Intro. Lecture 06: Initial Distributions*

Prof. Steven M. Lund
Physics and Astronomy Department
Facility for Rare Isotope Beams (FRIB)
Michigan State University (MSU)

US Particle Accelerator School (USPAS) Lectures On
"Self-Consistent Simulations of Beam and Plasma Systems"
Steven M. Lund, Jean-Luc Vay, Remi Lehe, and Daniel Winklehner

US Particle Accelerator School Summer Session
Colorado State U, Ft. Collins, CO, 13-17 June, 2016
(Version 20160615)

* Research supported by:

FRIB/MSU, 2014 On via: U.S. Department of Energy Office of Science Cooperative Agreement DE-SC0000661and National Science Foundation Grant No. PHY-1102511

and

LLNL/LBNL, Pre 2014 via: US Dept. of Energy Contract Nos. DE-AC52-07NA27344 and DE-AC02-05CH11231

Outline
Introductory Lectures on Self-Consistent Simulations

Initial Distributions and Particle Loading

Initial Distributions and Particle Loading

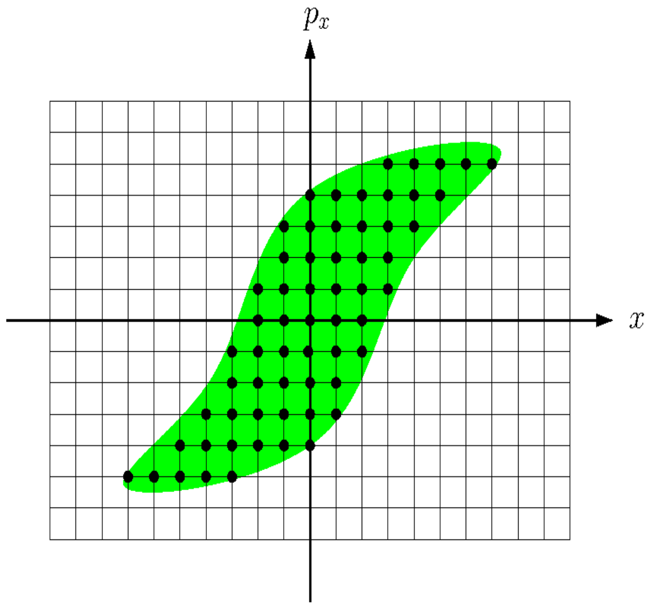
A: Overview

To start the large particle or distribution simulations, the initial distribution function of the beam must be specified.

- For direct Vlasov simulations, the distribution need simply be deposited on the phase-space grid

1D schematic

$$f(x_i, p_{xj}, t = t_{\mathrm{inital}})$$



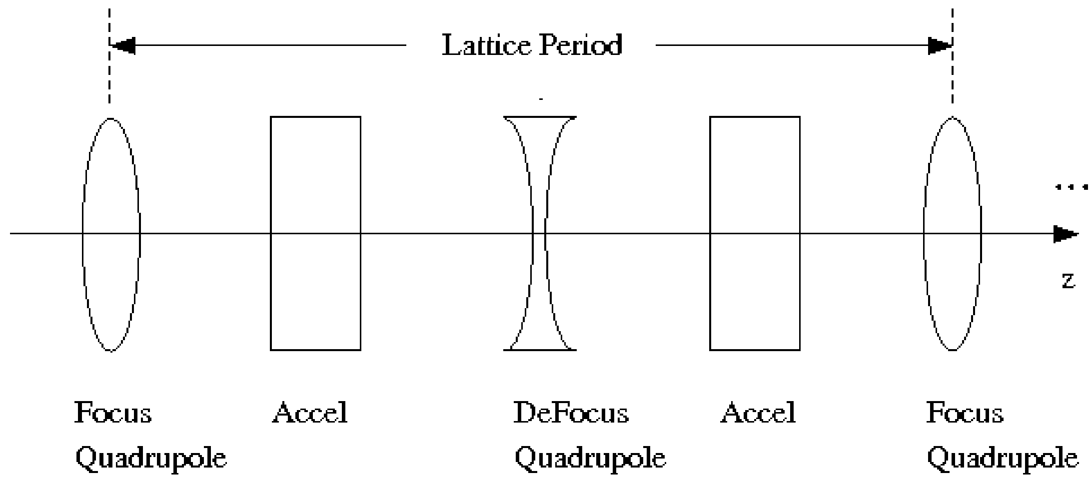
- For PIC simulations, an appropriate distribution of macro-particle phase-space coordinates must be generated or "loaded" to represent the Vlasov distribution f

$$\{\mathbf{x}_M, \ \mathbf{p}_M\} \qquad \text{Given at time } t = t_{\mathrm{inital}}$$

Discussion:

In realistic accelerators, focusing elements in the lattice are s-varying. In such situations there are no known smooth equilibrium distributions.

- The KV distribution is an exact equilibrium for linear focusing fields, but has unphysical (singular) structure in 4-dimensional transverse phase-space



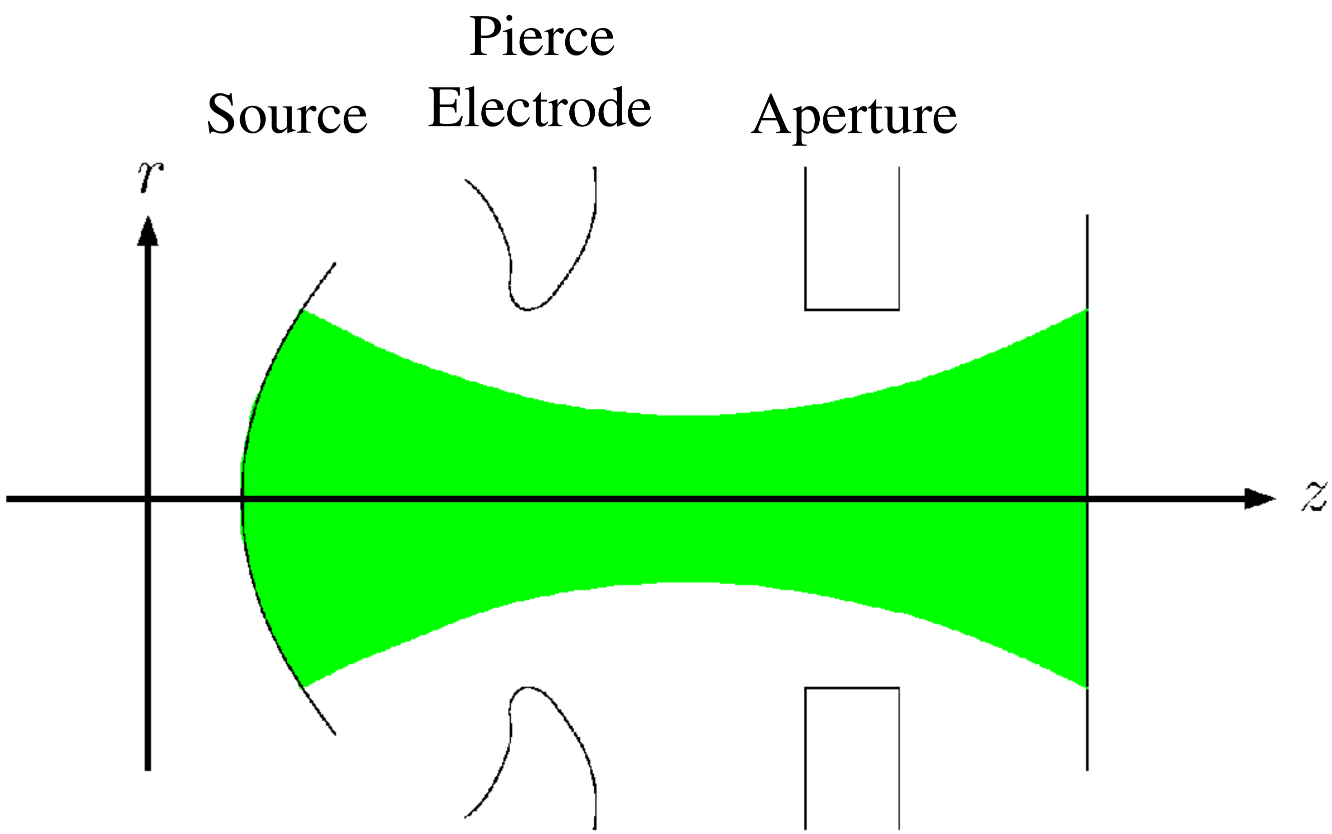
It is unclear in most cases if the beam is even best thought of as an equilibrium distribution as is typical in plasma physics.

- Neutral plasma: approx local thermal equilibrium
- Beam: Intense self fields and finite geometry complicate

In accelerators, the beam is injected from a source and may only reside in the machine (especially for a linac) for a small number of characteristic oscillation periods and may not fully relax to an equilibrium like state within the machine.

Initial Distributions: Source-to-Target Simulations

The lack of known, physically reasonable equilibria and the fact that the beams are injected from a source motivates so-called "source-to-target" simulations where particles are simulated off the source and tracked to the target. Such first principles simulations are most realistic if carried out with the actual focusing fields, accelerating waveforms, alignment errors, etc. Ideally, source-to-target simulations promise to predict expected machine performance.



However, ideal of source-to-target simulations can rarely be carried out due to:

- Source often contains much physics imperfectly modeled
  - Example: plasma injectors with complicated material physics, etc.
- Source is often incompletely described
  - Example: important alignment and material errors may not be known
- Computer limitations:
  - Memory required and simulation time
  - Convergence and accuracies
  - Limits of numerical methods applied

    Eample: singular description needed for Child-Langmuir model of space-charge limited injection

Initial Distributions: Types of Specified Loads

Due to the practical difficulty of carrying out simulations off the source, two alternative methods are commonly applied:

1) Load an idealized initial distribution

- Specify 6D phase space (or less for reduced model) at specific time
- Based on physically reasonable theory assumptions

2) Load experimentally measured distribution

- Construct/synthesize a distribution based on experimental measurements

Discussion:

The 2nd option of generating a distribution from experimental measurements, unfortunately, often has practical difficulties:

- Real diagnostics often are far from ideal 6D snapshots of beam phase-space
  - Distribution must be reconstructed from partial data on limited projection(s) of phase-space measured
  - Typically many assumptions must be made in the synthesis process
- Process of measuring the beam can itself change the beam

It can be helpful to understand processes and limitations starting from idealized initial beams with "equivalent" parameters to experimental measures

– Example: use measured value of beam energy and charge, rms beam sizes, and rms measure phase-space area

Discussion Continued:

Because of the practical difficulties of loading a distribution based exclusively on experimental measurements, idealized distributions are often loaded:

- Employ distributions based on reasonable, physical approximation
- Use limited experimental measures to initialize:
  - Energy, current, rms equivalent beam sizes and emittances
- Simpler initial state can often aid insight:
  - Fewer simultaneous processes can allow one to more clearly understand how limits arise
  - Seed perturbations of relevance when analyzing resonance effects, instabilities, halo, etc.

A significant complication: There are no known exact smooth equilibrium distribution functions valid for periodic focusing channels:

- Approximate theories valid for low phase advances may exist
  Startsev, Sonnad, Davidson, Struckmeier, and others

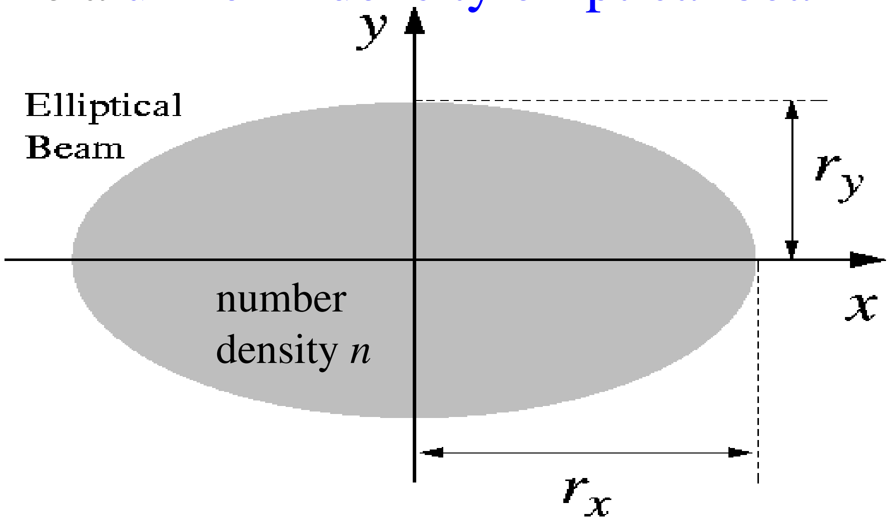
We will formulate a simple approximate procedure to load an initial distribution that reflects features one would expect of a quiescent high-intensity beam

6.B 1^{st} overview ideal KV case
6.D Semi-Gaussian case: less idealized
6.E Pseudo-Equilibrium case
6.F [Future notes: Source model]

B: The KV Equilibrium and the rms Equivalent Beam

[Kapchinskij and Vladimirskij, Proc. Int. Conf. On High Energy Accel., p. 274 (1959); and Review: Lund, Kikuchi, and Davidson, PRSTAB **12**, 114801 (2009)]

Assume a uniform density elliptical beam in a periodic focusing lattice



Line-Charge:

$$\lambda = qn(s)\pi r_x(s) r_y(s)$$
$$= \text{const} \qquad \text{(charge conservation)}$$

Beam Edge:

$$\frac{x^2}{r_x^2(s)} + \frac{y^2}{r_y^2(s)} = 1 \qquad \text{(ellipse)}$$

Free-space self-field solution within the beam (see: USPAS notes BPISC) is:

$$\phi = -\frac{\lambda}{2\pi\epsilon_0}\left[\frac{x^2}{(r_x+r_y)r_x} + \frac{y^2}{(r_x+r_y)r_y}\right] + \text{const}$$

$$\boxed{\begin{aligned} E_x^s &= -\frac{\partial\phi}{\partial x} = \frac{\lambda}{\pi\epsilon_0}\frac{x}{(r_x+r_y)r_x} \\ E_y^s &= -\frac{\partial\phi}{\partial y} = \frac{\lambda}{\pi\epsilon_0}\frac{y}{(r_x+r_y)r_y} \end{aligned}}$$

valid only within the beam!

- Nonlinear outside beam

The particle equations of motion:

$$x'' + \kappa_x x = -\frac{q}{m\gamma_b^3\beta_b^2 c^2}\frac{\partial \phi}{\partial x}$$

$$y'' + \kappa_y y = -\frac{q}{m\gamma_b^3\beta_b^2 c^2}\frac{\partial \phi}{\partial y}$$

become within the beam:

- Linear equations of motion!

$$\boxed{\begin{aligned} x''(s) + \left\{\kappa_x(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_x(s)}\right\} x(s) = 0 \\ y''(s) + \left\{\kappa_y(s) - \frac{2Q}{[r_x(s)+r_y(s)]r_y(s)}\right\} y(s) = 0 \end{aligned}}$$

Here, Q is the dimensionless perveance defined by:

$$\boxed{Q = \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2 c^2} = \text{const}}$$

- Q provides a dimensionless measure of space-charge intensity
  - Appears in same form in many different space-charge problems
  - See USPAS lectures on *Beam Physics with Intense Space-Charge*

If we regard the envelope radii $r_x, \ r_y$ as specified functions of *s*, then these equations of motion are Hill's equations familiar from elementary accelerator physics:

$$\boxed{\begin{aligned} x''(s) + \kappa_x^{\text{eff}}(s)x(s) &= 0 \\ y''(s) + \kappa_y^{\text{eff}}(s)y(s) &= 0 \\ \kappa_x^{\text{eff}}(s) = \kappa_x(s) &- \frac{2Q}{[r_x(s)+r_y(s)]r_x(s)} \\ \kappa_y^{\text{eff}}(s) = \kappa_y(s) &- \frac{2Q}{[r_x(s)+r_y(s)]r_y(s)} \end{aligned}}$$

Suggests Procedure:

- Calculate Courant-Snyder invariants under assumptions made
- Construct a distribution function of Courant-Snyder invariants that generates the uniform density elliptical beam projection assumed
  - Nontrivial step: guess and show that it works: KV construction

Resulting distribution will be an equilibrium that does not evolve in functional form, but phase-space projections will evolve in *s* when focusing functions vary in *s*

Review (1): The Courant-Snyder invariant of Hill's equation

[Courant and Snyder, Annl. Phys. **3**, 1 (1958)]

Hill's equation describes a zero space-charge particle orbit in linear applied focusing fields:

$$\boxed{x''(s) + \kappa(s)x(s) = 0}$$

As a consequence of Floquet's theorem, the solution can be cast in phase-amplitude form:

$$x(s) = A_i w(s) \cos\psi(s) \qquad\qquad \psi'(s) \equiv \frac{1}{w^2(s)}$$

where $w(s)$ is the periodic amplitude function satisfying

$$w''(s) + \kappa(s)w(s) - \frac{1}{w^3(s)} = 0$$

$$w(s + L_p) = w(s) \qquad w(s) > 0$$

$\psi(s)$ is a phase function given by

$$\psi(s) = \psi_i + \int_{s_i}^{s} \frac{d\tilde{s}}{w^2(\tilde{s})}$$

A_i and ψ_i are constants set by initial conditions at $s = s_i$

Review (2): The Courant-Snyder invariant of Hill's equation

From this formulation, it follows that

$$x(s) = A_i w(s) \cos\psi(s)$$

$$x'(s) = A_i w'(s) \cos\psi(s) - \frac{A_i}{w(s)} \sin\psi(s)$$

$$\psi'(s) \equiv \frac{1}{w^2(s)}$$

or

$$\frac{x}{w} = A_i \cos\psi$$

$$wx' - w'x = A_i \sin\psi$$

square and add equations to obtain the Courant-Snyder invariant

$$\boxed{\left(\frac{x}{w}\right)^2 + (wx' - w'x)^2 = A_i^2 = \text{const}}$$

- Simplifies interpretation of dynamics
- Extensively used in accelerator physics

Phase-amplitude description of particles evolving within a uniform density beam:

Phase-amplitude form of x-orbit equations:

$$x(s) = A_{xi} w_x(s) \cos\psi_x(s)$$

$$x'(s) = A_{xi} w'_x(s) \cos\psi_x(s) - \frac{A_{xi}}{w_x(s)} \sin\psi_x(s)$$

initial conditions yield:
$(s = s_i)$

$$A_{xi} = \text{const}$$

$$\psi_{xi} = \psi_x(s = s_i) = \text{const}$$

where

$$\boxed{w''_x(s) + \kappa_x(s) w_x(s) - \frac{2Q}{[r_x(s) + r_y(s)] r_x(s)} w_x(s) - \frac{1}{w_x^3(s)} = 0}$$

$$w_x(s + L_p) = w_x(s) \qquad w_x(s) > 0$$

$$\psi_x(s) = \psi_{xi} + \int_{s_i}^{s} \frac{d\tilde{s}}{w_x^2(\tilde{s})}$$

identifies the Courant-Snyder invariant

$$\boxed{\left(\frac{x}{w_x}\right)^2 + (w_x x' - w'_x x)^2 = A_{xi}^2 = \text{const}}$$

Analogous equations hold for the y-plane

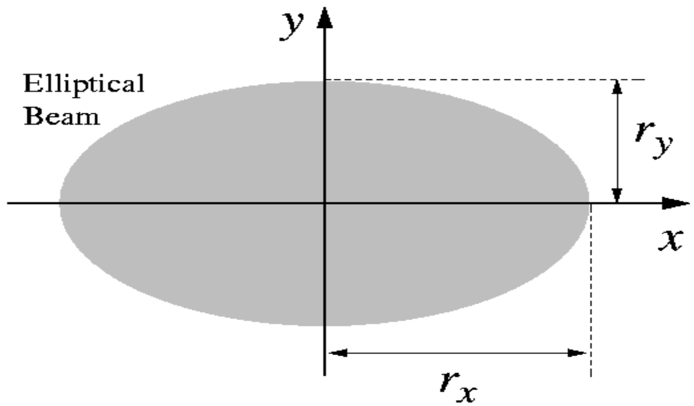
The KV envelope equations:

Define *maximum* Courant-Snyder invariants:

$$\boxed{\begin{aligned}\varepsilon_x &\equiv \text{Max}(A_{xi}^2) \\ \varepsilon_y &\equiv \text{Max}(A_{yi}^2)\end{aligned}}$$

$$x = A_{xi} w_x \cos\psi_x \xrightarrow{\cos\psi_x = 1} r_x = A_{x,\max} w_x$$

Values must correspond to the beam-edge radii:



$$\boxed{\begin{aligned}r_x(s) &= \sqrt{\varepsilon_x}\, w_x(s) \\ r_y(s) &= \sqrt{\varepsilon_y}\, w_y(s)\end{aligned}}$$

The equations for w_x and w_y can then be rescaled to obtain the familiar KV envelope equations for the matched beam envelope

$$\boxed{\begin{aligned}
&r_x''(s) + \kappa_x(s) r_x(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_x^2}{r_x^3(s)} = 0 \\
&r_y''(s) + \kappa_y(s) r_y(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_y^2}{r_y^3(s)} = 0 \\
&r_x(s + L_p) = r_x(s) \qquad r_x(s) > 0 \\
&r_y(s + L_p) = r_y(s) \qquad r_y(s) > 0
\end{aligned}}$$

Use variable rescalings to denote *x*- and *y*-plane Courant-Snyder invariants as:

$$\left(\frac{x}{w_x}\right)^2 + (w_x x' - w_x' x)^2 = A_{xi}^2 = \text{const}$$

$$\boxed{\begin{aligned} \left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r_x' x}{\varepsilon_x}\right)^2 &\equiv C_x = \text{const} \\ \left(\frac{y}{r_y}\right)^2 + \left(\frac{r_y y' - r_y' y}{\varepsilon_y}\right)^2 &\equiv C_y = \text{const} \end{aligned}}$$

Kapchinskij and Vladimirskij constructed a delta-function distribution of a linear combination of these Courant-Snyder invariants that generates the correct uniform density elliptical beam needed for consistency with the assumptions:

$$\boxed{f_\perp = \frac{\lambda}{q\pi^2 \varepsilon_x \varepsilon_y} \delta\left[C_x + C_y - 1\right]}$$

- Delta function means the sum of the *x*- and *y*-invariants is a constant
- Other forms cannot generate the needed uniform density elliptical beam projection (see: S9)
- Density inversion theorem covered later can be used to derive result

The KV equilibrium is constructed from the Courant-Snyder invariants:

KV equilibrium distribution write out full arguments in *x*, *x'* :

$$f_\perp(\mathbf{x}_\perp, \mathbf{x}'_\perp, s) = \frac{\lambda}{q\pi^2\varepsilon_x\varepsilon_y}\delta\left[\left(\frac{x}{r_x}\right)^2 + \left(\frac{r_x x' - r'_x x}{\varepsilon_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{r_y y' - r'_y y}{\varepsilon_y}\right)^2 - 1\right]$$

$$\delta(x) = \text{ Dirac delta function}$$

This distribution generates the correct uniform density elliptical beam:

- Show by direct calc: see USPAS notes, *Beam Physics with Intense Space-Charge*

$$n = \int d^2x'_\perp \; f_\perp \;\; = \begin{cases} \frac{\lambda}{q\pi r_x r_y}, & x^2/r_x^2 + y^2/r_y^2 < 1 \\ 0, & x^2/r_x^2 + y^2/r_y^2 > 1 \end{cases}$$

Obtaining this form consistent with the assumptions, thereby demonstrating full self-consistency of the KV equilibrium distribution.

- Full 4-D form of the distribution does not evolve in *s*
- Projections of the distribution can (and generally do!) evolve in *s*

Moments of the KV distribution can be calculated directly from the distribution to further aid interpretation:

Full 4D average:
$$\langle \cdots \rangle_\perp \equiv \frac{\int d^2x_\perp \int d^2x'_\perp \; \cdots f_\perp}{\int d^2x_\perp \int d^2x'_\perp \; f_\perp}$$

Restricted angle average:
$$\langle \cdots \rangle_{\mathbf{x}'_\perp} \equiv \frac{\int d^2x'_\perp \; \cdots f_\perp}{\int d^2x'_\perp \; f_\perp}$$

Envelope edge radius:
$$r_x = 2\langle x^2 \rangle_\perp^{1/2}$$

Envelope edge angle:
$$r'_x = 2\langle xx' \rangle_\perp / \langle x^2 \rangle_\perp^{1/2}$$

rms edge emittance (maximum Courant-Snyder invariant):
$$\varepsilon_x = 4[\langle x^2 \rangle_\perp \langle x'^2 \rangle_\perp - \langle xx' \rangle_\perp^2]^{1/2} = \text{const}$$

Coherent flows (within the beam, zero otherwise):
$$\langle x' \rangle_{\mathbf{x}'_\perp} = r'_x \frac{x}{r_x}$$

Angular spread (x-temperature, within the beam, zero otherwise):
$$T_x \equiv \langle (x' - \langle x' \rangle_{\mathbf{x}'_\perp})^2 \rangle_{\mathbf{x}'_\perp} = \frac{\varepsilon_x^2}{2r_x^2}\left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)$$

Summary of 1st and 2nd order moments of the KV distribution:

Moment	Value
$\int d^2x'_\perp \; x' f_\perp$	$r'_x \frac{x}{r_x} n$
$\int d^2x'_\perp \; y' f_\perp$	$r'_y \frac{y}{r_y} n$
$\int d^2x'_\perp \; x'^2 f_\perp$	$\left[r'^2_x \frac{x^2}{r_x^2} + \frac{\varepsilon_x^2}{2r_x^2}\left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)\right] n$
$\int d^2x'_\perp \; y'^2 f_\perp$	$\left[r'^2_y \frac{y^2}{r_y^2} + \frac{\varepsilon_y^2}{2r_y^2}\left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)\right] n$
$\int d^2x'_\perp \; xx' f_\perp$	$\frac{r'_x}{r_x} x^2 n$
$\int d^2x'_\perp \; yy' f_\perp$	$\frac{r'_y}{r_y} y^2 n$
$\int d^2x'_\perp \; (xy' - yx') f_\perp$	0
$\langle x^2 \rangle_\perp$	$\frac{r_x^2}{4}$
$\langle y^2 \rangle_\perp$	$\frac{r_y^2}{4}$
$\langle x'^2 \rangle_\perp$	$\frac{r'^2_x}{4} + \frac{\varepsilon_x^2}{4r_x^2}$
$\langle y'^2 \rangle_\perp$	$\frac{r'^2_y}{4} + \frac{\varepsilon_y^2}{4r_y^2}$
$\langle xx' \rangle_\perp$	$\frac{r_x r'_x}{4}$
$\langle yy' \rangle_\perp$	$\frac{r_y r'_y}{4}$
$\langle xy' - yx' \rangle_\perp$	0
$16[\langle x^2\rangle_\perp \langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2]$	ε_x^2
$16[\langle y^2\rangle_\perp \langle y'^2\rangle_\perp - \langle yy'\rangle_\perp^2]$	ε_y^2

All 1st and 2nd order moments not listed vanish, i.e.,

$$\int d^2x'_\perp \; xy f_\perp = 0$$

$$\langle xy \rangle_\perp = 0$$

see reviews by:

(limit of results presented)
Lund and Bukh, PRSTAB **7**, 024801 (2004), Appendix A

S.M. Lund, T. Kikuchi, and R.C. Davidson, PRSTAB **12,** 114801 (2009)

KV Envelope equation

The envelope equation reflects low-order force balances

$$r_x'' + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0$$

	Applied Focusing	Space-Charge Defocusing	Thermal Defocusing
Terms:	Lattice	Perveance	Emittance

Matched Solution:

$$r_x(s + L_p) = r_x(s)$$
$$r_y(s + L_p) = r_y(s)$$

$$\kappa_x(s + L_p) = \kappa_x(s)$$
$$\kappa_y(s + L_p) = \kappa_y(s)$$

Comments:

- Envelope equation is a projection of the 4D invariant distribution
  - Envelope evolution equivalently given by moments of the 4D equilibrium distribution
- Most important basic design equation for transport lattices with high space-charge intensity
  - Simplest consistent model incorporating applied focusing, space-charge defocusing, and thermal defocusing forces
  - Starting point of almost all practical machine design

Comments Continued:

- Beam envelope matching where the beam envelope has the periodicity of the lattice

$$r_x(s + L_p) = r_x(s)$$
$$r_y(s + L_p) = r_y(s)$$

will be covered in much more detail in Sec. 6.C. Envelope matching requires specific choices of initial conditions

$$r_x(s_i),\ r_y(s_i) \qquad\qquad r'_x(s_i),\ r'_y(s_i)$$

for periodic evolution.

- Instabilities of envelope equations are well understood and real
  - Must be avoided for reliable machine operation
  - See USPAS lecture notes on *Beam Physics with Intense Space Charge,* Transverse Centroid and Envelope Models

The matched solution to the KV envelope equations reflects the symmetry of the focusing lattice and must in general be calculated numerically

Matching Condition

$$\boxed{\begin{aligned} r_x(s+L_p) &= r_x(s) \\ r_y(s+L_p) &= r_y(s) \end{aligned}}$$

Example Parameters

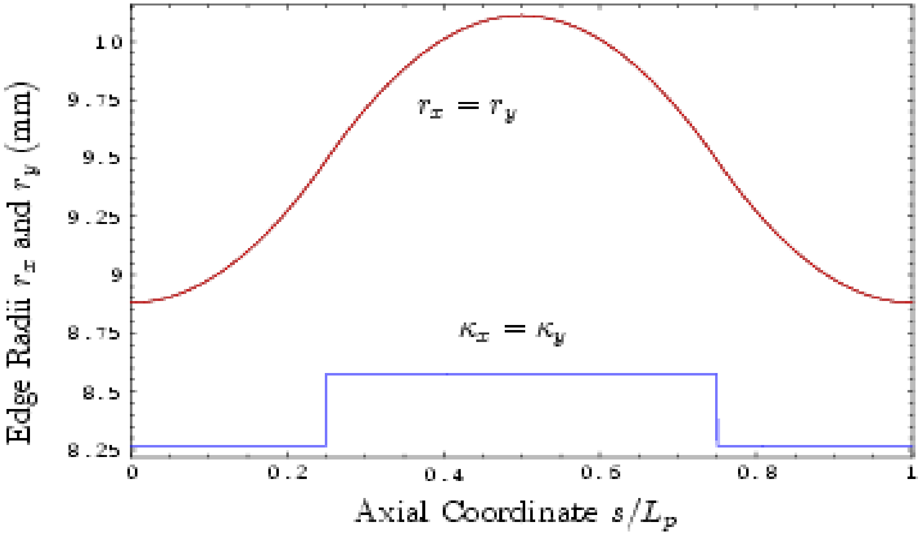
$$L_p = 0.5 \text{ m}, \quad \sigma_0 = 80^\circ, \quad \eta = 0.5$$

$$\varepsilon_x = \varepsilon_y = 50 \text{ mm-mrad}$$

$$\sigma/\sigma_0 = 0.2$$

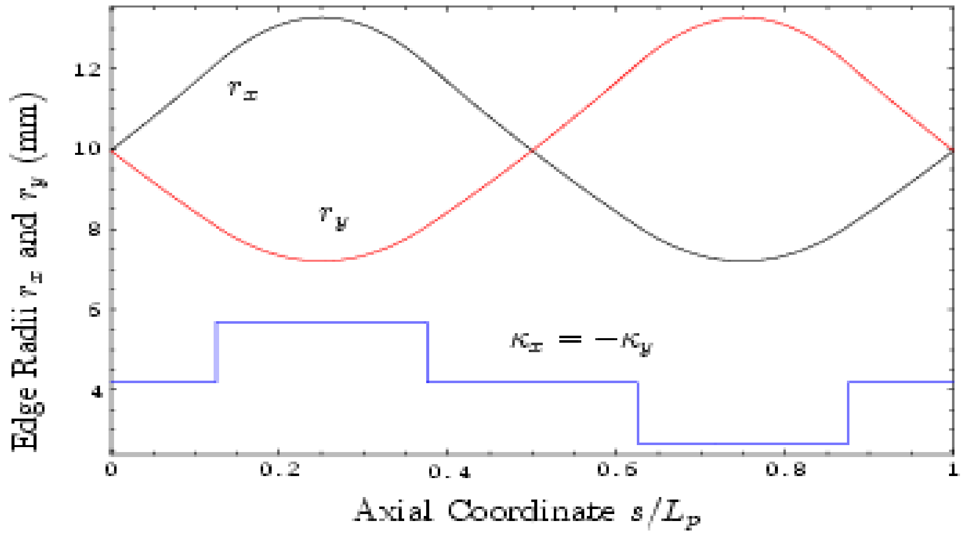
Solenoidal Focusing

$(Q = 6.6986 \times 10^{-4})$



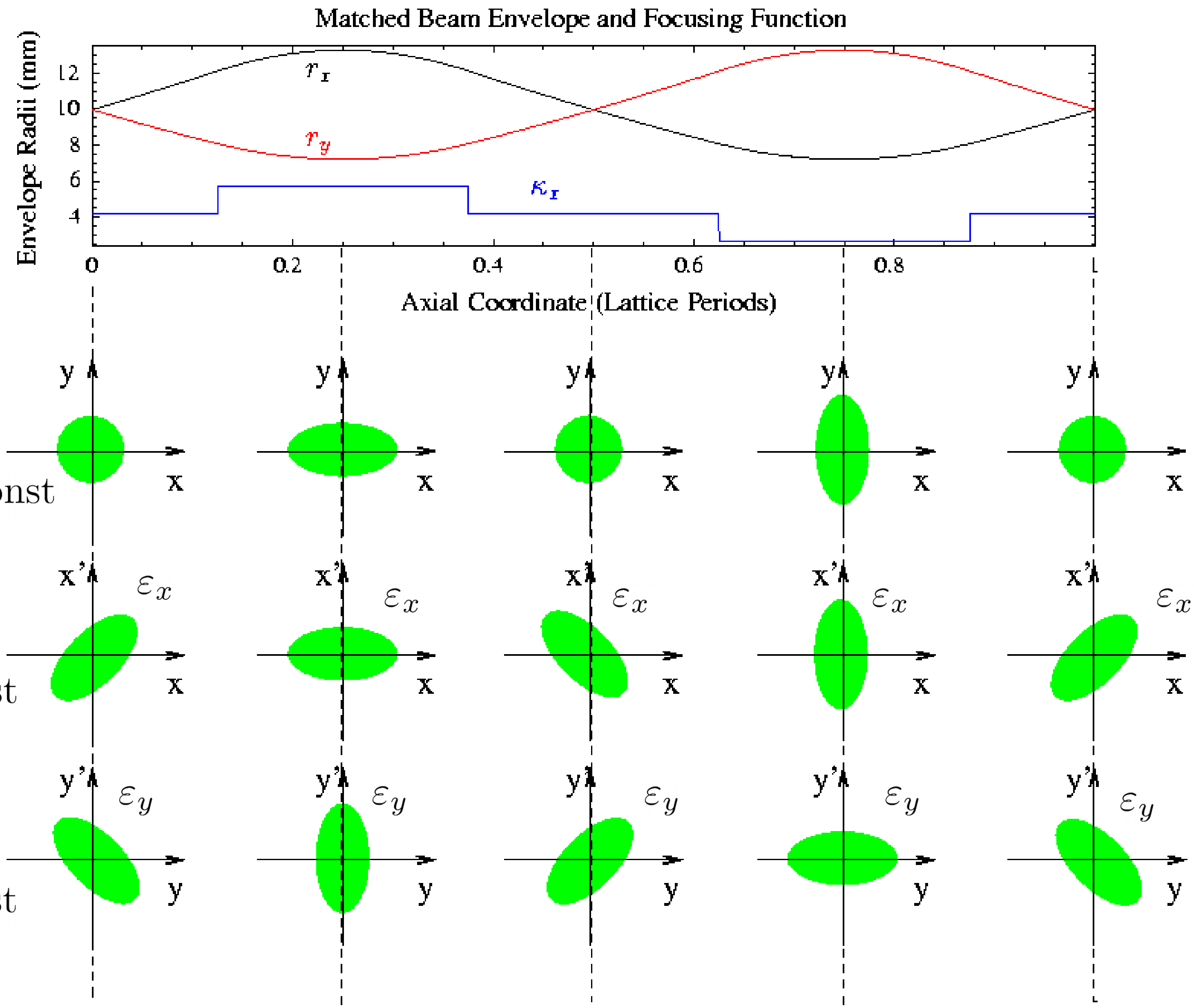
FODO Quadrupole Focusing

$(Q = 6.5614 \times 10^{-4})$



The matched beam is the most radially compact solution to the envelope equations rendering it highly important for beam transport

2D phase-space projections of a matched KV equilibrium beam in a periodic FODO quadrupole transport lattice



Projection

x-y
area: $\pi r_x r_y \neq \text{const}$

x-x'
area: $\pi \varepsilon_x = \text{const}$
(CS Invariant)

y-y'
area: $\pi \varepsilon_y = \text{const}$
(CS Invariant)

KV model shows that particle orbits in the presence of space-charge can be strongly modified – space charge slows the orbit response:

Matched envelope:

$$r_x''(s) + \kappa_x(s) r_x(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_x^2}{r_x^3(s)} = 0$$

$$r_y''(s) + \kappa_y(s) r_y(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_y^2}{r_y^3(s)} = 0$$

$$r_x(s + L_p) = r_x(s) \qquad\qquad r_x(s) > 0$$

$$r_y(s + L_p) = r_y(s) \qquad\qquad r_y(s) > 0$$

Equation of motion for x-plane "depressed" orbit in the presence of space-charge:

$$x''(s) + \kappa_x(s) x(s) - \frac{2Q}{[r_x(s) + r_y(s)] r_x(s)} x(s) = 0$$

All particles have the *same value* of depressed phase advance (similar Eqns in *y*):

$$\sigma_x \equiv \psi_x(s_i + L_p) - \psi_x(s_i) = \varepsilon_x \int_{s_i}^{s_i + L_p} \frac{ds}{r_x^2(s)}$$

Contrast: Review, the undepressed particle phase advance calculated in the lectures on Transverse Particle Dynamics

The undepressed phase advance is defined as the phase advance of a particle in the absence of space-charge ($Q = 0$):

- Denote by σ_{0x} to distinguished from the "depressed" phase advance σ_x in the presence of space-charge

$$w''_{0x} + \kappa_x w_{0x} - \frac{1}{w_{0x}^3} = 0 \qquad w_{0x}(s + L_p) = w_{0x}(s)$$

$$w_{0x} > 0$$

$$\sigma_{0x} = \int_{s_i}^{s_i+L_p} \frac{ds}{w_{0x}^2}$$

This can be equivalently calculated from the matched envelope with $Q = 0$:

$$r''_{0x} + \kappa_x r_{0x} - \frac{\varepsilon_x^2}{r_{0x}^3} = 0 \qquad r_{0x}(s + L_p) = r_{0x}(s)$$

$$r_{0x} > 0$$

$$\sigma_{0x} = \varepsilon_x \int_{s_i}^{s_i+L_p} \frac{ds}{r_{0x}^2}$$

- Value of ε_x is arbitrary (answer for σ_{0x} is independent)

Depressed particle phase advance provides a convenient measure of space-charge strength

For simplicity take (plane symmetry in average focusing and emittance)

$$\sigma_{0x} = \sigma_{0y} \equiv \sigma_0 \qquad\qquad \varepsilon_x = \varepsilon_y \equiv \varepsilon$$

Depressed phase advance of particles moving within a matched beam envelope:

$$\boxed{\sigma = \varepsilon \int_{s_i}^{s_i+L_p} \frac{ds}{r_x^2(s)} = \varepsilon \int_{s_i}^{s_i+L_p} \frac{ds}{r_y^2(s)}}$$

Limits:

1) $\lim_{Q \to 0} \sigma = \sigma_0$ Envelope just rescaled amplitude: $r_x^2 = \varepsilon w_{0x}^2$

2) $\lim_{\varepsilon \to 0} \sigma = 0$ Matched envelope exists with $\varepsilon = 0$
Then $\varepsilon = 0$ multiplying phase advance integral

Normalized space charge strength

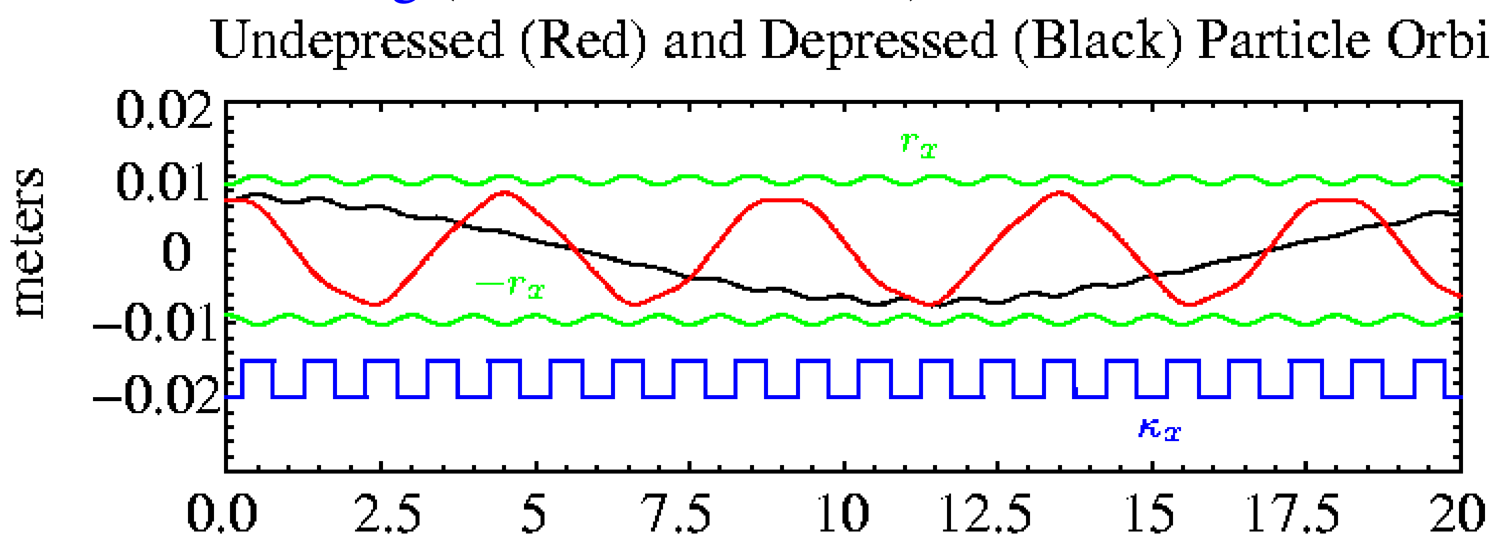
$$\boxed{0 \leq \sigma/\sigma_0 \leq 1}$$

$\sigma/\sigma_0 \to 0$ Cold Beam
(space-charge dominated)
$\varepsilon \to 0$

$\sigma/\sigma_0 \to 1$ Warm Beam
(kinetic dominated)
$Q \to 0$

Depressed particle x-plane orbits within a matched KV beam in a periodic FODO quadrupole channel for the matched beams previously shown

Solenoidal Focusing (Larmor frame orbit):



x-plane orbit:

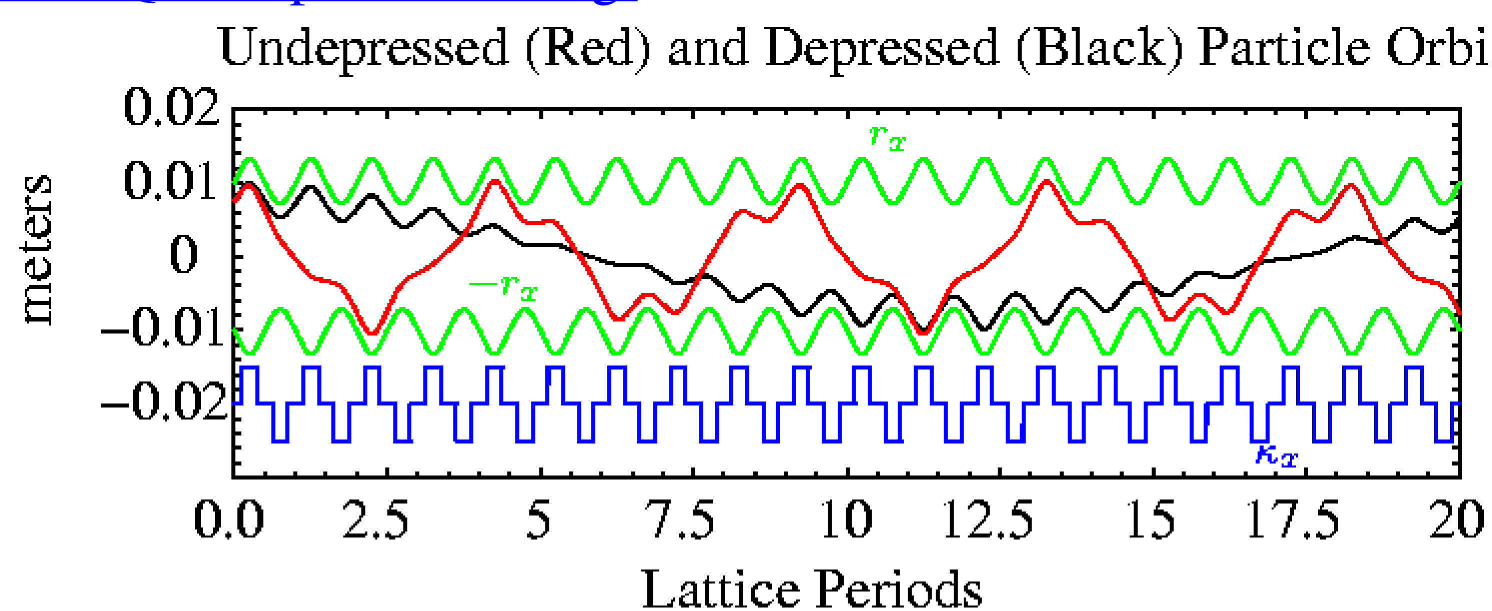
$$y = 0 = y'$$

Both Problems Tuned for:

$$\sigma_0 = 80^\circ$$

$$\frac{\sigma}{\sigma_0} = 0.2$$

FODO Quadrupole Focusing:



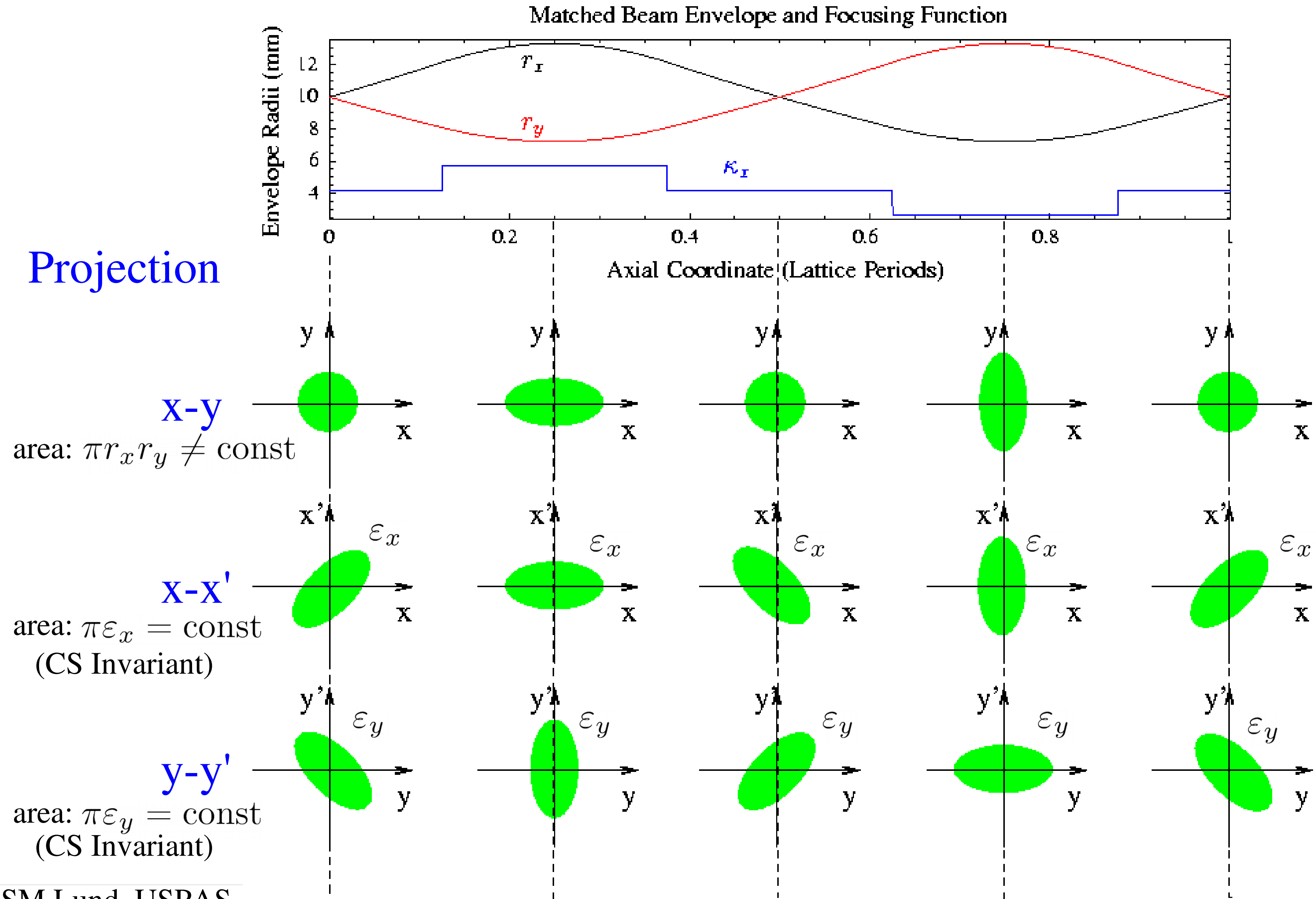
x-plane orbit:

$$y = 0 = y'$$

Comment: All particles in the distribution move in response to both applied and self-fields. You cannot turn off space-charge for an undepressed orbit.

- Cannot really turn space-charge off: but it helps interpretation

Orbit bundles are fully consistent with uniformly filled phase-space projections for all s



The rms equivalent beam model helps interpret general beam evolution in terms of an "equivalent" local KV distribution

Real beams distributions in the lab will not be KV form. But the KV model can be applied to interpret arbitrary distributions via the concept of *rms equivalence*.

For the same focusing lattice, replace any beam charge $\rho(x, y)$ density by a uniform density KV beam of the same species (q, m) and energy (β_b) in each axial slice (s) using averages calculated from the actual "real" beam distribution with:

$$\langle \cdots \rangle_\perp \equiv \frac{\int d^2x_\perp \int d^2x'_\perp \; \cdots f_\perp}{\int d^2x_\perp \int d^2x'_\perp \; f_\perp} \qquad f_\perp = \text{ real distribution}$$

rms equivalent beam (identical 1st and 2nd order moments):

Quantity	KV Equiv.	Calculated from Distribution
Perveance	Q	$= q^2 \int d^2x_\perp \int d^2x'_\perp \; f_\perp \,/[2\pi\epsilon_0 \gamma_b^3 \beta_b^2 c^2]$
x-Env Rad	r_x	$= 2\langle x^2 \rangle_\perp^{1/2}$
y-Env Rad	r_y	$= 2\langle y^2 \rangle_\perp^{1/2}$
x-Env Angle	r'_x	$= 2\langle xx' \rangle_\perp / \langle x^2 \rangle_\perp^{1/2}$
y-Env Angle	r'_y	$= 2\langle yy' \rangle_\perp / \langle y^2 \rangle_\perp^{1/2}$
x-Emittance	ε_x	$= 4[\langle x^2 \rangle_\perp \langle x'^2 \rangle_\perp - \langle xx' \rangle_\perp]^{1/2}$
y-Emittance	ε_y	$= 4[\langle y^2 \rangle_\perp \langle y'^2 \rangle_\perp - \langle yy' \rangle_\perp]^{1/2}$

Comments on rms equivalent beam concept:

- The emittances will generally evolve in s
  - Means that the equivalence must be recalculated in every slice as the emittances evolve
  - This evolution is often small for well behaved (stable) beams which we design for and emittance for beams with strong space-charge is small and has only limited impact on the envelope
- Concept is highly useful
  - KV equilibrium properties well understood and are approximately correct to model lowest order “real” beam properties
  - See, Reiser, *Theory and Design of Charged Particle Beams* (1994, 2008) for a detailed and instructive discussion of rms equivalence

Sacherer expanded the concept of rms equivalency by showing that the equivalency works exactly for beams with elliptic symmetry space-charge
[Sacherer, IEEE Trans. Nucl. Sci. 18, 1101 (1971)]

For any beam with elliptic symmetry charge density in each transverse slice:

$$\boxed{\rho = \rho\left(\frac{x^2}{r_x^2} + \frac{y^2}{r_y^2}\right)}$$

Based on very nontrivial result:

$$\langle x \frac{\partial \phi}{\partial x} \rangle_\perp = -\frac{\lambda}{4\pi\epsilon_0} \frac{r_x}{r_x + r_y}$$

the KV envelope equations

$$\boxed{\begin{aligned} r_x''(s) + \kappa_x(s) r_x(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_x^2(s)}{r_x^3(s)} &= 0 \\ r_y''(s) + \kappa_y(s) r_y(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_y^2(s)}{r_y^3(s)} &= 0 \end{aligned}}$$

remain valid when (averages taken with the full distribution):

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m \gamma_b^3 \beta_b^2 c^2} = \text{const} \qquad \lambda = q\int d^2x_\perp\, \rho \;=\; \text{const}$$

$$r_x = 2\langle x^2 \rangle_\perp^{1/2} \qquad \varepsilon_x = 4[\langle x^2 \rangle_\perp \langle x'^2 \rangle_\perp - \langle x x' \rangle_\perp^2]^{1/2}$$

$$r_y = 2\langle y^2 \rangle_\perp^{1/2} \qquad \varepsilon_y = 4[\langle y^2 \rangle_\perp \langle y'^2 \rangle_\perp - \langle y y' \rangle_\perp^2]^{1/2}$$

The emittances may evolve in s under this model
(see USPAS lectures on Transverse Kinetic Stability)

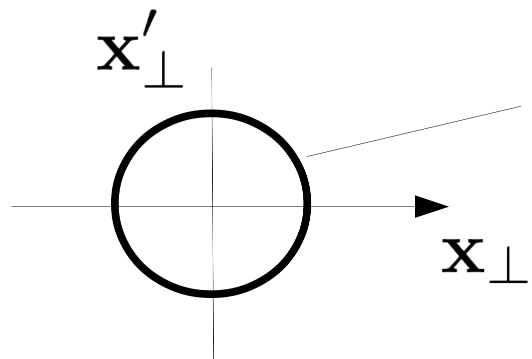
Further comments on the KV equilibrium: Distribution Structure

KV equilibrium distribution:

$$\boxed{f_\perp \sim \delta[\text{Courant-Snyder invariants}]}$$

Forms a highly singular hyper-shell in 4D phase-space

Schematic:

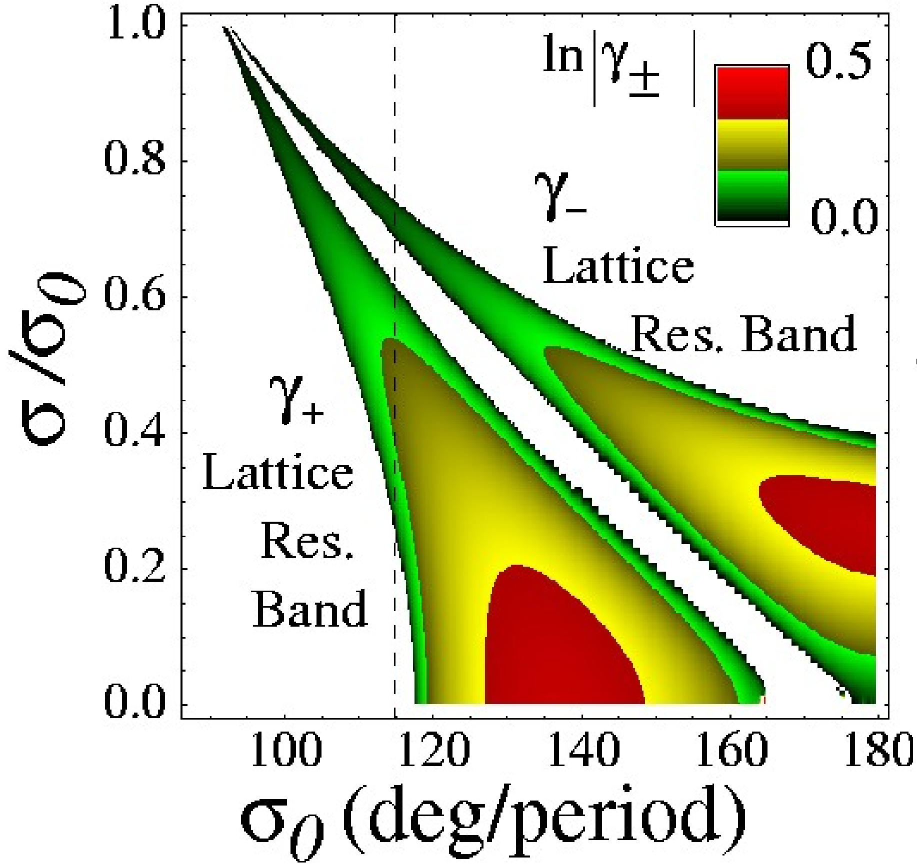


- Singular distribution has large "Free-Energy" to drive many instabilities
  - Low order envelope modes are physical and highly important (see: USPAS lectures on Centroid and Envelope Descriptions of Beams)
- Perturbative analysis shows strong collective instabilities
  - Hofmann, Laslett, Smith, and Haber, Part. Accel. **13**, 145 (1983)
  - Higher order instabilities (collective modes) have unphysical aspects due to (delta-function) structure of distribution and must be applied with care (see: USPAS lectures on Kinetic Stability of Beams)
  - Instabilities can cause problems if the KV distribution is employed as an initial beam state in self-consistent simulations

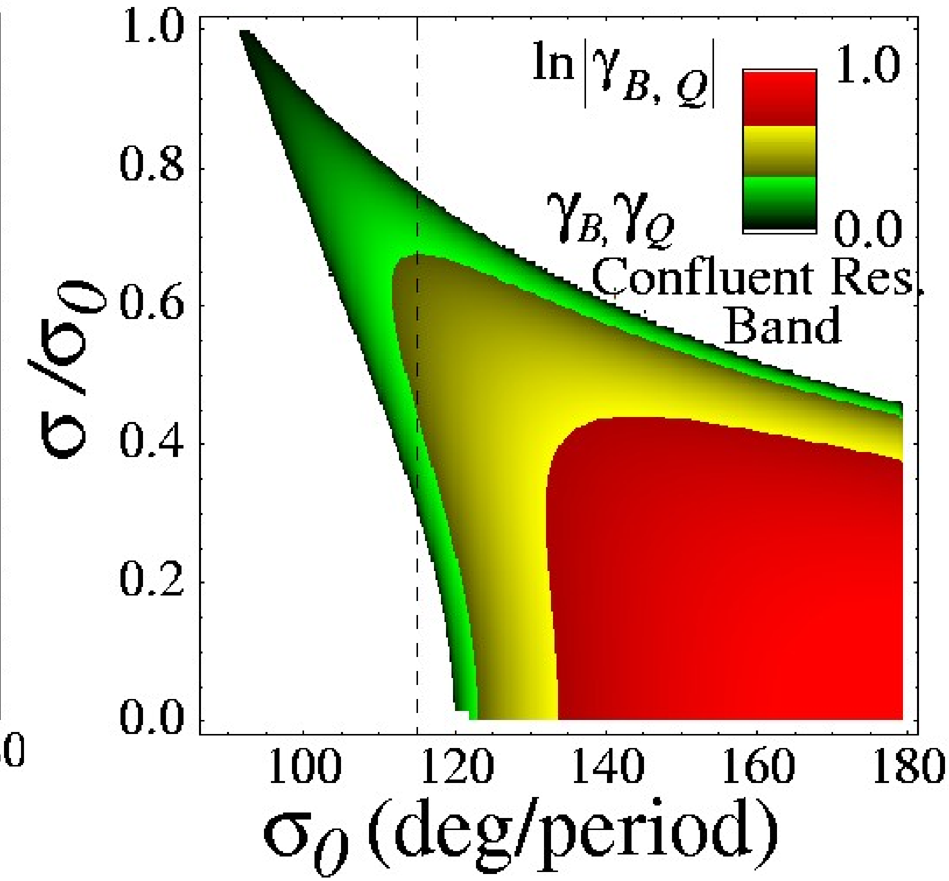
Instability bands of the KV envelope equation are well understood in periodic focusing channels and must be avoided in machine operation

Envelope Mode Instability Growth Rates

Solenoid (η _ = 0.25)



Quadrupole FODO (η _ = 0.70)



[S.M. Lund and B. Bukh, PRSTAB **7** 024801 (2004)]

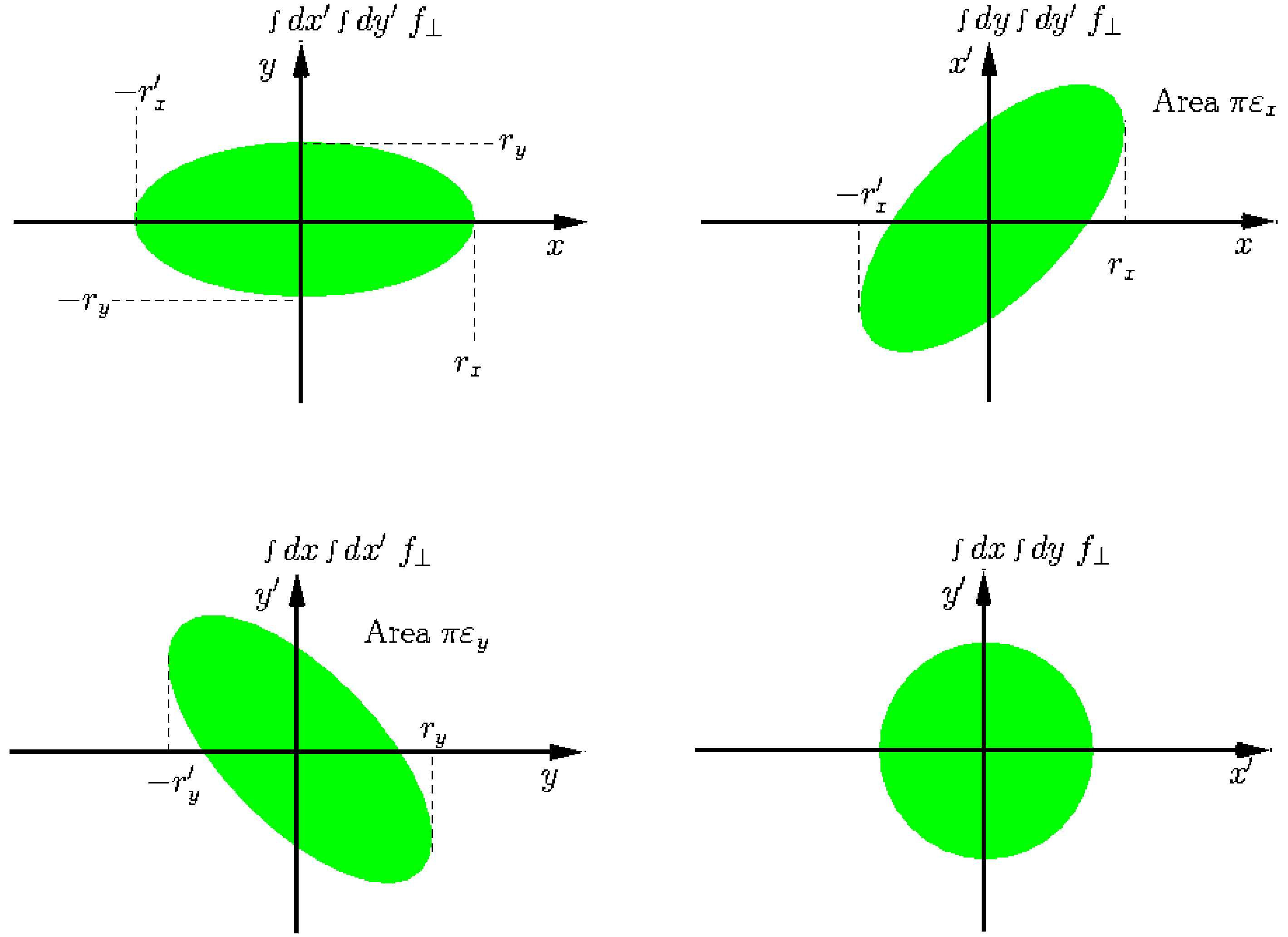
Cannot operate a machine in the instability bands: beam wildly unstable

- Avoid on load unless you want "exciting" simulations!

Further comments on the KV equilibrium: 2D Projections

All 2D projections of the KV distribution are uniformly filled ellipses

- Not very different from what is often observed in experimental measurements and self-consistent simulations of stable beams with strong space-charge
- Falloff of distribution at "edges" can be rapid, but smooth, for strong space-charge

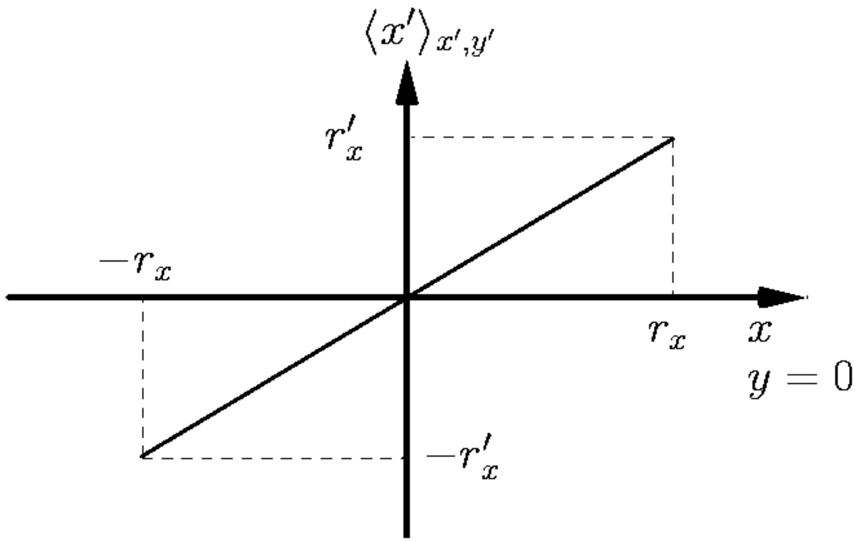


Further comments on the KV equilibrium:
Angular Spreads: Coherent and Incoherent

Angular spreads within the beam:

Coherent (flow):

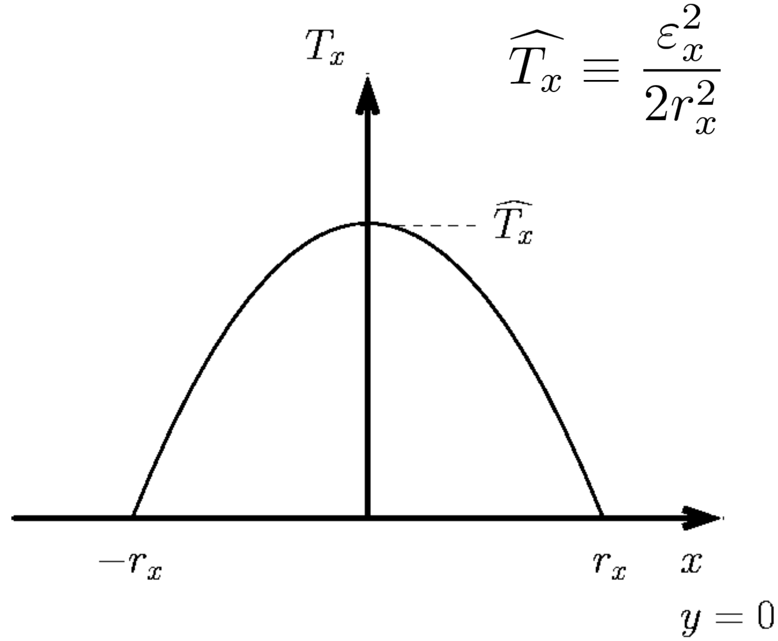
$$\langle x' \rangle_{\mathbf{x}'_\perp} \equiv \frac{\int d^2x'_\perp \; x'_\perp \; f_\perp}{\int d^2x'_\perp \; f_\perp} = r'_x \frac{x}{r_x}$$



Incoherent (temperature):

$$\langle (x' - r'_x x/r_x)^2 \rangle_{\mathbf{x}'_\perp} = \frac{\varepsilon_x^2}{2r_x^2}\left(1 - \frac{x^2}{r_x^2} - \frac{y^2}{r_y^2}\right)$$

$$\widehat{T_x} \equiv \frac{\varepsilon_x^2}{2r_x^2}$$



- Coherent flow required for periodic focusing to conserve charge
- Temperature must be zero at the beam edge since the distribution edge is sharp
- Parabolic temperature profile is consistent with linear grad P pressure forces in a fluid model interpretation of the (kinetic) KV distribution

Further comments on the KV equilibrium:

The KV distribution is the *only* exact equilibrium distribution formed from Courant-Snyder invariants of linear forces valid for periodic focusing channels:

- Low order properties of the distribution are physically appealing
- Illustrates relevant Courant-Snyder invariants in simple form
  - Arguments demonstrate that these invariants should be a reasonable approximation for beams with strong space charge
- KV distribution does not have a 3D generalization [see F. Sacherer, Ph.d. thesis, 1968]

Strong Vlasov instabilities associated with the KV model render the distribution inappropriate for use in evaluating machines at high levels of detail:

- Instabilities are not all physical and render interpretation of results difficult
  - Difficult to separate physical from nonphysical effects in simulations

Possible Research Problem (unsolved in 40+ years!):
Can an *exact* Vlasov equilibrium be constructed for a *smooth* (non-singular), nonuniform density distribution in a linear, periodic focusing channel?

- Not clear what invariants can be used or if any can exist
  - Nonexistence proof would also be significant
- Recent perturbation theory and simulation work suggest prospects
  - Self-similar classes of distributions
- Lack of a smooth equilibrium does not imply that real machines cannot work!

Because of a lack of theory for a smooth, self-consistent distribution that would be more physically appealing than the KV distribution we will examine smooth distributions in the idealized continuous focusing limit (after an analysis of the continuous limit of the KV theory):

- Allows more classic "plasma physics" like analysis
- Illuminates physics of intense space charge
- Lack of continuous focusing in the laboratory will prevent over generalization of results obtained

A 1D analog to the KV distribution called the "Neuffer Distribution" is useful in longitudinal physics

- Based on linear forces with a "g-factor" model
- Distribution not singular in 1D and is fully stable in continuous focusing
- See: USPAS lectures on Longitudinal Physics

In spite of idealizations, the KV distribution is very useful in simulations

- Provides basis with rms equivalency for understanding beam matching
- Useful for simulation benchmarking

C: Beam Envelope Matching

An rms matched beam has correct flow symmetry in a periodic transport lattice to repeat every lattice period when modeled in an rms equivalent beam sense by the KV envelope equations. Almost all accelerator lattices are periodic focusing. Typical setup of initial simulation distributions will involve finding envelope matching conditions for the lattice.

- Matched beam most compact (min excursions) for efficient transport
  - Makes best use of focusing strength
- Matching suppresses potential for instabilities to the extent possible
  - Less "free energy" to drive waves, instabilities, and halo production
- Terminology: A beam not rms matched to a periodic focusing lattice is called "Mismatched"

An extensive review paper contains much information on envelope matching and properties and instabilities of mismatch oscillations

> Lund and Bukh, PRSTAB **7**, Stability properties of the transverse envelope equations describing intense ion beam transport, 024801 (2004)

Matched Envelope Constraints

Neglect acceleration $(\gamma_b \beta_b = \text{const})$ or use transformed variables:

$$\boxed{\begin{aligned} r_x''(s) + \kappa_x(s) r_x(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_x^2}{r_x^3(s)} &= 0 \\ r_y''(s) + \kappa_y(s) r_y(s) - \frac{2Q}{r_x(s) + r_y(s)} - \frac{\varepsilon_y^2}{r_y^3(s)} &= 0 \\ r_x(s + L_p) = r_x(s) \qquad\qquad & r_x(s) > 0 \\ r_y(s + L_p) = r_y(s) \qquad\qquad & r_y(s) > 0 \end{aligned}}$$

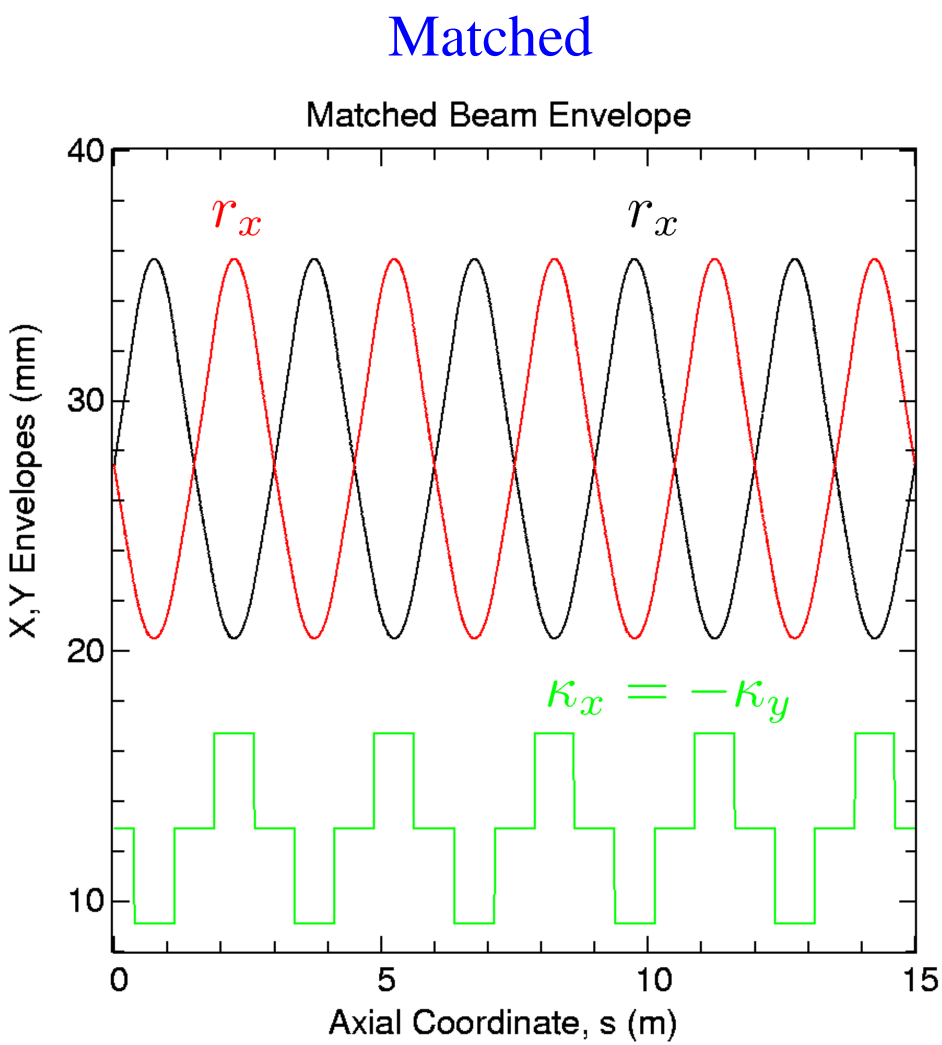
Matching involves finding specific initial conditions for the envelope to have the periodicity of the lattice:

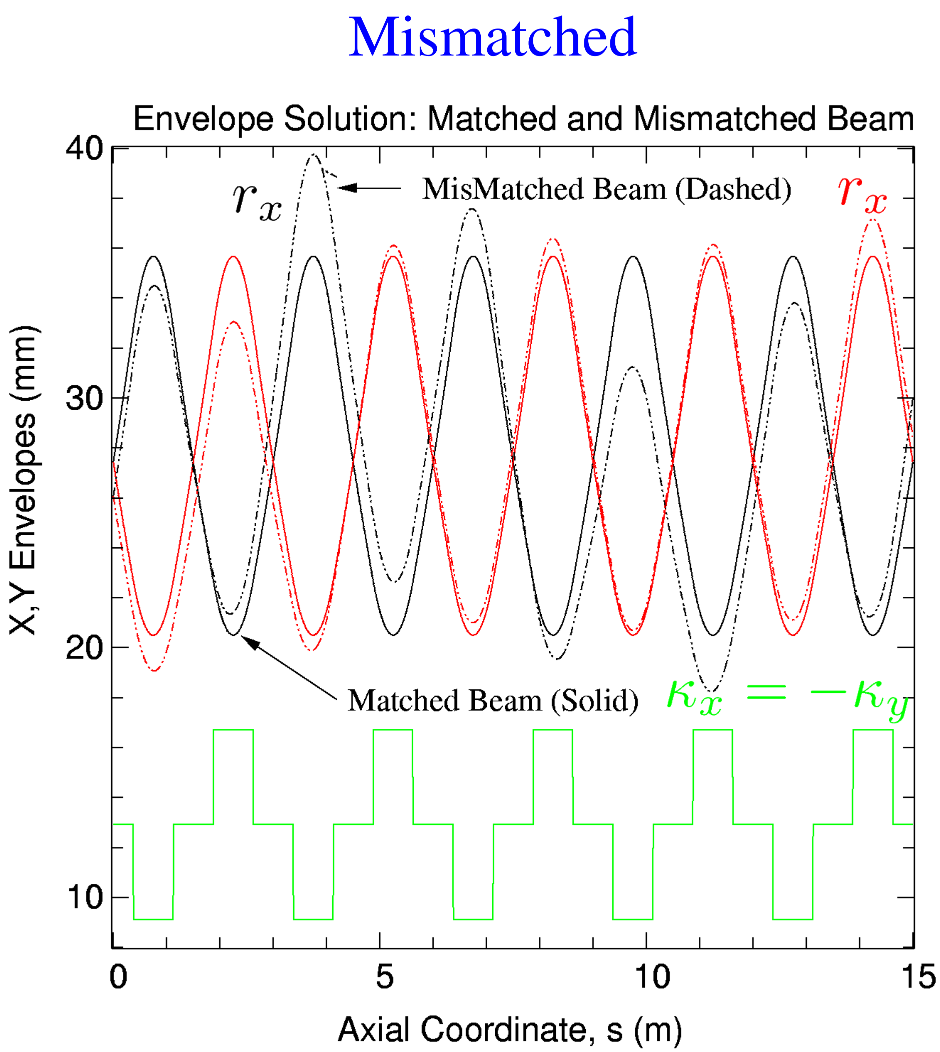
Find Values of: Such That: (periodic)

$$\boxed{\begin{matrix} r_x(s_i) & r_x'(s_i) \\ r_y(s_i) & r_y'(s_i) \end{matrix}} \longrightarrow \boxed{\begin{matrix} r_x(s_i + L_p) = r_x(s_i) & r_x'(s_i + L_p) = r_x'(s_i) \\ r_y(s_i + L_p) = r_y(s_i) & r_y'(s_i + L_p) = r_y'(s_i) \end{matrix}}$$

- Typically constructed with numerical root finding from estimated/guessed values
  - Can be surprisingly difficult for complicated lattices (high σ_0) with strong space-charge
- Iterative technique developed to numerically calculate without root finding;
  Lund, Chilton and Lee, PRSTAB **9**, 064201 (2006)
  - Method exploits Courant-Snyder invariants of depressed orbits within the beam

Typical Matched vs Mismatched solution for FODO channel:





The matched beam is the most radially compact solution to the envelope equations rendering it highly important for beam transport

- Matching uses optics most efficiently to maintain radial beam confinement
- Mismatch provides extra "free energy" to drive instabilities/halo

The matched solution to the KV envelope equations reflects the symmetry of the focusing lattice and must, in general, be calculated numerically

Envelope equation very nonlinear

$$\boxed{\begin{aligned} r_x(s+L_p) &= r_x(s) \\ r_y(s+L_p) &= r_y(s) \\ \varepsilon_x &= \varepsilon_y \end{aligned}}$$

Parameters

$$L_p = 0.5\ \text{m}, \quad \sigma_0 = 80^\circ, \quad \eta = 0.5$$

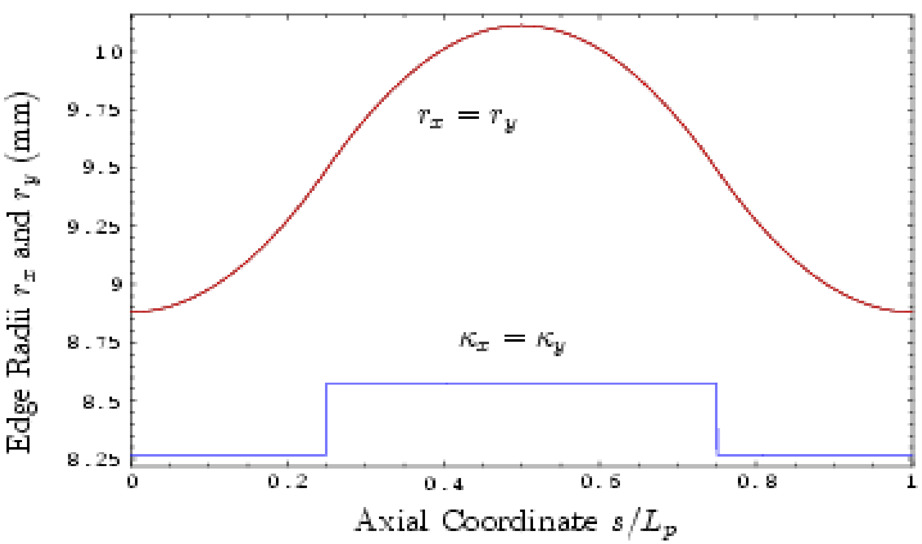
$$\varepsilon_x = 50\ \text{mm-mrad}$$

$$\sigma/\sigma_0 = 0.2$$

Perveance Q iterated to obtain matched solution with this tune depression

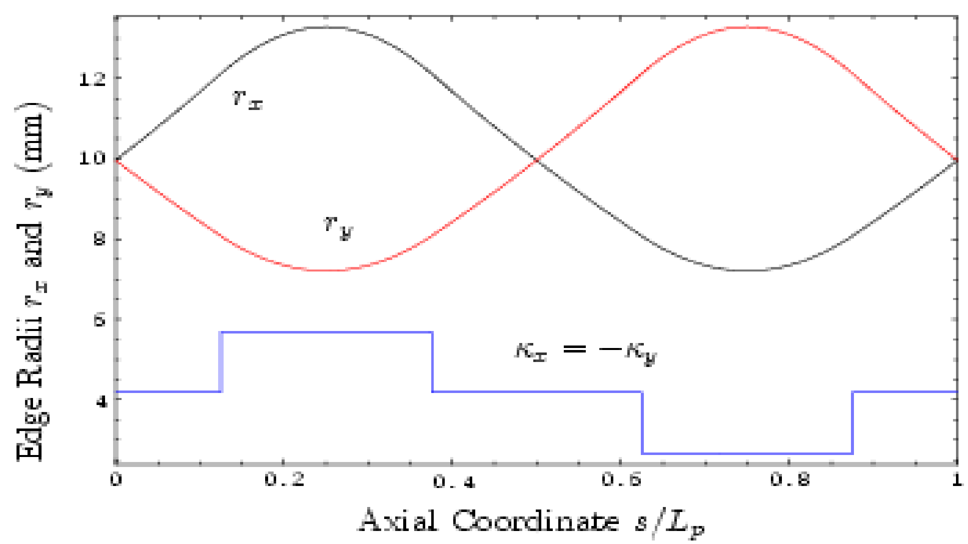
Solenoidal Focusing

$$(Q = 6.6986 \times 10^{-4})$$



FODO Quadrupole Focusing

$$(Q = 6.5614 \times 10^{-4})$$



Iterative Numerical Matching Code implemented in Mathematica provided

Lund, Chilton, and Lee, PRSTAB **9**, 064201 (2006)

IM (Iterated Matching) Method

- IM Method uses fail-safe numerical iteration technique without root finding to construct matched envelope solutions in periodic focusing lattices
  - Based on projections of Courant-Snyder invariants of depressed orbits in beam
  - Applies to arbitrarily complicated lattices (with user input focusing functions)
  - Works even where matched envelope is unstable
- Can find matched solutions under a variety of parameterizations:

  Case 0: (standard) $\kappa_x, \kappa_y, L_p \quad (\sigma_{0x}, \sigma_{0y}) \quad Q, \ \varepsilon_x, \ \varepsilon_y$

  Case 1: $\kappa_x, \kappa_y, L_p \quad (\sigma_{0x}, \sigma_{0y}) \quad Q, \ \sigma_x, \ \sigma_y \quad$ (find consit: $\varepsilon_x, \ \varepsilon_y$)

  Case 2: $\kappa_x, \kappa_y, L_p \quad (\sigma_{0x}, \sigma_{0y}) \quad \varepsilon_x = \varepsilon_y, \ \sigma_x = \sigma_y$ (find consit: Q)
- Mathematica code on github:

  % git clone https://github.com/smlund/iterative_match
- Included as a warp package: Match() function in envmatch_Kvinvariant.py
- Optional features in Mathematica version include additional information:
  - Characteristic particle orbits in beam
  - Matched envelope stability properties

To Run Mathematica code:

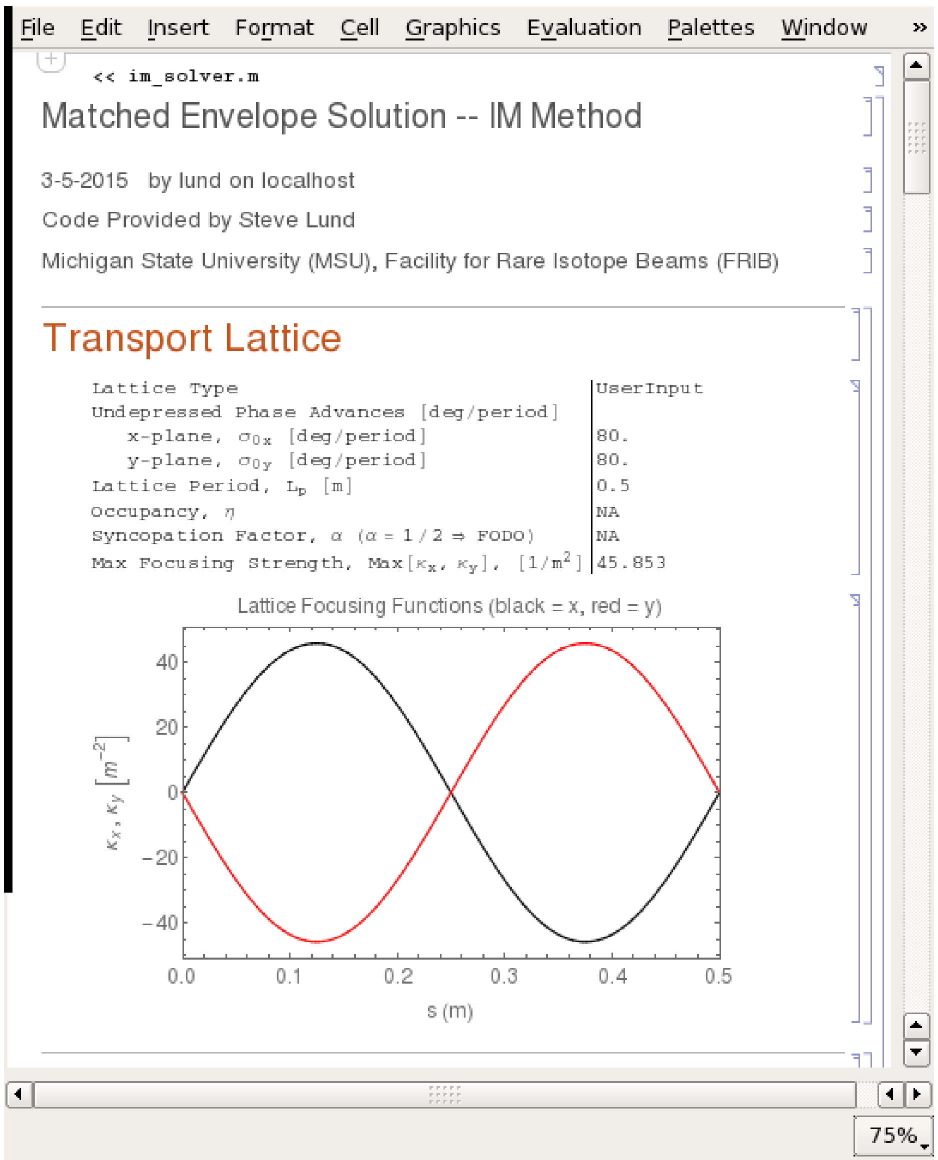
see readme.txt file with source code for details

1) Place “im_*.m” program files in directory and set parameters (text editor) in “im_inputs.m”
2) Open Mathematica Notebook in directory
3) Run in notebook by typing: << im_solver.m [shift-return]

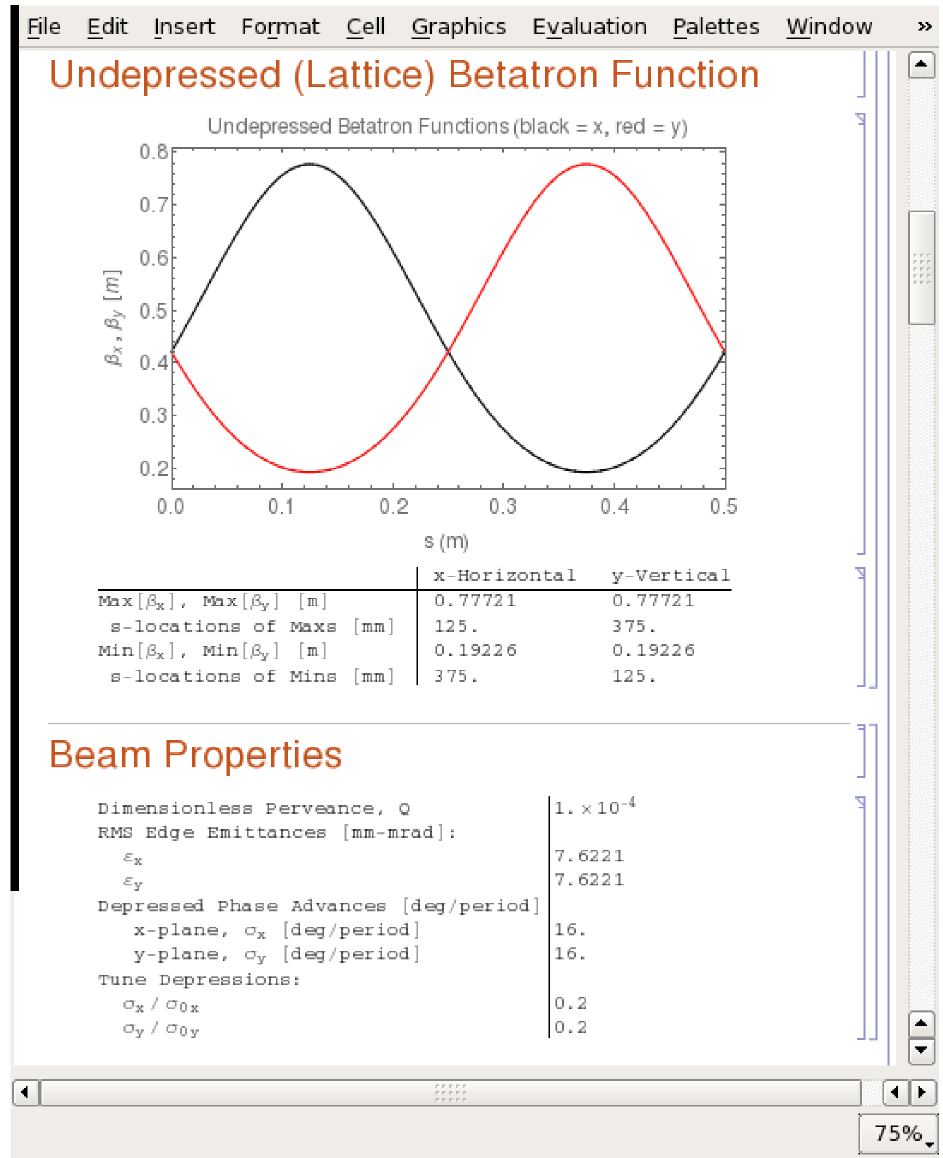
Example Run: sinusoidally varying quadrupole lattice with $\kappa_x = -\kappa_y$

- See "examples/user" subdirectory in source code distribution (other examples also)

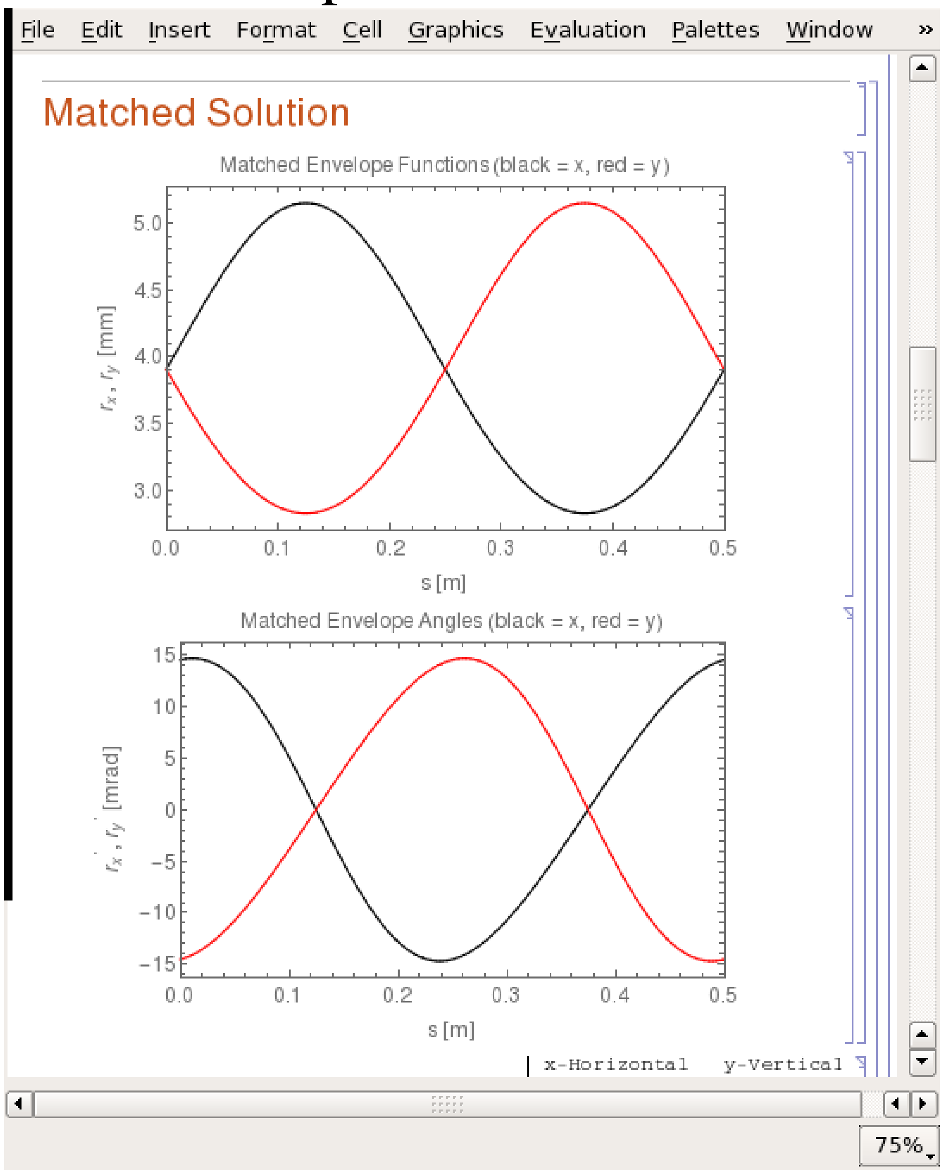
Output: 1st



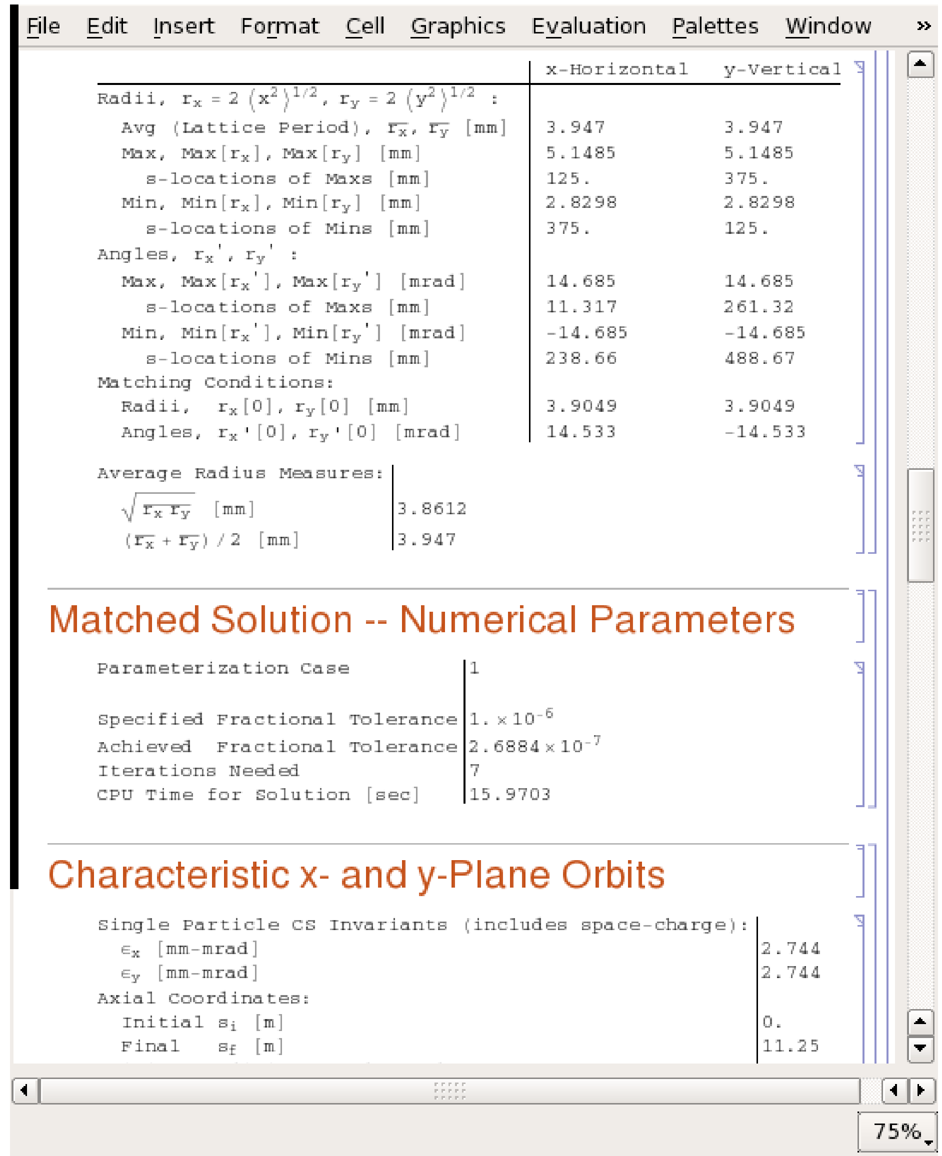
Output: 2ndt



Output: 3rd



Output: 4th



	x-Horizontal	y-Vertical
Radii, $r_x = 2\langle x^2\rangle^{1/2}$, $r_y = 2\langle y^2\rangle^{1/2}$:		
Avg (Lattice Period), $\overline{r_x}$, $\overline{r_y}$ [mm]	3.947	3.947
Max, Max[r_x], Max[r_y] [mm]	5.1485	5.1485
s-locations of Maxs [mm]	125.	375.
Min, Min[r_x], Min[r_y] [mm]	2.8298	2.8298
s-locations of Mins [mm]	375.	125.
Angles, r_x', r_y' :		
Max, Max[r_x'], Max[r_y'] [mrad]	14.685	14.685
s-locations of Maxs [mm]	11.317	261.32
Min, Min[r_x'], Min[r_y'] [mrad]	-14.685	-14.685
s-locations of Mins [mm]	238.66	488.67
Matching Conditions:		
Radii, r_x[0], r_y[0] [mm]	3.9049	3.9049
Angles, r_x'[0], r_y'[0] [mrad]	14.533	-14.533

Average Radius Measures:	
$\sqrt{\overline{r_x}\,\overline{r_y}}$ [mm]	3.8612
$(\overline{r_x} + \overline{r_y})/2$ [mm]	3.947

Matched Solution -- Numerical Parameters

Parameterization Case	1
Specified Fractional Tolerance	$1. \times 10^{-6}$
Achieved Fractional Tolerance	2.6884×10^{-7}
Iterations Needed	7
CPU Time for Solution [sec]	15.9703

Characteristic x- and y-Plane Orbits

Single Particle CS Invariants (includes space-charge):	
ϵ_x [mm-mrad]	2.744
ϵ_y [mm-mrad]	2.744
Axial Coordinates:	
Initial s_i [m]	0.
Final s_f [m]	11.25

+ More on particle orbits and matched envelope stability

D: Semi-Gaussian Beam

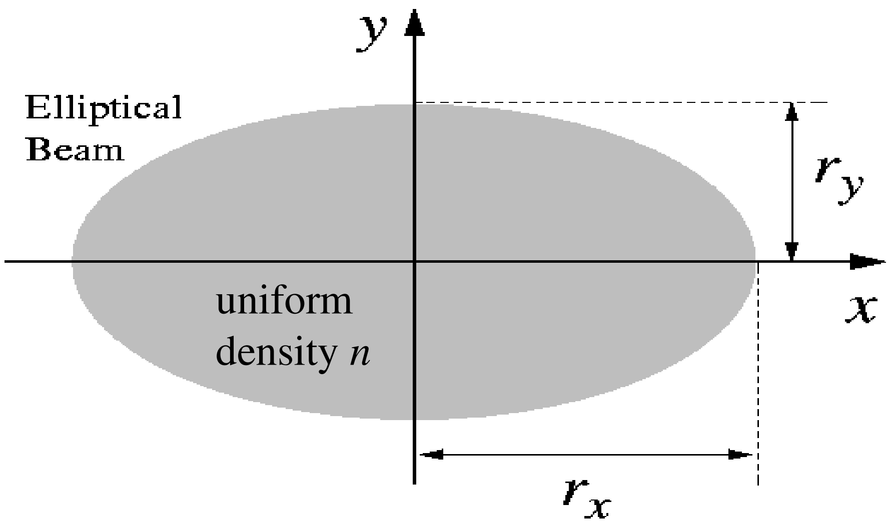
It is not necessary to load an equilibrium distribution to get a reasonably quiescent initial condition:

- For high space-charge intensity, expect Debye screening to lead to a beam more or less uniform out to an edge where the density drops rapidly to low values
- If beam is injected off a uniform temperature source or has relaxed, expect (roughly) spatially uniform thermal velocity spread across the core of the beam

  See USPAS course notes, *Beam Physics with Intense Space-Charge*

These properties suggest the so-called semi-Gaussian load:

Uniform Density within an elliptical beam envelope



Macroparticle x,y uniformly distributed within

$$\left(\frac{x}{r_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 \leq 1$$

Spatially uniform Gaussian Distributed Angles

Can achieve this by taking macroparticle angles

$$\begin{aligned} x' &= \underbrace{r'_x \frac{x}{r_x}}_{\text{Coherent Flow}} + \underbrace{\frac{\varepsilon_x}{2r_x} g_x}_{\text{Thermal Spread}} \\ y' &= r'_y \frac{y}{r_y} + \frac{\varepsilon_y}{2r_y} g_y \end{aligned}$$

$g_x, \; g_y$ Gaussian dist random numbers (unit variance)

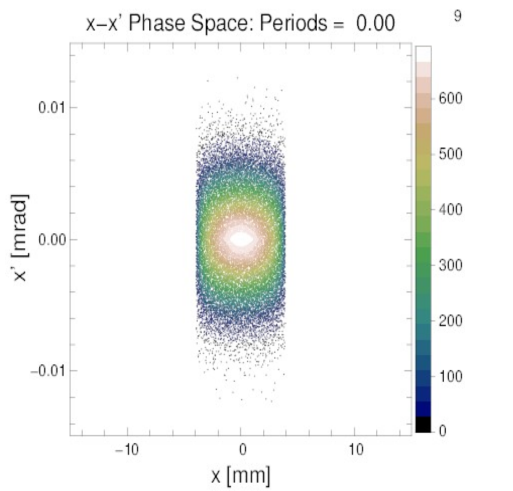
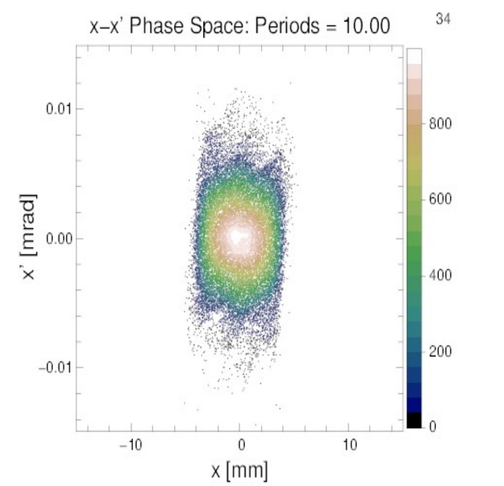
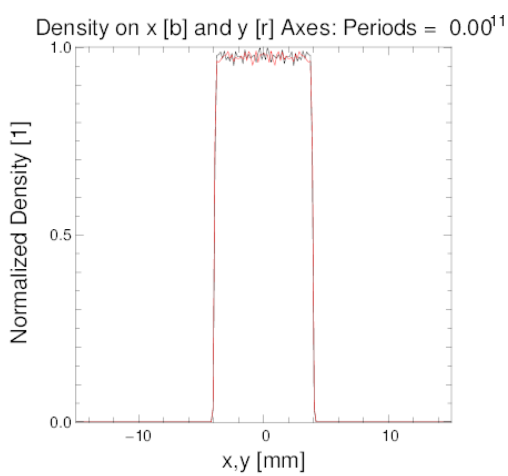
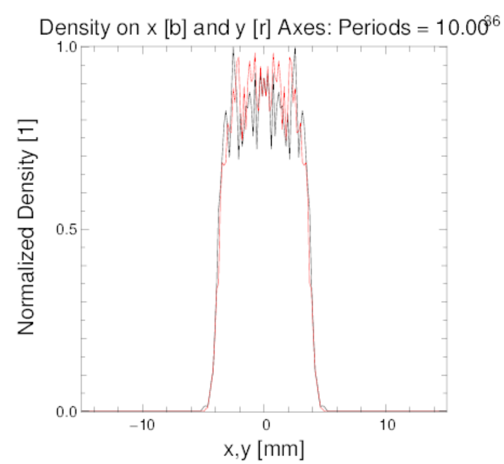
Coherent Flow

Thermal Spread

- Coherent term for envelope angle

Simulation of an initial semi-Gaussian load:

Interactively examine results from Warp script xy-quad-mag-mg.py for a periodic quadrupole lattice

	Initial	10 Lattice Periods
x-x' Phase Space		
Density (along x,y axes)		

Comments:

- Note quiescent equilibrium but not bad: unphysical features of distribution appear to rapidly relax with little growth in rms phase space area in spite of an initial spectrum of waves launched
- Shows you can take reasonable approximate initial states to roughly represent physical beams in well behaved situations where there is not pronounced instability
  - Part of reason why real machines work: there can be a relative insensitivity to details

E: Initial PseudoEquilbrium Distributions Based on Continuous Focusing Equilibria

Simple psudo-equilibrium initial distribution:

1) Use rms equivalent measures to specify the beam
   - Natural set of parameters for accelerator applications
2) Map rms equivalent beam to a smooth, continuous focused matched beam
   - Use smooth core models that are stable in continuous focusing:

     Waterbag Equilibrium
     Parabolic Equilibrium
     Thermal Equilibrium
     ⋮

     See USPAS *Beam Physics with Intense Space Charge* *n*otes on: Transverse Equilibrium Distributions
3) Transform continuous focused beam for rms equivalency with original beam specification
   - Use KV transforms to preserve uniform beam Courant-Snyder invariants

Procedure will apply to any s-varying focusing channel

- Focusing channel need not be periodic
- Beam can be initially rms equivalent matched or mismatched if launched in a periodic transport channel
- Can apply to both 2D transverse and 3D beams: illustrate for 2D transverse

4-Step Procedure for Initial Distribution Specification

[Lund, Kikuchi, Davidson, PRSTAB **12**, 114801 (2009)]

Assume focusing lattice is given:

$$\kappa_x(s), \quad \kappa_y(s) \qquad \text{specified}$$

Strength usually set by specifying undpepressed phase advances

$$\sigma_{0x}, \quad \sigma_{0y}$$

Step 1:

For each particle (3D) or slice (2D) specify 2nd order rms properties at axial coordinate s

Envelope coordinates/angles: (specify beam envelope)

$$r_x(s) = 2\langle x^2\rangle_\perp^{1/2} \qquad r_x'(s) = 2\langle xx'\rangle_\perp/\langle x^2\rangle_\perp^{1/2}$$

$$r_y(s) = 2\langle y^2\rangle_\perp^{1/2} \qquad r_y'(s) = 2\langle yy'\rangle_\perp/\langle y^2\rangle_\perp^{1/2}$$

RMS Emittances: (specify phase-space area)

$$\varepsilon_x(s) = 4[\langle x^2\rangle_\perp\langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2]^{1/2}$$

$$\varepsilon_y(s) = 4[\langle y^2\rangle_\perp\langle y'^2\rangle_\perp - \langle yy'\rangle_\perp^2]^{1/2}$$

Perveance: (specify space-charge intensity)

$$Q = \frac{q\lambda(s)}{2\pi\epsilon_0 m\gamma_b^3(s)\beta_b^2(s)c^2}$$

Procedure for Initial Distribution Specification (2)

If the beam is rms matched, we take:

$$r_x'' + \kappa_x r_x - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_x^2}{r_x^3} = 0$$

$$r_y'' + \kappa_y r_y - \frac{2Q}{r_x + r_y} - \frac{\varepsilon_y^2}{r_y^3} = 0$$

$$\kappa_x(s + L_p) = \kappa_x(s)$$
$$\kappa_y(s + L_p) = \kappa_y(s)$$

$$r_x(s + L_p) = r_x(s)$$
$$r_y(s + L_p) = r_y(s)$$

- Not necessary even for periodic lattices
  - Procedure applies to mismatched beams

Procedure for Initial Distribution Specification (3)

Step 2:

Define an rms matched, continuously focused beam in each transverse *s*-slice:

Continuous	s-Varying	
$r_b(s) =$	$\sqrt{r_x(s) r_y(s)}$	Envelope Radius
$\varepsilon_b(s) =$	$\sqrt{\varepsilon_x(s)\varepsilon_y(s)}$	Emittance
$Q(s) =$	$Q(s)$	Perveance

Define a (local) matched beam focusing strength in continuous focusing consistent with the rms beam envelope:

$$\cancel{r_b''} + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon_b^2}{r_b^2} = 0$$

$$\boxed{k_{\beta 0}^2(s) = \frac{Q(s)}{r_b^2(s)} + \frac{\varepsilon_b^2(s)}{r_b^4(s)}}$$

Procedure for Initial Distribution Specification (4)

Step 3:

Specify an rms matched continuously focused equilibrium consistent with step 2:

Specify an equilibrium function:

$$\boxed{f_\perp(x, y, x', y') = f_\perp(H_\perp)} \qquad H_\perp = \frac{1}{2}\mathbf{x}_\perp'^2 + \frac{1}{2}k_{\beta 0}^2\mathbf{x}_\perp^2 + \frac{q\phi}{m\gamma_b^3\beta_b^2c^2}$$

and constrain parameters used to define the equilibrium function $f_\perp(H_\perp)$ with:

$$\lambda = q\int d^2x \int d^2x' \; f_\perp(H_\perp)$$

Line Charge <--> Perveance

$$r_b^2 = \frac{4\int d^2x \int d^2x' \; x^2 f_\perp(H_\perp)}{\int d^2x \int d^2x' \; f_\perp(H_\perp)}$$

rms edge radius

$$\frac{\varepsilon_b^2}{r_b^2} = \frac{4\int d^2x \int d^2x' \; x'^2 f_\perp(H_\perp)}{\int d^2x \int d^2x' \; f_\perp(H_\perp)}$$

rms edge emittance

- This can be rms equivalence with a *smooth* distribution NOT a KV distribution!
- Constraint equations are generally highly nonlinear and must be solved numerically
  - Allows specification of beam with natural accelerations variables
  - Procedures to implement this can be involved (research problem)

Procedure for Initial Distribution Specification (5)

Step 4:

Transform the continuous focused beam coordinates to rms equivalency in the system with *s*-varying focusing:

$$x = \frac{r_x}{r_b} x_i \qquad y = \frac{r_y}{r_b} y_i$$

$$x' = \frac{\varepsilon_x}{\varepsilon_b}\frac{r_b}{r_x} x'_i + \frac{r'_x}{r_b} x_i \qquad y' = \frac{\varepsilon_y}{\varepsilon_b}\frac{r_b}{r_y} y'_i + \frac{r'_y}{r_b} y_i$$

Here, $\{x_i\}$, $\{y_i\}$, $\{x'_i\}$, $\{y'_i\}$ are coordinates of the continuous equilibrium

- Transform reflects structure of linear field Courant-Snyder invariants but applied to the nonuniform beam
  - Approximation effectively treats Hamiltonian as Courant-Snyder invariant
  - Properties of beam nonuniform distribution retained in transform
  - Expect errors to be largest near beam radial "edge" at high space-charge intensity
- If applied to simulations using macroparticles (e.g., PIC codes), then details of transforms must be derived to weight macroparticles
  - Details in: Lund, Kikuchi, Davidson, PRSTAB **12**, 114801 (2009)

Procedure for Initial Distribution Specification (6)

Load N particles in x,y,x',y' phase space consistent with continuous focusing equilibrium distribution $f_\perp(H_\perp)$

Step A (set particle coordinates):

Calculate beam radial number density $n(r)$ by (generally numerically) solving the Poisson/stream equation and load particle x,y coordinates:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

- Radial coordinates r: Set by transforming uniform deviates consistent with $n(r)$
- Azimuthal angles θ: Distribute randomly or space for low noise

Step B (set particle angles):

Evaluate $f_\perp(U,r)$ with $U = \sqrt{x'^2 + y'^2}$ at the particle x, y coordinates loaded in step A to calculate the angle probability distribution function and load x', y' coordinates:

$$x' = U\cos\xi$$

$$y' = U\sin\xi$$

- Radial coordinate U: Set by transforming uniform deviates consistent with $f_\perp(U,r)$
- Azimuthal coordinate ξ: Distribute randomly or space for low noise

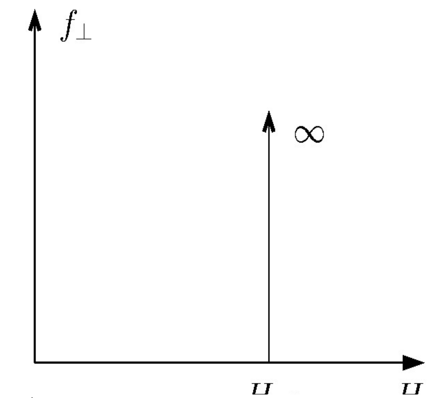
Broad range of choices of continuous focusing equilibrium distributions to apply procedure too: want smooth for physical

Common choices for $f_\perp(H_\perp)$ analyzed in the literature:

1) KV (already covered)

$$\boxed{f_\perp \propto \delta(H_\perp - H_{\perp b})}$$

$$H_{\perp b} = \text{const}$$

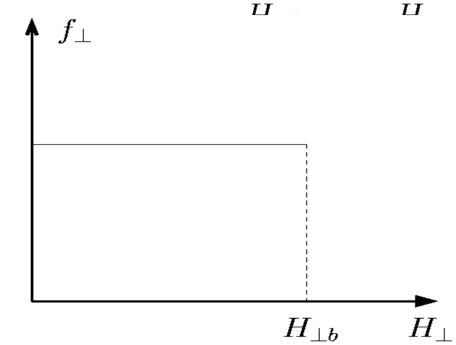


2) Waterbag

[see USPAS + M. Reiser, *Charged Particle Beams*, (1994, 2008)]

$$\boxed{f_\perp \propto \Theta(H_{\perp b} - H_\perp)}$$

$$\Theta(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x \end{cases}$$

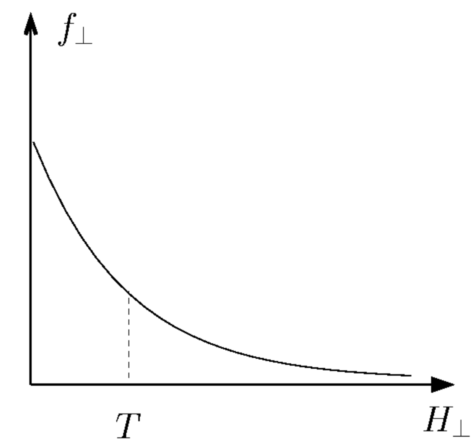


3) Thermal

[see USPAS + M. Reiser; Davidson, *Nonneutral Plasmas*, 1990]

$$\boxed{f_\perp \propto \exp(-H_\perp/T)}$$

$$T = \text{const} > 0$$



Infinity of choices can be made!

- Fortunately, range of behavior can be understood with a few reasonable choices

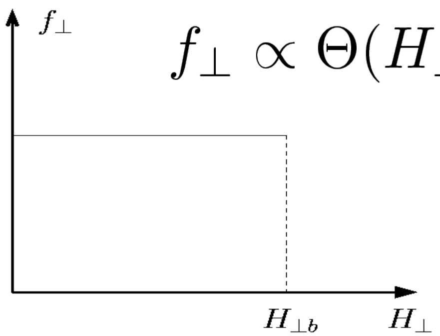
When relative space-charge is strong, all smooth equilibrium distributions expected to look similar out to far edge

Constant charge and focusing: $Q = 10^{-4}$ $\quad k_{\beta 0}^2 = \text{const}$

Vary relative space-charge strength: $\sigma/\sigma_0 = 0.1,\ 0.2,\ \cdots,\ 0.9$

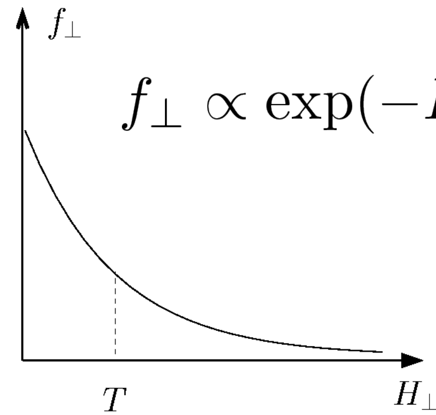
Waterbag Distribution

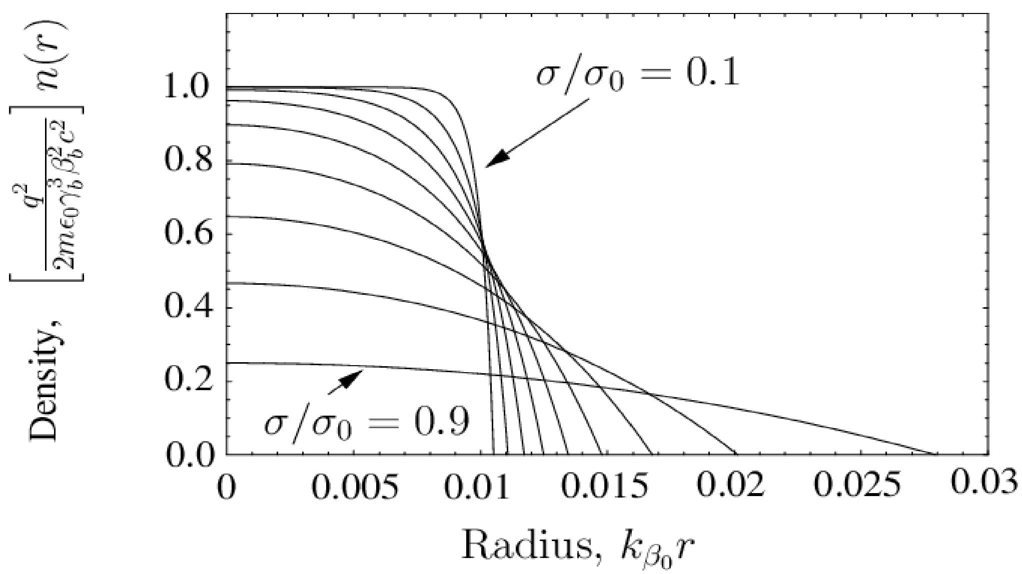
$$f_\perp \propto \Theta(H_{\perp b} - H_\perp)$$

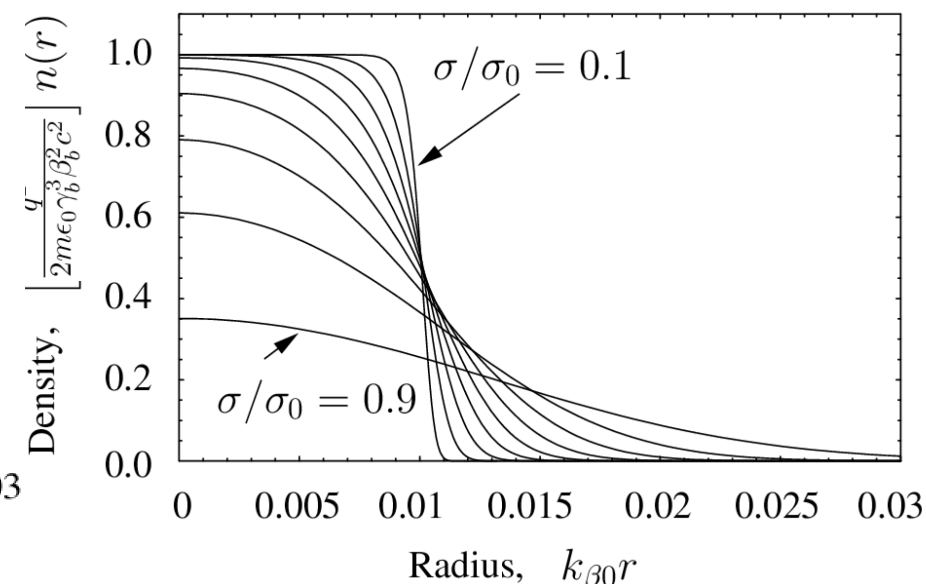


Thermal Distribution

$$f_\perp \propto \exp(-H_\perp/T)$$



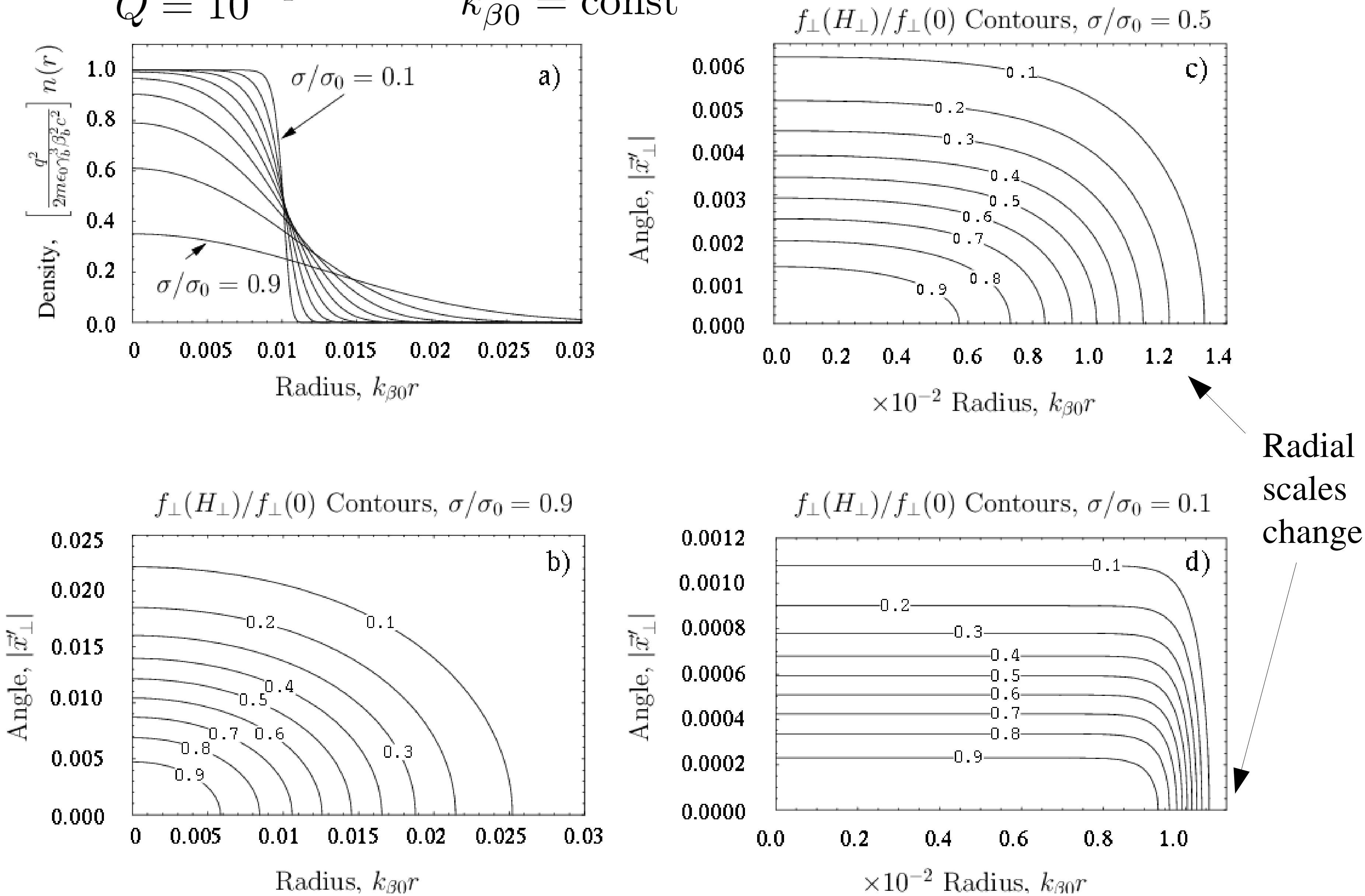




Edge shape varies with distribution choice, but cores similar when σ/σ_0 small

Thermal distribution contours at fixed line charge and focusing strength

$Q = 10^{-4}$ $\qquad$ $k_{\beta 0}^2 = \text{const}$

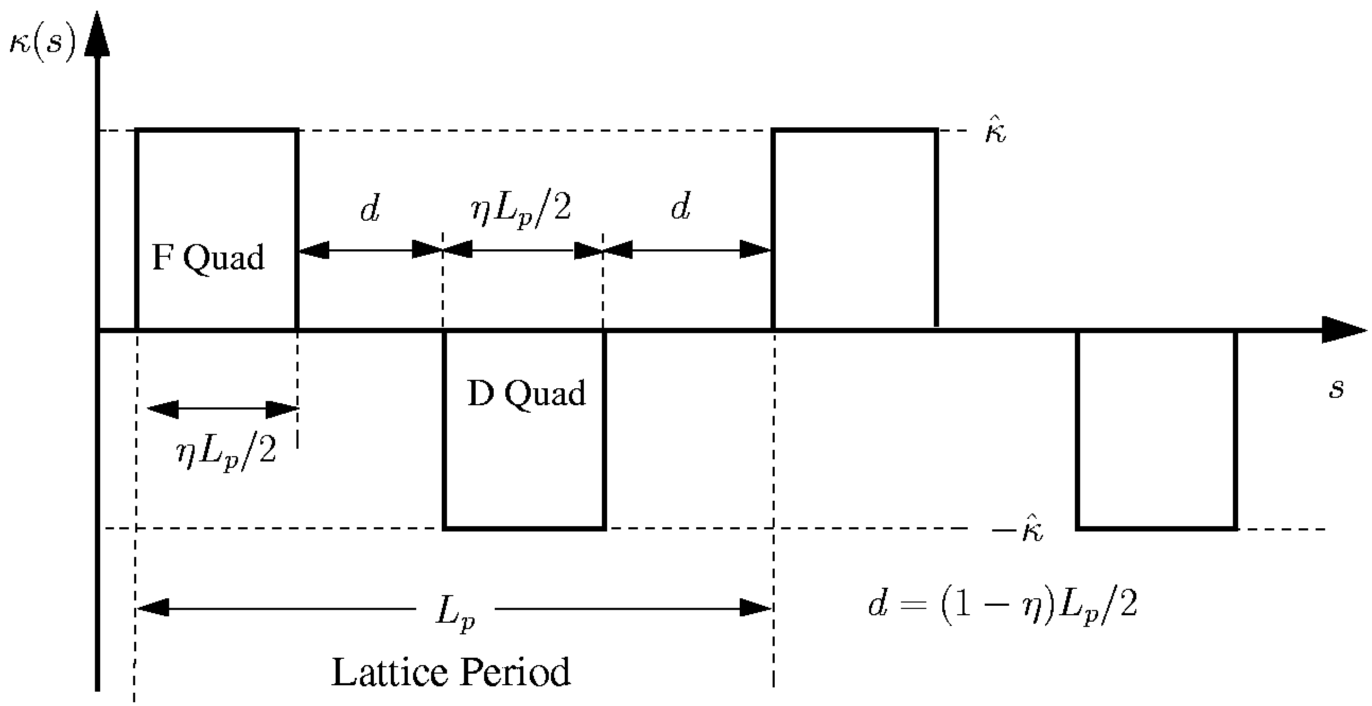


- Particles will move approximately force-free till approaching the edge where it is rapidly bent back: Highly nonlinear particle orbits.

Carry out numerical Vlasov simulations of the initial Pseudoequlibrium distributions to check how procedure works

Use the Warp (PIC) Vlasov code to advance an initial pseudoequilibrium distribution in a periodic FODO lattice to check how significant transient evolutions are period by period:

- Little evolution => suggests near relaxed equilibrium structure



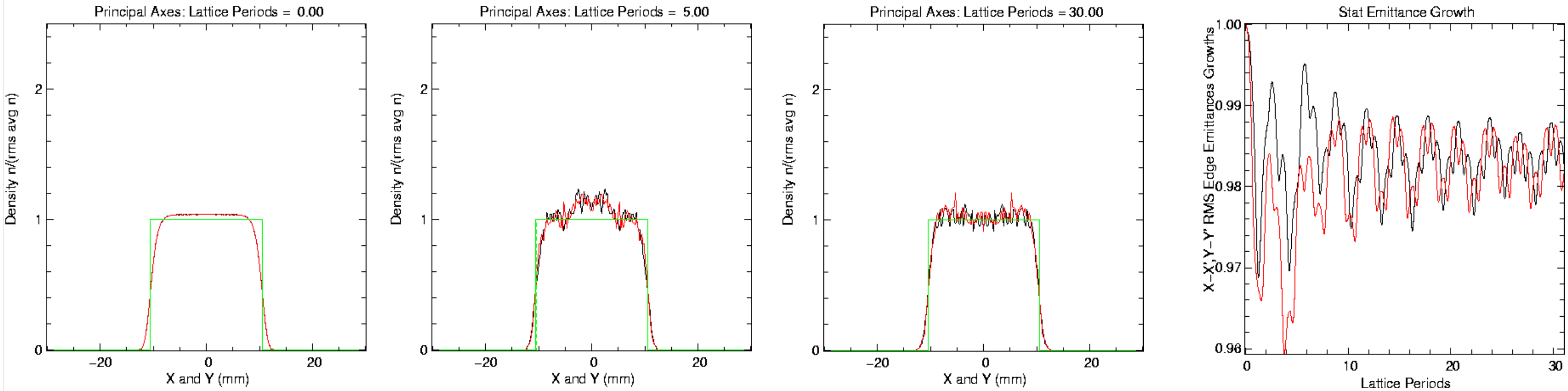
$$\sigma_0 = \text{specified}, \quad L_p = 0.5 \text{ m}, \quad \varepsilon_x = \varepsilon_y = 50 \text{ mm-mrad}$$

$$\sigma/\sigma_0 \;\; \text{adjusted to fix } Q$$

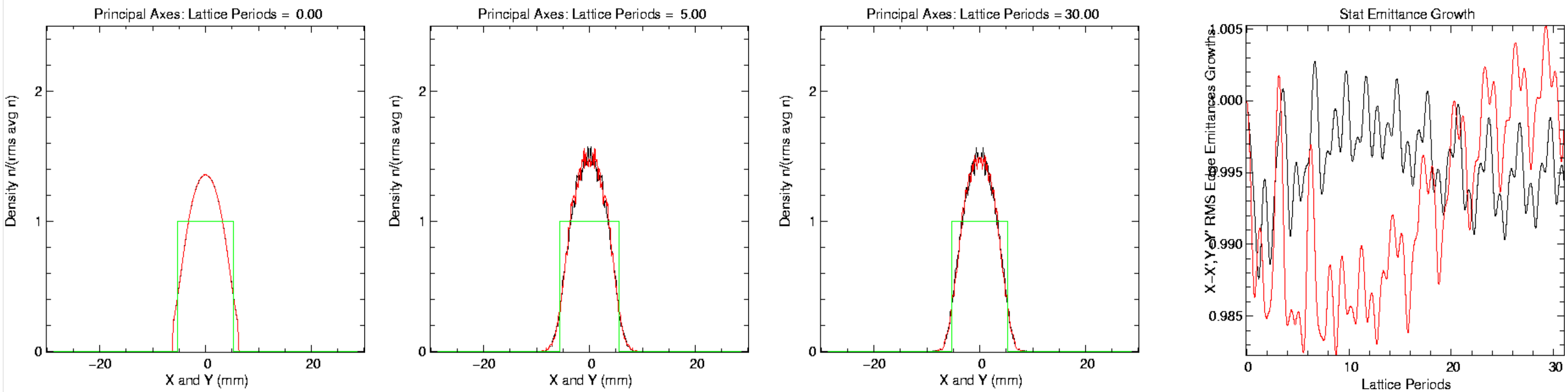
Warp PIC Simulation (see S9) Results – Pseudo Thermal Equilibrium

$$\sigma_0 = 70^\circ, \quad L_p = 0.5 \text{ m}, \quad \varepsilon_x = \varepsilon_y = 50 \text{ mm-mrad}$$

$$\sigma/\sigma_0 = 0.2$$



$$\sigma/\sigma_0 = 0.7$$

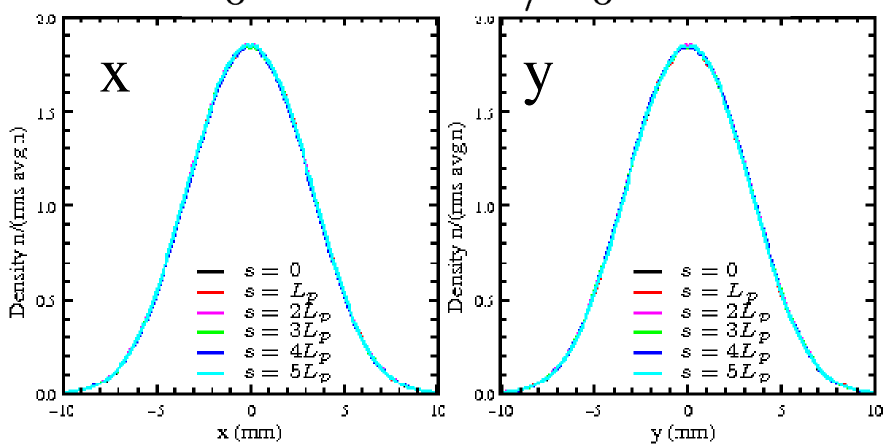


Transient evolution of initial pseudo-equilibrium distributions with thermal core form in a FODO quadrupole focusing lattice

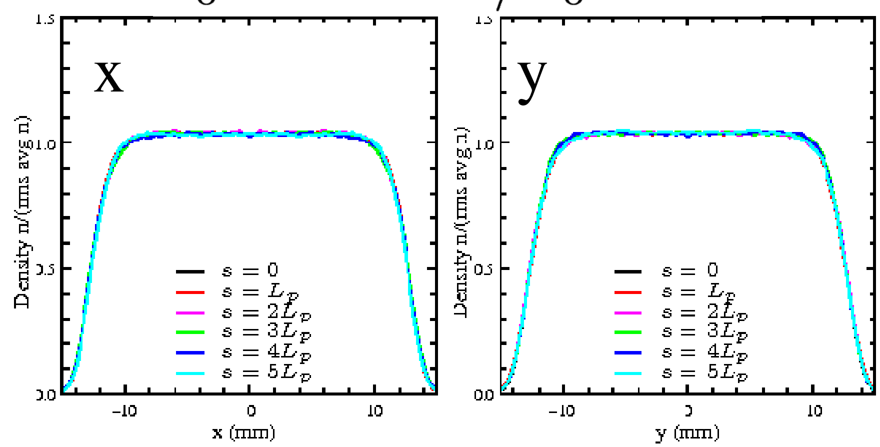
Density profiles along x and y axes

Snapshots at lattice period intervals over 5 periods

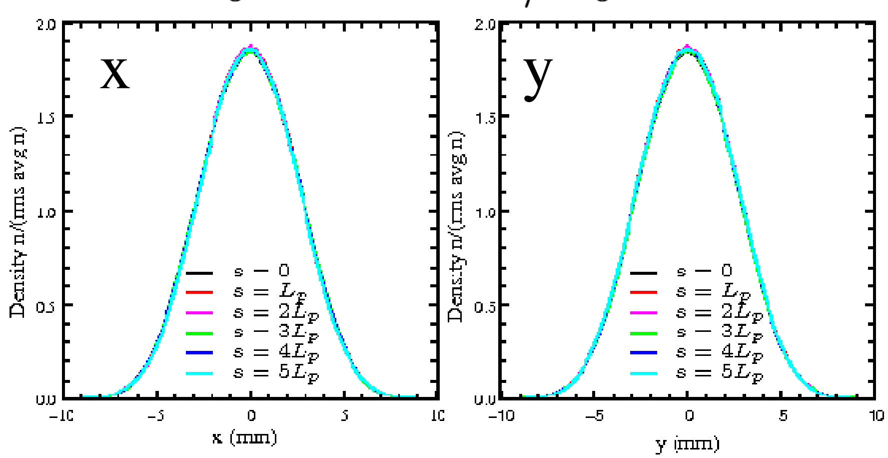
$\sigma_0 = 45^\circ \quad \sigma/\sigma_0 = 0.9$



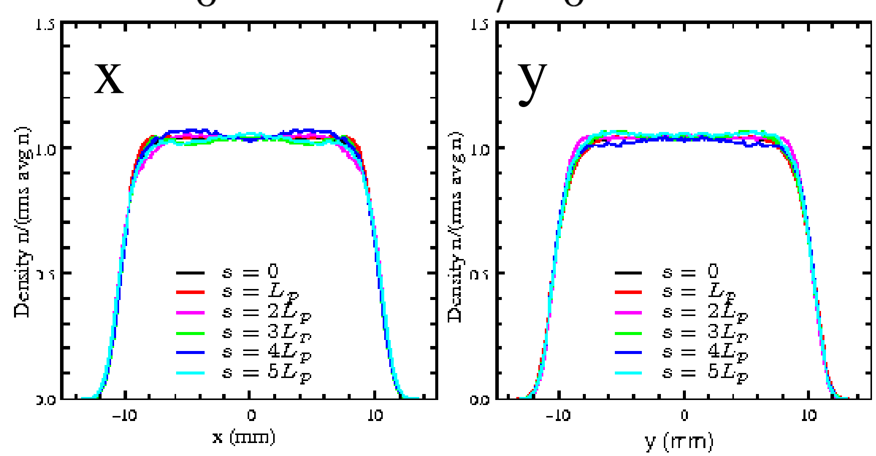
$\sigma_0 = 45^\circ \quad \sigma/\sigma_0 = 0.2$



$\sigma_0 = 70^\circ \quad \sigma/\sigma_0 = 0.9$



$\sigma_0 = 70^\circ \quad \sigma/\sigma_0 = 0.2$

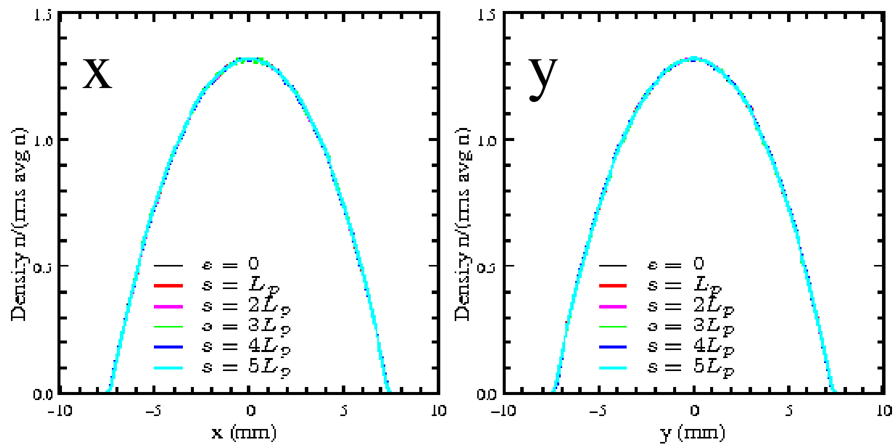


Transient evolution of initial pseudo-equilibrium distributions with waterbag core form in a FODO quadrupole focusing lattice

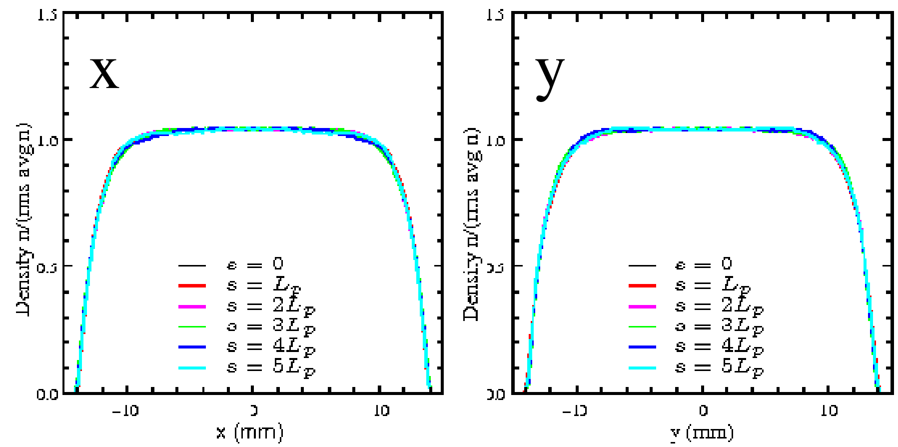
Density profiles along x and y axes

Snapshots at lattice period intervals over 5 periods

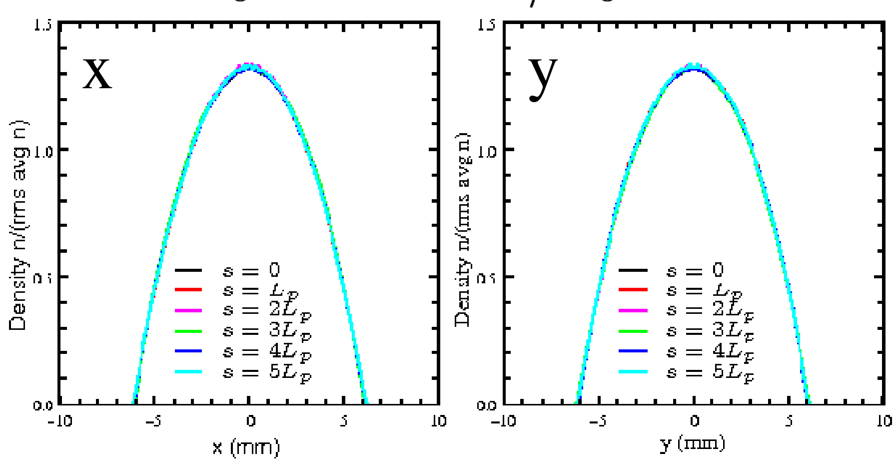
$\sigma_0 = 45^\circ \quad \sigma/\sigma_0 = 0.9$



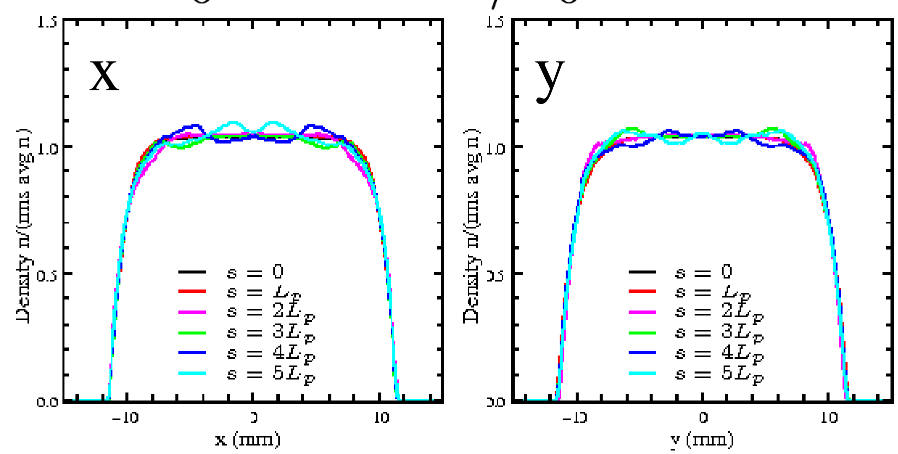
$\sigma_0 = 45^\circ \quad \sigma/\sigma_0 = 0.2$



$\sigma_0 = 70^\circ \quad \sigma/\sigma_0 = 0.9$



$\sigma_0 = 70^\circ \quad \sigma/\sigma_0 = 0.2$

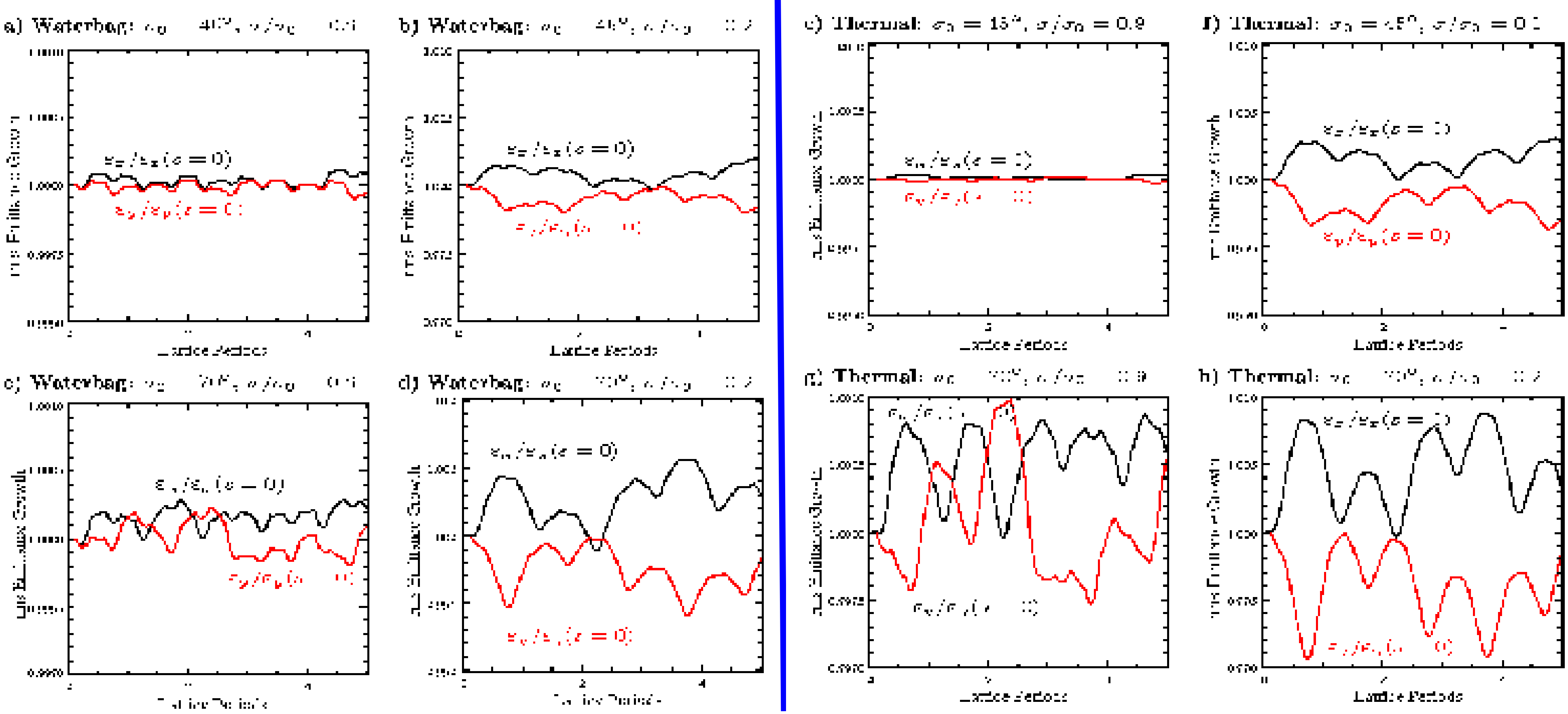


The beam phase-space area (rms emittance measure) changes little during the evolutions indicating near equilibrium form

$$\varepsilon_x = 4\left[\langle x^2\rangle_\perp \langle x'^2\rangle_\perp - \langle xx'\rangle_\perp^2\right]^{1/2} \qquad \text{Plot: } \varepsilon_x(s)/\varepsilon_x(s_i)$$

$$\varepsilon_y = 4\left[\langle y^2\rangle_\perp \langle y'^2\rangle_\perp - \langle yy'\rangle_\perp^2\right]^{1/2} \qquad \text{Plot: } \varepsilon_y(s)/\varepsilon_y(s_i)$$

Waterbag Form | Gaussian/Thermal Form



Max Ranges 0.1 % Variation!!

Compare pseudo-equilibrium loads with other accelerator loads

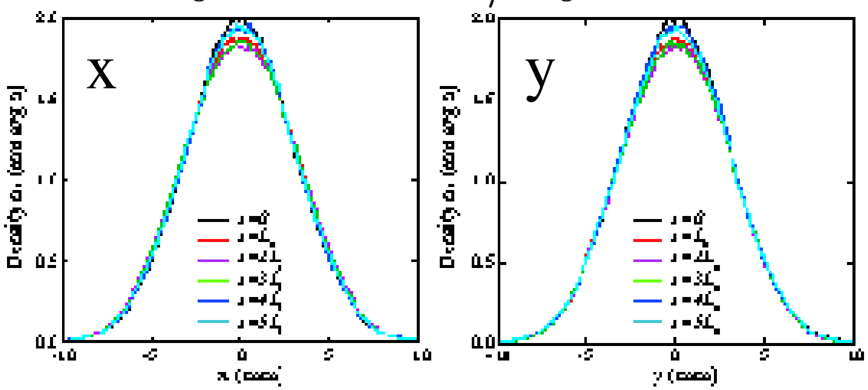
Comparison distribution from linear-field Courant-Snyder invariants

Batygin, Nuc. Inst. Meth. A **539**, 455 (2005)

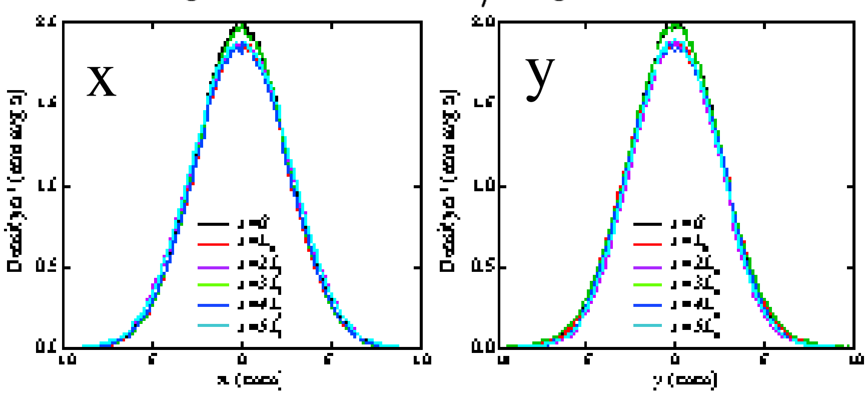
Thermal/Gaussian forms with weak space-charge

Linear-Field Courant-Snyder:

$\sigma_0 = 45^\circ \quad \sigma/\sigma_0 = 0.9$

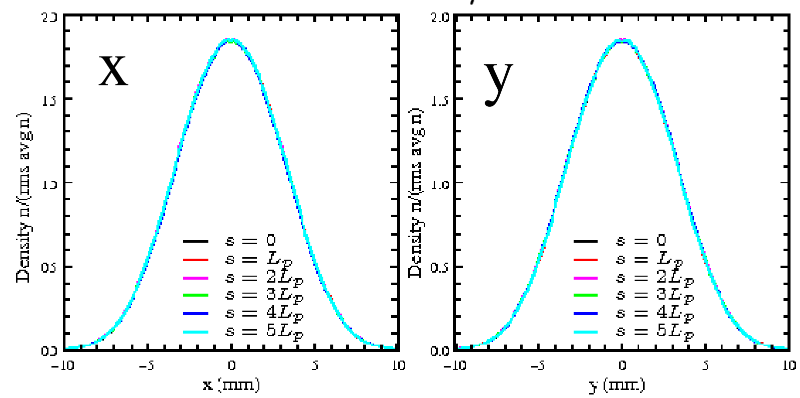


$\sigma_0 = 70^\circ \quad \sigma/\sigma_0 = 0.9$

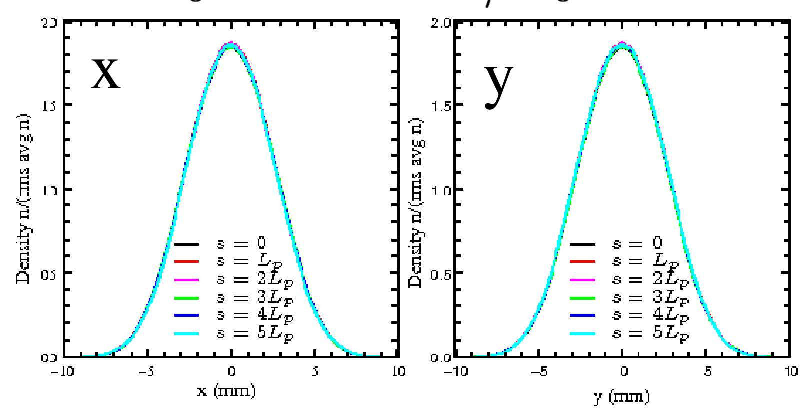


Pseudo-Equilibrium

$\sigma_0 = 45^\circ \quad \sigma/\sigma_0 = 0.9$



$\sigma_0 = 70^\circ \quad \sigma/\sigma_0 = 0.9$



Compare pseudo-equilibrium loads with other accelerator loads

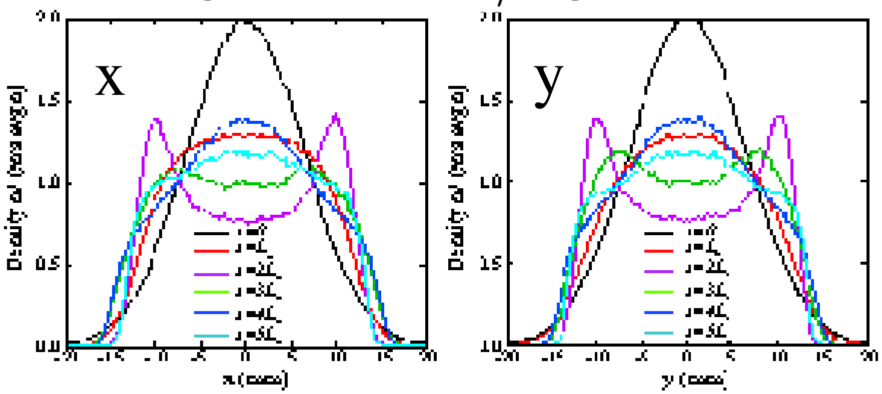
Comparison distribution from linear-field Courant-Snyder invariants

Batygin, Nuc. Inst. Meth. A **539**, 455 (2005)

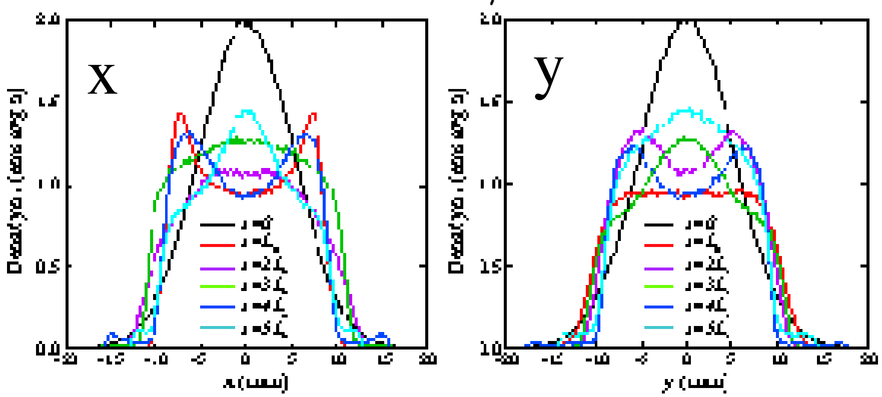
Thermal/Gaussian forms with strong space-charge

Linear-Field Courant-Snyder:

$\sigma_0 = 45^\circ \quad \sigma/\sigma_0 = 0.2$

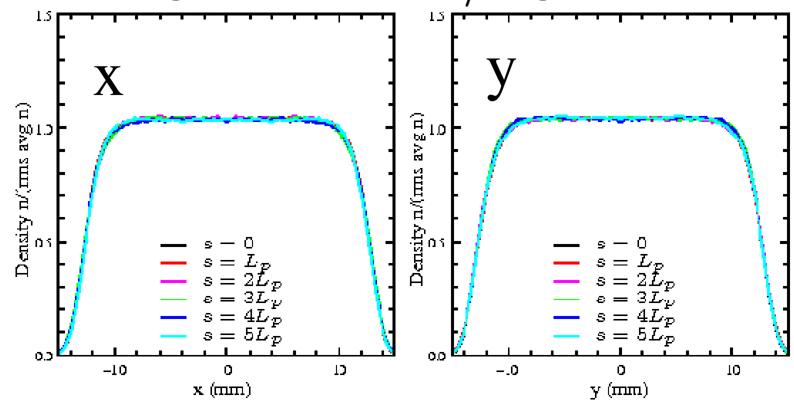


$\sigma_0 = 70^\circ \quad \sigma/\sigma_0 = 0.2$

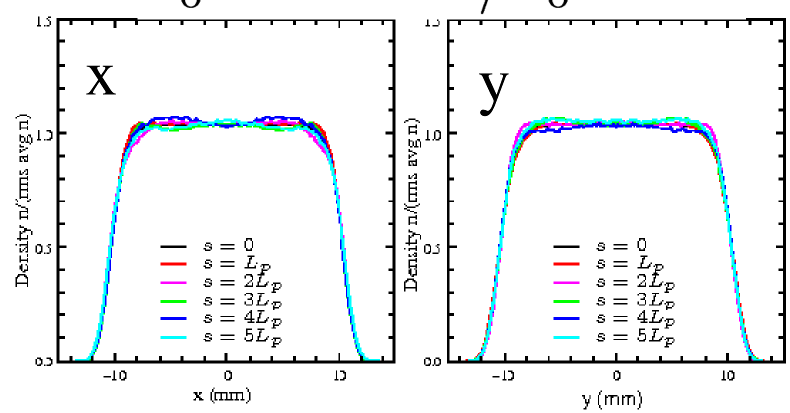


Pseudo-Equilibrium

$\sigma_0 = 45^\circ \quad \sigma/\sigma_0 = 0.2$



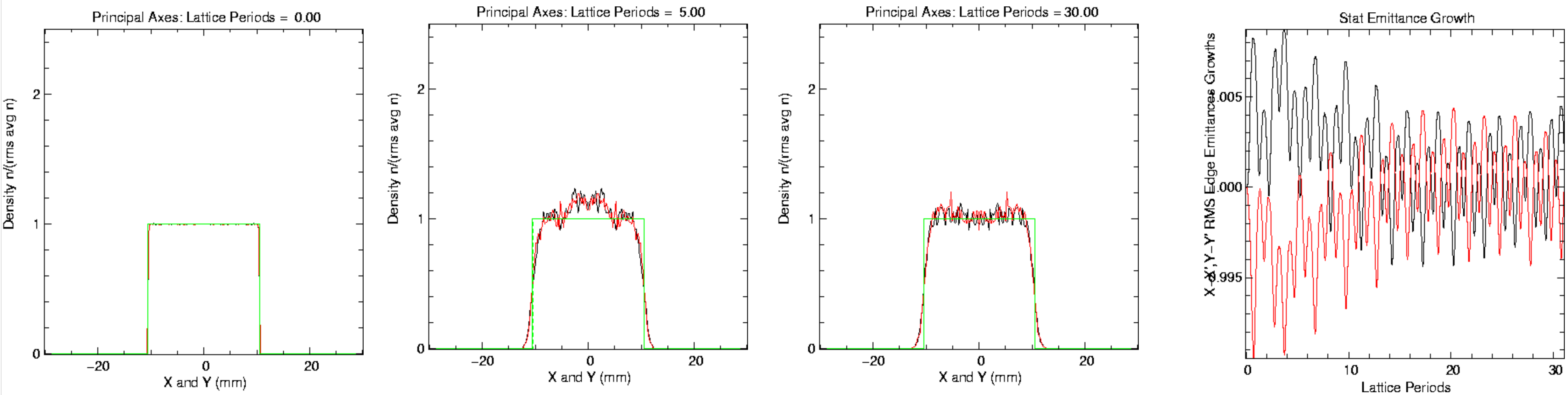
$\sigma_0 = 70^\circ \quad \sigma/\sigma_0 = 0.2$



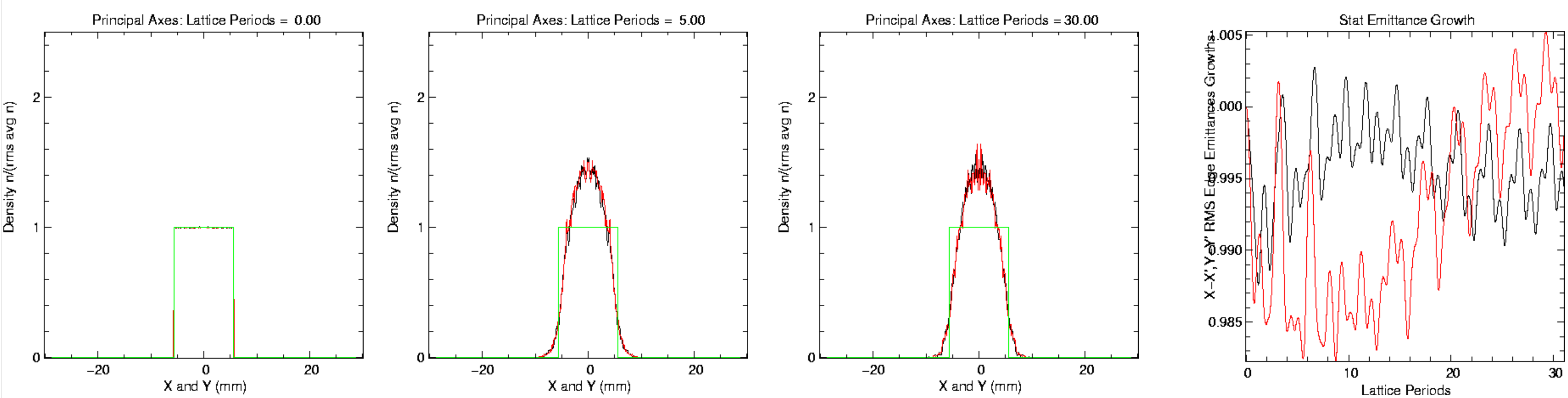
Warp PIC Simulation (see S9) Results – Semi-Gaussian (for contrast)

$$\sigma_0 = 70^\circ, \quad L_p = 0.5\ \mathrm{m}, \quad \varepsilon_x = \varepsilon_y = 50\ \mathrm{mm\text{-}mrad}$$

$$\sigma/\sigma_0 = 0.2$$



$$\sigma/\sigma_0 = 0.7$$



Summary: Results suggest near equilibrium structure with good quiescent transport can be obtained for a broad range of beam parameters with a smooth distribution core loaded using the pseudoequilibrium construction

Find:

- Works well for quadrupole transport for $\sigma_0 \lesssim 85^\circ$
  - Should not work where beam is unstable and all distributions are expected to become unstable for $\sigma_0 >\sim 85^\circ$ see lectures on Transverse Kinetic Stability:

    Experiment: Tiefenback, Ph.D. Thesis, U.C. Berkeley (1986)

    Theory: Lund and Chawla, Proc. 2005 Part. Accel. Conf.

- Works better when matched envelope has less "flutter":
  - Solenoids: larger lattice occupancy η
  - Quadrupoles: smaller σ_0
  - Not surprising since less flutter" corresponds to being closer to continuous focusing

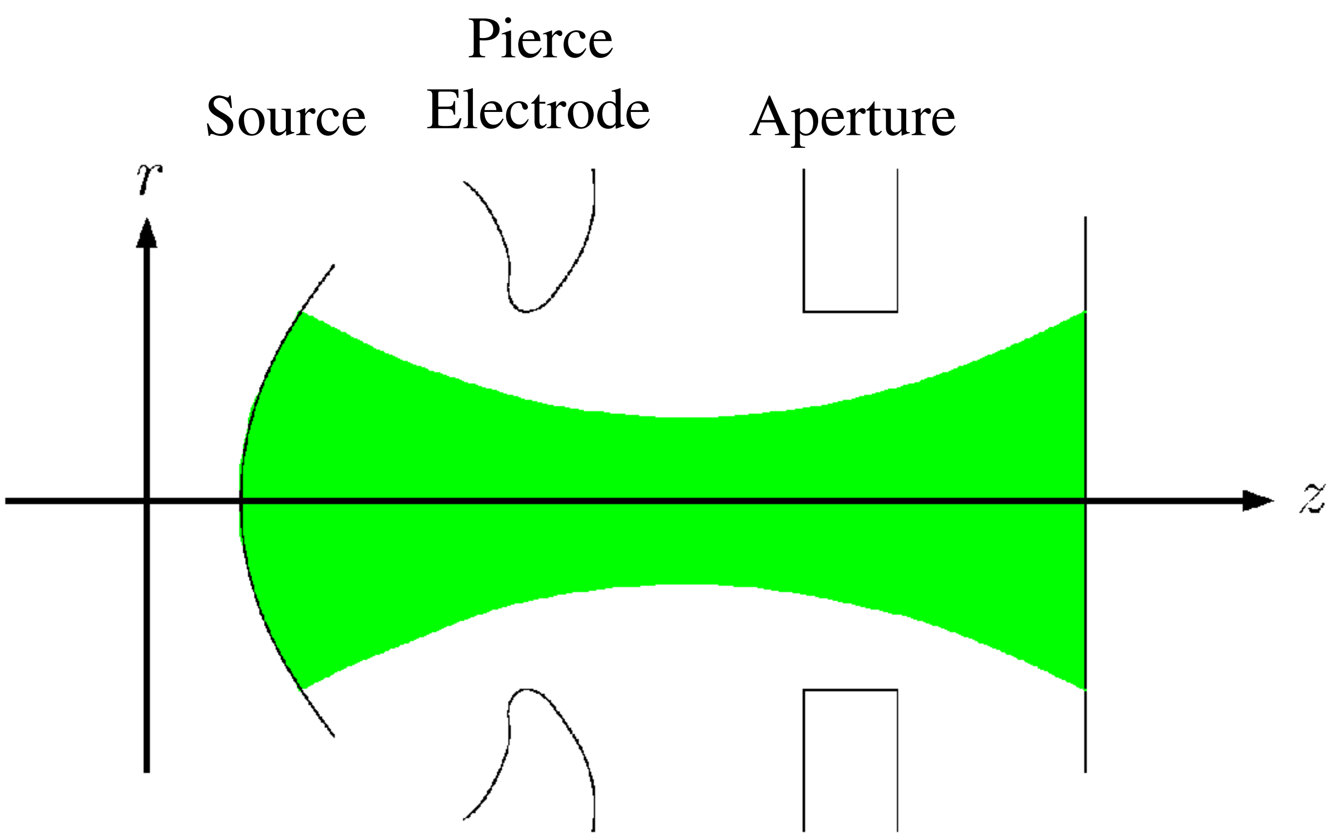
Comments on Procedure for Initial Distribution Specification

- Applies to both 2D transverse and 3D beams
- Easy to generalize procedure for beams with centroid offsets
- Generates a charge distribution with elliptical symmetry
  - Sacherer's results on rms equivalency apply
  - Distribution will reflect self-consistent Debye screening
- Equilibria are only pseudo-equilibria since transforms are not exact
  - Nonuniform space-charge results in errors
  - Transform consistent with preserved Courant-Snyder invariants for uniform density beams
  - Errors largest near the beam edge - expect only small errors for very strong space charge where Debye screening leads to a flat density profile with rapid fall-off at beam edge
- Many researchers have presented or employed aspects of the improved loading prescription presented here, including:

I. Hofmann, GSI	M. Reiser, U. Maryland	K. Sonnadi, KEK
E. Startsev, PPPL	Y. Batygin, SLAC	Y. Struckmeir, GSI

F: Injection of Distribution off a Source

Sorry, no time. To be added in future versions



Corrections and suggestions for improvements welcome!

These notes will be corrected and expanded for reference and for use in future editions of US Particle Accelerator School (USPAS) and Michigan State University (MSU) courses. Contact:

Prof. Steven M. Lund
Facility for Rare Isotope Beams
Michigan State University
640 South Shaw Lane
East Lansing, MI 48824

lund@frib.msu.edu
(517) 908 – 7291 office
(510) 459 - 4045 mobile

Please provide corrections with respect to the present archived version at:

https://people.nscl.msu.edu/~lund/uspas/scs_2016

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C.K. Birdsall and A.B. Langdon, *Plasma Physics via Computer Simulation*, McGraw-Hill Book Company (1985)

R.W. Hockney and J.W. Eastwood, *Computer Simulation using Particles*, Institute of Physics Publishing (1988)

Review of Initial Distribution Loads

S. Lund, T. Kikuchi, and R. Davidson, “Generation of initial kinetic distributions for simulation of long-pulse charged particle beams with high space-charge intensity,” PRSTAB **12**, 114801 (2009)

Review of Envelope Equations and Stability Properties

S.M. Lund and B. Bukh, “Properties of the transverse envelope equations describing intense ion beam transport, PRSTAB **7**, 024801 (2004)

Intense Beam Stability in Quadrupole and Solenoid Periodic Focusing Channels

Tiefenback, Ph.D. Thesis, U.C. Berkeley (1986)

Lund and Chawla, NIMA **561**, 203 (2006)