$ are the linear field components incorporated in $\kappa_x,\ \kappa_y$

to express the equations of motion as:

$$\boxed{\begin{aligned} x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \kappa_x x &= \frac{q}{m\gamma_b\beta_b^2c^2}\delta E^a_x - \frac{q}{m\gamma_b\beta_b c}\delta B^a_y + \frac{q}{m\gamma_b\beta_b c}\delta B^a_z y' \\ &\quad - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x} \\ y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \kappa_y y &= \frac{q}{m\gamma_b\beta_b^2c^2}\delta E^a_y + \frac{q}{m\gamma_b\beta_b c}\delta B^a_x - \frac{q}{m\gamma_b\beta_b c}\delta B^a_z x' \\ &\quad - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}\end{aligned}}$$

This formulation can be most useful to understand the effect of deviations from the usual linear model where intuition is developed

Comments:

- Best suited to non-solenoidal focusing
  - Simplified Larmor frame analysis for solenoidal focusing is only valid for axisymmetric potentials $\phi = \phi(r)$ which may not hold in the presence of non-ideal perturbations.
  - Applied field perturbations $\delta\mathbf{E}^a_\perp,\ \delta\mathbf{B}^a$ would also need to be projected into the Larmor frame
- Applied field perturbations $\delta\mathbf{E}^a_\perp,\ \delta\mathbf{B}^a$ will not necessarily satisfy the 3D Maxwell Equations by themselves
  - Follows because the linear field components $\mathbf{E}^a_\perp|_L,\ \mathbf{B}^a|_L$ will not, in general, satisfy the 3D Maxwell equations by themselves

C.12 Macroscopic Fluid Models

Fluid Models

- Obtained by taking statistical averages of kinetic model over velocity/momentum degrees of freedom
- Described in terms of "macroscopic" variables (density, flow velocity, pressure...) that vary in ***x*** and *t*
- Models must be closed (truncated) at some order via physically motivated assumptions (cold, negligible heat flow, ...)

Moments:

Density $\quad n: \quad n(\mathbf{x},t) = \int d^3p\ f(\mathbf{x},\mathbf{p},t)$

Flow velocity $\quad \mathbf{V}: \quad n\mathbf{V}(\mathbf{x},t) = \int d^3p\ \mathbf{v}f(\mathbf{x},\mathbf{p},t)$

Flow momentum $\quad \mathbf{P}: \quad n\mathbf{P}(\mathbf{x},t) = \int d^3p\ \mathbf{p}f(\mathbf{x},\mathbf{p},t)$

Pressure tensor $\quad \mathcal{P}_{ij}: \quad n\mathcal{P}_{ij}(\mathbf{x},t) = \int d^3p\ [p_i - P_i(\mathbf{x},t)] \times [v_j - V_j(\mathbf{x},t)]f(\mathbf{x},\mathbf{p},t)$

Higher rank objects $\quad \vdots \quad \vdots \quad \vdots$

C.13 Fluid Models: Equations of Motion

Equations of Motion (Eulerian perspective)

Continuity:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial\mathbf{x}}\cdot[n\mathbf{V}] = 0$$

Force: ith component

$$n\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\frac{\partial}{\partial\mathbf{x}}\right)P_i + \sum_j \frac{\partial}{\partial x_j}\mathcal{P}_{ij} = qn[\mathbf{E} + \mathbf{V}\times\mathbf{B}]_i$$

Pressure: tensor component

$$\frac{\partial}{\partial t}\mathcal{P}_{ij} \cdots$$

$\vdots$

Infinite chain of equations to reproduce kinetic Vlasov model

Will clarify this more in a homework problem

Comments:

- Takes infinite chain of complicated nonlinear macroscopic equations for "equivalence" to kinetic theory (even Vlasov model). So why do it?
- Simpler and easy to interpret at low order: so if low order works it can provide insight on physics

Fields:

Maxwell Equations are the same as for the particle and kinetic cases with charge and current density coupling to fluid variables given by:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Charge Density

$$\rho(\mathbf{x}, t) = \rho_{\text{ext}}(\mathbf{x}, t) + qn(\mathbf{x}, t)$$

external (applied)

particle beam

Current Density

$$\mathbf{J}(\mathbf{x}, t) = \mathbf{J}_{\text{ext}}(\mathbf{x}, t) + qn(\mathbf{x}, t)\mathbf{V}(\mathbf{x}, t)$$

+ boundary conditions on $\mathbf{E}, \ \mathbf{B}$

C.14 Fluid Models: Multispecies Generalization

Subscript species with j. Then in the continuity, force, pressure, ... equations replace

Particle Properties	Moments
$m \longrightarrow m_j$	$n \longrightarrow n_j$
$q \longrightarrow q_j$	$\mathbf{V} \longrightarrow \mathbf{V}_j$
	$\vdots$

Replace the charge and current density couplings in the Maxwell Equations with

$$\rho(\mathbf{x}, t) = \rho_{\text{ext}}(\mathbf{x}, t) + \sum_j q_j n_j(\mathbf{x}, t)$$

$$\mathbf{J}(\mathbf{x}, t) = \mathbf{J}_{\text{ext}}(\mathbf{x}, t) + \sum_j q_j n_j(\mathbf{x}, t)\mathbf{V}_j(\mathbf{x}, t)$$

C.15 Lagrangian Formulation of Distribution Methods

In kinetic and especially fluid models it can be convenient to adopt *Lagrangian* methods. For fluid models these can be distinguished as follows:

Eulerian Fluid Model:

Flow quantities are functions of space (**x**) and and evolve in time (t)

- Example: density $n(\mathbf{x}, t)$ and flow velocity $\mathbf{V}(\mathbf{x}, t)$

Lagrangian Fluid Model:

Identify parts of evolution (flow) with objects (material elements) and follow the flow in time (t)

- Shape and position of elements must generally evolve to represent flow
- Example: envelope model edge radii $r_x(s), \ r_y(s)$

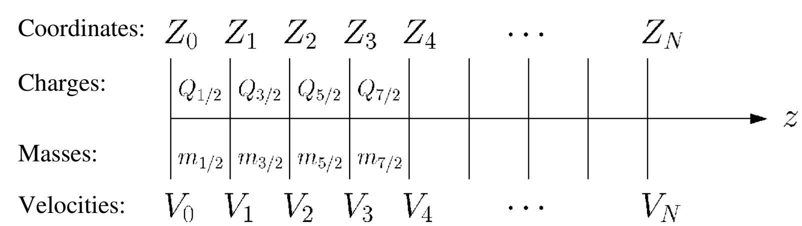
Many distribution methods for Vlasov's Equation are hybrid Lagrangian methods

- Macro particle "shapes" in PIC (Particle in Cell) method to be covered can be thought of as Lagrangian elements representing a *Vlasov flow*

C.16 Example Lagrangian Fluid Model

1D Lagrangian model of the longitudinal evolution of a cold beam

- Discretize fluid into longitudinal elements with boundaries
  - Continuity equation automatically satisfied by the structure of elements
- Derive equations of motion for elements (defined by boundaries)



$z = Z_i$ slice boundaries

$\dfrac{dZ_i}{dt} = V_i$ velocities of slice boundaries

$Q_{i+1/2}$ fixed

$m_{i+1/2}$ fixed

$\dfrac{Q_{i+1/2}}{m_{i+1/2}} = \dfrac{q}{m} = \text{const}$ for single species (set initial coordinates)

Structure of the macro-elements automatically solves the continuity equation (i.e., local charge conservation consistent with flow)

- Charge fixed in elements but density will vary due to relative motion of the slice boundaries.
  - Varying local density will be reflected in value of electric field from solution of Poisson's equation.
- Effective flow (current) results from motion of element

Kinematics of elements given by the motion of the slice boundaries which can be expressed as ODEs.

- Assume nonrelativisitic

$$\boxed{\begin{aligned} \frac{dZ_i(t)}{dt} &= V_i(t) \\ \frac{dV_i(t)}{dt} &= \frac{q}{m} E_z(Z_i, t) \end{aligned}}$$

- Several methods might be used to calculate E_z as summarized on the next slide

Methods to calculate the electric field

- Appropriate versions used for specific class of problem

1) Take "slices" to have some radial extent modeled by a perpendicular envelope etc. and deposit the $Q_{i+1/2}$ onto a grid and solve:

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0} \qquad E_z = -\frac{\partial \phi}{\partial z}$$

subject to $E_z \to 0$ as $|z| \to \infty$

2) Employ a "g-factor" model

$$E_z = -\frac{g}{4\pi\epsilon_0}\frac{\partial \lambda}{\partial z}$$

λ calculated from $Q_{i+1/2}$ and extent of the elements etc.

3) Pure 1D model using Gauss' Law (1D: charge to left – charge to right gives E)

- Derivation in Appendix A

$$E_z = \frac{1}{2\epsilon_0}\left\{ \int_{-\infty}^{z} d\tilde{z}\; \rho(\tilde{z}, t) \; - \int_{z}^{\infty} d\tilde{z}\; \rho(\tilde{z}, t) \right\}$$

$$\sim \text{charge downstream} - \text{charge upstream}$$

Appendix A: Solution of 1D Electric Field in Free-Space

The 1D electric field $E_z(z,t)$ satisfies

$$\frac{\partial}{\partial z} E_z(z,t) = \frac{1}{\epsilon_0}\rho(z,t)$$

Integrate "downstream":

$$\int_{-\infty}^{z} d\tilde{z}\; \frac{\partial}{\partial \tilde{z}} E_z(\tilde{z},t) = \frac{1}{\epsilon_0}\int_{-\infty}^{z} d\tilde{z}\; \rho(\tilde{z},t)$$

$$E_z(z,t) - E_z(\infty) = \frac{1}{\epsilon_0}\int_{-\infty}^{z} d\tilde{z}\; \rho(\tilde{z},t) \qquad (1)$$

Similarly, integrate "upstream":

$$E_z(\infty) - E_z(z,t) = \frac{1}{\epsilon_0}\int_{z}^{\infty} d\tilde{z}\; \rho(\tilde{z},t) \qquad (2)$$

Subtract (2) from (1) and use symmetry $E_z(-\infty) = -E_z(\infty)$ to obtain:

$$\boxed{\begin{gathered} E_z = \frac{1}{2\epsilon_0}\left\{ \int_{-\infty}^{z} d\tilde{z}\; \rho(\tilde{z},t) \; - \int_{z}^{\infty} d\tilde{z}\; \rho(\tilde{z},t) \right\} \\ \sim \text{charge downstream} - \text{charge upstream} \end{gathered}}$$

- Coulomb field long range in 1D
- Gridded solution easy to implement numerically: just sums on grid

A1

D. Moment Methods
D.1 Overview

Moment Methods

- Most reduced description of an intense beam
  - Often employed in lattice designs
- Beam represented by a finite (closed and truncated) set of moments that are advanced from initial values
  - Here by moments, we mean functions of a single variable *s* or *t*
- Such models are *not* generally self-consistent
  - Some special cases such as a stable transverse KV equilibrium distribution (see: USPAS lectures in *Beam Physics with Intense Space Charge* on Transverse Equilibrium Distributions) are consistent with truncated moment description (rms envelope equation)
  - Typically derived from assumed distributions with self-similar evolution
- See: USPAS lectures in *Beam Physics with Intense Space Charge* on Transverse Centroid and Envelope Descriptions of Beam Evolution for more details on moment methods applied to transverse beam physics

D.2 Moment Methods: 1st Order Moments

Many moment models exist. Illustrate with examples for transverse beam evolution

Moment definition:

$$\langle \ldots \rangle_\perp \equiv \frac{\int d^2x_\perp \int d^2x'_\perp \ldots f}{\int d^2x_\perp \int d^2x'_\perp f}$$

Averages over the transverse degrees of freedom in the distribution

1st order moments:

$$\mathbf{X} = \langle \mathbf{x} \rangle_\perp \qquad \text{Centroid coordinate}$$

$$\mathbf{X}' = \langle \mathbf{x}' \rangle_\perp \qquad \text{Centroid angle}$$

$$\Delta = \left\langle \frac{\delta p_s}{p_s} \right\rangle_\perp \equiv \langle \delta \rangle_\perp \qquad \text{Off momentum}$$

$$\vdots \qquad\qquad \vdots$$

D.2 Moment Methods: 2nd and Higher Order Moments

2nd order moments:

x moments	y moments	x-y cross moments	dispersive moments
$\langle x^2 \rangle_\perp$	$\langle y^2 \rangle_\perp$	$\langle xy \rangle_\perp$	$\langle x\delta \rangle_\perp$, $\langle y\delta \rangle_\perp$
$\langle xx' \rangle_\perp$	$\langle yy' \rangle_\perp$	$\langle x'y \rangle_\perp$, $\langle xy' \rangle_\perp$	$\langle x'\delta \rangle_\perp$, $\langle y'\delta \rangle_\perp$
$\langle x'^2 \rangle_\perp$	$\langle y'^2 \rangle_\perp$	$\langle x'y' \rangle_\perp$	$\langle \delta^2 \rangle_\perp$

It is typically convenient to subtract centroid from higher-order moments

$$\tilde{x} \equiv x - X \qquad \tilde{x}' \equiv x' - X'$$

$$\tilde{y} \equiv y - Y \qquad \tilde{y}' \equiv y' - Y'$$

$$\langle \tilde{x}^2 \rangle_\perp = \langle (x - X)^2 \rangle_\perp = \langle x^2 \rangle_\perp - X^2, \text{ etc.}$$

3rd order moments: Analogous to 2nd order case, but more for each order

$$\langle x^3 \rangle_\perp , \ \langle x^2 y \rangle_\perp , \ \ldots$$

D.3 Moment Methods: Common 2nd Order Moments

Many quantities of physical interest are expressed in terms of moments

Statistical beam size: (rms edge measure)

$$r_x = 2\langle \tilde{x}^2 \rangle_\perp^{1/2}$$

$$r_y = 2\langle \tilde{y}^2 \rangle_\perp^{1/2}$$

Measures effective transverse beam size

Statistical emittances: (rms edge measure)

$$\varepsilon_x = 4\left[\langle \tilde{x}^2 \rangle_\perp \langle \tilde{x}'^2 \rangle_\perp - \langle \tilde{x}\tilde{x}' \rangle_\perp^2\right]^{1/2}$$

$$\varepsilon_y = 4\left[\langle \tilde{y}^2 \rangle_\perp \langle \tilde{y}'^2 \rangle_\perp - \langle \tilde{y}\tilde{y}' \rangle_\perp^2\right]^{1/2}$$

Measures effective transverse phase-space volume of beam

D.4 Moment Methods: Equations of Motion

Equations of Motion

- Express in terms of a function of moments
- Moments are advanced from specified initial conditions

Form equations:

$$\frac{d}{ds}\mathbf{M} = \mathbf{F}(\mathbf{M})$$

M = vector of moments, generally infinite
F = vector function of **M**, generally nonlinear

Moment methods generally form an infinite chain of equations that do *not* truncate. To be useful the system must be truncated. Truncations are usually carried out by assuming a specific form of the distribution that can be described by a finite set of moments

- Self-similar evolution: form of distribution assumed not to change
  - Analytical solutions often employed
- Neglect of terms

A simple example will be employed to illustrate these points

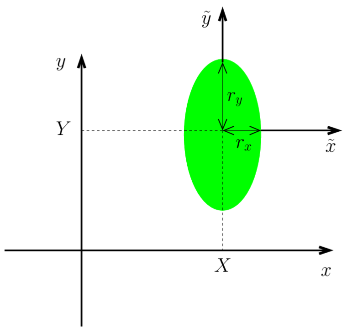
D.5 Moment Methods: Example: Transverse Envelope Eqns.

Truncation assumption: unbunched uniform density elliptical beam in free space

- $\delta = 0$, no axial velocity spread
- All cross moments zero, i.e. $\langle \tilde{x}\tilde{y} \rangle_\perp = 0$

Centroid:
$$X = \langle x \rangle_\perp \qquad Y = \langle y \rangle_\perp$$

Envelope:
$$r_x = 2\langle \tilde{x}^2 \rangle_\perp^{1/2} \qquad r_y = 2\langle \tilde{y}^2 \rangle_\perp^{1/2}$$



For: $\dfrac{\tilde{x}^2}{r_x^2} + \dfrac{\tilde{y}^2}{r_y^2} < 1$

$$E_{\tilde{x}} = \frac{\lambda}{\pi\epsilon_0} \frac{\tilde{x}}{(r_x + r_y) r_x}$$

$$E_{\tilde{y}} = \frac{\lambda}{\pi\epsilon_0} \frac{\tilde{y}}{(r_x + r_y) r_y}$$

$\lambda =$ line charge density

These results are employed to derive the moment equations of motion (See S.M. Lund lectures on Transverse Centroid and Envelope Models)

Example Continued (2) - Equations of Motion in Matrix Form

$$\frac{d}{ds}\begin{bmatrix} \langle x \rangle_\perp \\ \langle x' \rangle_\perp \\ \langle y \rangle_\perp \\ \langle y' \rangle_\perp \end{bmatrix} = \begin{bmatrix} \langle x' \rangle_\perp \\ -\kappa_x(s)\langle x \rangle_\perp \\ \langle y' \rangle_\perp \\ -\kappa_y(s)\langle y \rangle_\perp \end{bmatrix}$$

$$\frac{d}{ds}\begin{bmatrix} \langle \tilde{x}^2 \rangle_\perp \\ \langle \tilde{x}\tilde{x}' \rangle_\perp \\ \langle \tilde{x}'^2 \rangle_\perp \\ \langle \tilde{y}^2 \rangle_\perp \\ \langle \tilde{y}\tilde{y}' \rangle_\perp \\ \langle \tilde{y}'^2 \rangle_\perp \end{bmatrix} = \begin{bmatrix} 2\langle \tilde{x}\tilde{x}' \rangle_\perp \\ \langle \tilde{x}'^2 \rangle_\perp - \kappa_x(s)\langle \tilde{x}^2 \rangle_\perp + \frac{Q\langle \tilde{x}^2 \rangle_\perp^{1/2}}{2[\langle \tilde{x}^2 \rangle_\perp^{1/2} + \langle \tilde{y}^2 \rangle_\perp^{1/2}]} \\ -2\kappa_x(s)\langle \tilde{x}\tilde{x}' \rangle_\perp + \frac{Q\langle \tilde{x}\tilde{x}' \rangle_\perp}{\langle \tilde{x}^2 \rangle_\perp^{1/2}[\langle \tilde{x}^2 \rangle_\perp^{1/2} + \langle \tilde{y}^2 \rangle_\perp^{1/2}]} \\ 2\langle \tilde{y}\tilde{y}' \rangle_\perp \\ \langle \tilde{y}'^2 \rangle_\perp - \kappa_y(s)\langle \tilde{y}^2 \rangle_\perp + \frac{Q\langle \tilde{y}^2 \rangle_\perp^{1/2}}{2[\langle \tilde{x}^2 \rangle_\perp^{1/2} + \langle \tilde{y}^2 \rangle_\perp^{1/2}]} \\ -2\kappa_y(s)\langle \tilde{y}\tilde{y}' \rangle_\perp + \frac{Q\langle \tilde{y}\tilde{y}' \rangle_\perp}{\langle \tilde{y}^2 \rangle_\perp^{1/2}[\langle \tilde{x}^2 \rangle_\perp^{1/2} + \langle \tilde{y}^2 \rangle_\perp^{1/2}]} \end{bmatrix}$$

- Form truncates due to assumed distribution form
- Self-consistent with the KV distribution. See: S.M. Lund lectures on Transverse Equilibrium Distributions

Example Continued (3) - Reduced Form Equations of Motion

Using 2nd order moment equations we can show that

$$\frac{d}{ds}\varepsilon_x^2 = 0 = \frac{d}{ds}\varepsilon_y^2$$

$$\Longrightarrow \quad \boxed{\begin{aligned} \varepsilon_x^2 &= 16\left[\langle x^2 \rangle_\perp \langle x'^2 \rangle_\perp - \langle xx' \rangle_\perp^2\right] = \text{const} \\ \varepsilon_y^2 &= 16\left[\langle y^2 \rangle_\perp \langle y'^2 \rangle_\perp - \langle yy' \rangle_\perp^2\right] = \text{const} \end{aligned}}$$

The 2nd order moment equations can be equivalently expressed as

$$\boxed{\begin{aligned} \frac{dr_x}{ds} &= r_x' \; ; \quad \frac{d}{ds}r_x' + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0 \\ \frac{dr_y}{ds} &= r_y' \; ; \quad \frac{d}{ds}r_y' + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0 \end{aligned}}$$

Example Continued (4) : Contrast Form of Matrix and Reduced Form Moment Equations

Relative advantages of the use of coupled matrix form versus reduced equations can depend on the problem/situation

Coupled Matrix Equations

$$\frac{d}{ds}\mathbf{M} = \mathbf{F}(\mathbf{M})$$

M = Moment Vector
F = Force Vector

- Easy to formulate
  - Straightforward to incorporate additional effects
- Natural fit to numerical routine
  - Easy to code

Reduced Equations

$$X'' + \kappa_x X = 0$$

$$r_x'' + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

etc.

Reduction based on identifying invariants such as
$\varepsilon_x^2 = 16\left[\langle \tilde{x}^2 \rangle_\perp \langle \tilde{x}'^2 \rangle_\perp - \langle \tilde{x}\tilde{x}' \rangle_\perp^2\right]$
helps understand solutions

- Compact expressions

E. Hybrid Methods

Beyond the three levels of modeling outlined earlier:
0) Particle methods
1) Distribution methods
2) Moment methods

there exist numerous "hybrid" methods that combine features of several methods. Hybrid methods may be the most common in detailed simulations.

Examples of Common Hybrid Methods:

- Particle-in-Cell (PIC) models
  - Shaped (Lagrangian) macro-particles represent the distribution
  - Macro-particles evolved using particle equations of motion
  - Interactions via self-field are smoothed to represent continuum mechanics
- Gyro-kinetic models
  - Average over fast gyro motion in strong magnetic fields: common in plasma physics
- Delta-f models
  - Evolve perturbed distribution with marker particles evolving about a core "equilibrium" distribution

⋮

Hybrid Methods Continued (2)

Comments on selecting methods:

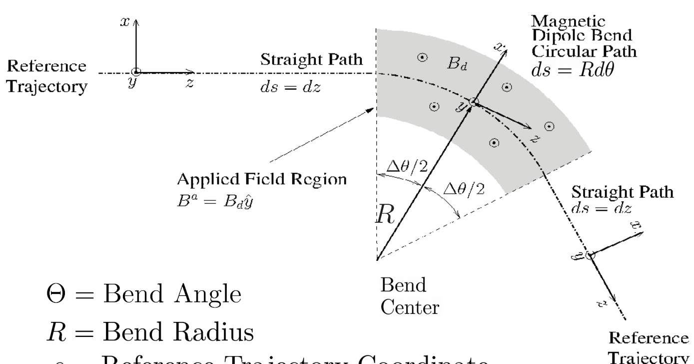
- Particle and distribution methods are appropriate for higher levels of detail
- Moment methods are used for rapid iteration of machine design
  - Moments also typically calculated as diagnostics in particle and distribution methods
- Even within one (e.g. particle method) there are many levels of description:
  - Electromagnetic and electrostatic, with many field solution methods
  - 1D, 2D, 3D

  ⋮
- Employing a hierarchy of models with full diagnostics allows cross-checking (both in numerics and physics) and aids understanding
  - No single method is best in all cases

F Bent Coordinate System and Particle Equations of Motion with Dipole Bends and Axial Momentum Spread

Accelerators lattices often employ dipole bends in rings and transfer lines. In simulations, it can prove more convenient to employ coordinates that follow the beam in a bend. Here we outline modifications to formulations for bends.

- Orthogonal system employed called Frenet-Serret coordinates



$\Theta = $ Bend Angle

$R = $ Bend Radius

$s = $ Reference Trajectory Coordinate

In this perspective, dipoles are adjusted given the design momentum of the reference particle to bend the orbit through a radius *R*.

- Bends usually only in one plane (say *x*)
  - Implemented by a dipole applied field: E_x^a or B_y^a
- Easy to apply material analogously for *y*-plane bends, if necessary

Denote:

$$\boxed{p_0 = m\gamma_b\beta_b c = \text{design momentum}}$$

Then a magnetic *x*-bend through a radius *R* is specified by:

$$\boxed{\begin{aligned} \mathbf{B}^a &= B_y^a \hat{\mathbf{y}} = \text{const in bend} \\ \frac{1}{R} &= \frac{qB_y^a}{p_0} \end{aligned}}$$

Analogous formula for Electric Bend will be derived in problem set

The particle rigidity is defined as ($[B\rho]$ read as one symbol called "B-Rho"):

$$\boxed{[B\rho] \equiv \frac{p_0}{q} = \frac{m\gamma_b\beta_b c}{q}}$$

is often applied to express the bend result as:

$$\boxed{\frac{1}{R} = \frac{B_y^a}{[B\rho]}}$$

Comments on bends:

- R can be positive or negative depending on sign of $B_y^a/[B\rho]$
- For straight sections, $R \rightarrow \infty \quad (\text{ or equivalently, } B_y^a = 0)$
- Lattices often made from discrete element dipoles and straight sections with separated function optics
  - Bends can provide "edge focusing"
  - Sometimes elements for bending/focusing are combined
- For a ring, dipoles strengths are tuned with particle rigidity/momentum so the reference orbit makes a closed path lap through the circular machine
  - Dipoles adjusted as particles gain energy to maintain closed path
  - In a Synchrotron dipoles and focusing elements are adjusted together to maintain focusing and bending properties as the particles gain energy. This is the origin of the name "Synchrotron."
- Total bending strength of a ring in Tesla-meters limits the ultimately achievable particle energy/momentum in the ring

For a magnetic field over a path length S , the beam will be bent through an angle:

$$\Theta = \frac{S}{R} = \frac{SB_y^a}{[B\rho]}$$

To make a ring, the bends must deflect the beam through a total angle of 2π :

- Neglect any energy gain changing the rigidity over one lap

$$\boxed{2\pi = \sum_{i,\text{Dipoles}} \Theta_i = \sum_i \frac{S_i}{R_i} = \sum_i \frac{S_i B_{y,i}^a}{[B\rho]}}$$

For a symmetric ring, N dipoles are all the same, giving for the bend field:

- Typically choose parameters for dipole field as high as technology allows for a compact ring

$$\boxed{B_y^a = 2\pi \frac{[B\rho]}{NS}}$$

For a symmetric ring of total circumference C with straight sections of length L between the bends:

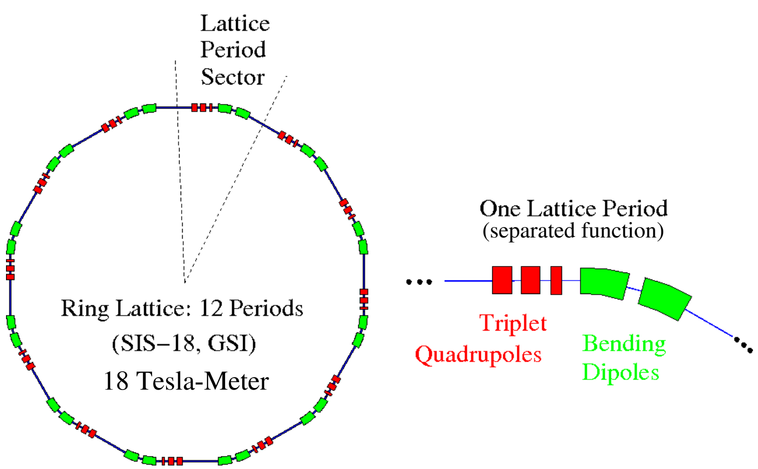
- Features of straight sections typically dictated by needs of focusing, acceleration, and dispersion control

$$\boxed{C = NS + NL}$$

Example: Typical separated function lattice in a Synchrotron

Focus Elements in Red

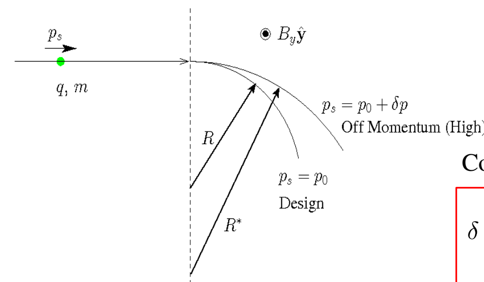
Bending Elements in Green



For "off-momentum" errors:

$$\boxed{\begin{aligned} p_s &= p_0 + \delta p \\ p_0 &= m\gamma_b\beta_b c = \text{design momentum} \\ \delta p &= \text{off- momentum} \end{aligned}}$$

This will modify the particle equations of motion, particularly in cases where there are bends since particles with different momenta will be bent at different radii



Common notation:

$$\boxed{\delta \equiv \frac{\delta p}{p_0} = \text{Fractional Momentum Error}}$$

- Not usual to have acceleration in bends
  - Dipole bends and quadrupole focusing are sometimes combined

Derivatives in accelerator Frenet-Serret Coordinates

Summarize results only needed to transform the Maxwell equations, write field derivatives, etc.

- Reference: Chao and Tigner, *Handbook of Accelerator Physics and Engineering*

$$\Psi(x,y,s) \qquad = \text{Scalar}$$

$$\mathbf{V}(x,y,s) = V_x(x,y,s)\hat{\mathbf{x}} + V_y(x,y,s)\hat{\mathbf{y}} + V_s(x,y,s)\hat{\mathbf{s}} \qquad = \text{Vector}$$

Vector Dot and Cross-Products: $(\mathbf{V}_1,\ \mathbf{V}_2\ \ \text{Two Vectors})$

$$\mathbf{V}_1 \cdot \mathbf{V}_2 = V_{1x}V_{2x} + V_{1y}V_{2y} + V_{1s}V_{2s}$$

$$\mathbf{V}_1 \times \mathbf{V}_2 = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{s}} \\ V_{1x} & V_{1y} & V_{1s} \\ V_{2x} & V_{2y} & V_{2s} \end{vmatrix}$$

$$= (V_{1x}V_{2s} - V_{1s}V_{2x})\hat{\mathbf{x}} + (V_{1s}V_{2x} - V_{1x}V_{2s})\hat{\mathbf{y}} + (V_{1x}V_{2y} - V_{1y}V_{2x})\hat{\mathbf{s}}$$

Elements:

$$d^2x_\perp = dxdy$$

$$d^3x_\perp = \left(1 + \frac{x}{R}\right) dxdyds$$

$$d\vec{\ell} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{s}}\left(1 + \frac{x}{R}\right) ds$$

Gradient:

$$\nabla\Psi = \hat{\mathbf{x}}\frac{\partial \Psi}{\partial x} + \hat{\mathbf{y}}\frac{\partial \Psi'}{\partial y} + \hat{\mathbf{s}}\frac{1}{1+x/R}\frac{\partial \Psi}{\partial s}$$

Divergence:

$$\nabla \cdot \mathbf{V} = \frac{1}{1+x/R}\frac{\partial}{\partial x}\left[(1+x/R)V_x\right] + \frac{\partial V_y}{\partial y} + \frac{1}{1+x/R}\frac{\partial V_s}{\partial s}$$

Curl:

$$\nabla \times \mathbf{V} = \hat{\mathbf{x}}\left(\frac{\partial V_s}{\partial y} - \frac{1}{1+x/R}\frac{\partial V_y}{\partial s}\right) + \hat{\mathbf{y}}\frac{1}{1+x/R}\left(\frac{\partial V_x}{\partial s} - \frac{\partial}{\partial x}\left[(1+x/R)V_s\right]\right)$$

$$+ \hat{\mathbf{s}}(1+x/R)\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

Laplacian:

$$\nabla^2\Psi = \frac{1}{1+x/R}\frac{\partial}{\partial x}\left[\left(1+\frac{x}{R}\right)\frac{\partial \Psi}{\partial x}\right] + \frac{\partial^2 \Psi}{\partial y^2} + \frac{1}{1+x/R}\frac{\partial}{\partial s}\left[\frac{1}{1+x/R}\frac{\partial \Psi}{\partial s}\right]$$

Transverse particle equations of motion including bends and "off-momentum" effects

- See texts such as Edwards and Syphers for guidance on derivation steps
- Full derivation is beyond needs/scope of this class

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' + \left[\frac{1}{R^2(s)}\frac{1-\delta}{1+\delta}\right]x = \frac{\delta}{1+\delta}\frac{1}{R(s)} + \frac{q}{m\gamma_b\beta_b^2c^2}\frac{E_x^a}{(1+\delta)^2}$$

$$- \frac{q}{m\gamma_b\beta_b c}\frac{B_y^a}{1+\delta} + \frac{q}{m\gamma_b\beta_b c}\frac{B_s^a}{1+\delta}y' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{1}{1+\delta}\frac{\partial\phi}{\partial x}$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' = \frac{q}{m\gamma_b\beta_b^2c^2}\frac{E_y^a}{(1+\delta)^2} + \frac{q}{m\gamma_b\beta_b c}\frac{B_x^a}{1+\delta}$$

$$- \frac{q}{m\gamma_b\beta_b c}\frac{B_s^a}{1+\delta}x' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{1}{1+\delta}\frac{\partial\phi}{\partial y}$$

$$p_0 = m\gamma_b\beta_b c = \text{Design Momentum} \qquad \frac{1}{R(s)} = \frac{B_y^a(s)|_{\text{Dipole}}}{[B\rho]} \qquad [B\rho] = \frac{p_0}{q}$$

$$\delta \equiv \frac{\delta p}{p_0} = \text{Fractional Momentum Error}$$

Comments:

- Design bends only in *x* and $B_y^a,\ E_x^a$ contain <u>no</u> dipole terms (design orbit)
  - Dipole components set via the design bend radius *R*(*s*)
- Equations contain only low-order terms in momentum spread δ

Comments continued:

- Equations are often applied linearized in δ
- Achromatic focusing lattices are often designed using equations with momentum spread to obtain focal points independent of δ to some order
  x and *y* equations differ significantly due to bends modifying the *x*-equation when *R*(*s*) is finite
- It will be shown in the problems that for electric bends:

$$\frac{1}{R(s)} = \frac{E_x^a(s)}{\beta_b c[B\rho]}$$

- Applied fields for focusing: $\mathbf{E}_\perp^a,\ \mathbf{B}_\perp^a,\ B_s^a$ must be expressed in the bent *x,y,s* system of the reference orbit
  - Includes error fields in dipoles
- Self fields may also need to be solved taking into account bend terms
  - Often can be neglected in Poisson's Equation

$$\left\{\frac{1}{1+x/R}\frac{\partial}{\partial x}\left[\left(1+\frac{x}{R}\right)\frac{\partial}{\partial x}\right] + \frac{\partial^2}{\partial y^2} + \frac{1}{1+x/R}\frac{\partial}{\partial s}\left[\frac{1}{1+x/R}\frac{\partial}{\partial s}\right]\right\}\phi = -\frac{\rho}{\epsilon_0}$$

if $R \to \infty$ reduces to familiar:

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial s^2}\right\}\phi = -\frac{\rho}{\epsilon_0}$$

Corrections and suggestions for improvements welcome!

These notes will be corrected and expanded for reference and for use in future editions of US Particle Accelerator School (USPAS) and Michigan State University (MSU) courses. Contact:

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Please provide corrections with respect to the present archived version at:

https://people.nscl.msu.edu/~lund/uspas/scs_2016

Redistributions of class material welcome. Please do not remove author credits.

References: For more information see:

These US Particle Accelerator School (USPAS) course notes are posted with updates, corrections, and supplemental material at:

https://people.nscl.msu.edu/~lund/bpisc_2014

This course evolved from material originally presented in a related USPAS course :

JJ Barnard and SM Lund, *Beam Physics with Intense Space-Charge,* USPAS:
http://people.nscl.msu.edu/~lund/bpisc_2011 2015 Lecture Notes + Info

Also taught at the USPAS in 2011, 2008, 2006, 2004, and 2001
and a similar version at UC Berkeley in 2009

This course serves as a reference for physics discussed in this course from a numerical modeling perspective.

References: Continued (2):

Magnet Effective Length

S.M. Lund and B. Bukh , "Stability properties of the transverse envelope equations describing intense ion beam transport," PRSTAB **7**, 204801 (2004)

Plasma/Fluid Models with much material relevant to beams

Nicholson, *Introduction to Plasma Physics,* Wiley, 1982

Davidson, *Physics of Nonneutral Plasmas*,
World Scientific, 1990 (2001 republished)

Davidson, *Physics of Intense Charged Particle Beams*,
World Scientific 2001

Miller, *Physics of Intense Charged Particle Beams*, Plenum Press, 1985

Krall, *Principles of Plasma Physics*, San Francisco Press, 1986

Chen, *Introduction to Plasma Physics and Controlled Fusion*, Springer, 1984