Intro. Lecture 07: Numerical Convergence*

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US Particle Accelerator School (USPAS) Lectures On
"Self-Consistent Simulations of Beam and Plasma Systems"
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Detailed Outline
Introductory Lectures on Self-Consistent Simulations

Numerical Convergence

A. Overview

B. Resolution: Advance Step
- Courant Conditions
- Applied Field Structures
- Collective Waves

C. Resolution: Spatial Grid
- Beam Edge
- Collective Waves
- Electrostatic Structures on Mesh

D. Statistics
- Debye Screening
- Classes of Particle Simulations

E. Illustrative Examples with the Warp Code
- Weak Space-Charge
- Intermediate Space-Charge
- Strong Space-Charge
- Strong Space-Charge with Instability

Numerical Convergence
A: Overview

Numerical simulations must be checked for proper resolution and statistics to be confident that answers obtained are correct and physical:

Resolution of discretized quantities

- Time t or axial s step of advance
- Spatial grid of fieldsolve
- For direct Vlasov: the phase-space grid

Statistics for PIC

- Number of macroparticles used to represent Vlasov flow to control noise
  - Vlasov flow represented by markers a finite number results in deviations from continuum model

Increased resolution and statistics generally require more computer resources (time and memory) to carry out the required simulation. It is usually desirable to carry out simulations with the minimum resources required to achieve correct, converged results that are being analyzed. Unfortunately, there are no set rules on adequate resolution and statistics. What is required generally depends on:

- What quantity is of interest
- How long an advance is required
- What numerical methods are being employed .....

General Guidance on Numerical Convergence Issues

Although it is not possible to give detailed rules on numerical convergence issues, useful general guidance can be given:

- Find results from similar problems using similar methods when possible
- Analyze quantities that are easy to interpret and provide good measures of convergence for the use of the simulation
  - Some moments like rms emittances:

$$\varepsilon_x = 4\left[\langle \tilde{x}^2 \rangle_\perp \langle \tilde{x}'^2 \rangle_\perp - \langle \tilde{x}\tilde{x}' \rangle_\perp^2\right]^{1/2} \qquad \begin{aligned} \tilde{x} &\equiv x - \langle x \rangle_\perp \\ \tilde{x}' &\equiv x' - \langle x' \rangle_\perp \end{aligned}$$

    can provide relatively sensitive and easy to interpret measures of relative phase-space variations induced by numerical effects when plotted as overlaid time (or *s*) evolution "histories"
- Compare to simulations of similar problems using similar methods
- Benchmark code against problems with known analytical solutions
  - Apply a variety of numerical methods to judge which applies best
- Benchmark code against established, well verified simulation tools
  - Use different numerical methods expected to be more or less accurate

- Recheck convergence whenever runs differ significantly or when different quantities are analyzed
  - What is adequate for one problem/measure may not be for another
  - Ex: rms envelope evolution easier to converge than collective modes
- Although it is common to increase resolution and statistics till quantities do not vary, it is *also* useful to purposefully analyze poor convergence so characteristics of unphysical errors can be recognized
  - Learn characteristic signature of failures to resolve effects so subtle onset issues can be recognized more easily
- Expect to make *many* setup, debugging, and convergence test runs for each useful series of simulations carried out

B: Resolution: Advance Step

Discussion is applicable to advancing particles in the axial machine coordinate (s) or time (*t*). We will present the discussion in terms of the timestep Δ_t

Courant Conditions

- Particles should not move more than one spatial mesh cell in a single timestep

$$v_j \Delta_t < \Delta_j$$

2D *xy*: $j = x, y$
2D *rz*: $j = r, z$
3D *xyz*: $j = x, y, z$

Essence of condition is that data should have time to propagate to the spatial range of relevance on which the numerical method is formulated.

- EM waves should not propagate more than one spatial mesh cell in a single timestep

$$v_g \Delta_t < \Delta_j$$

$$v_g = |\text{wave group velocity}|$$

Condition is demanding for fast electromagnetic waves with $v_g \sim c$

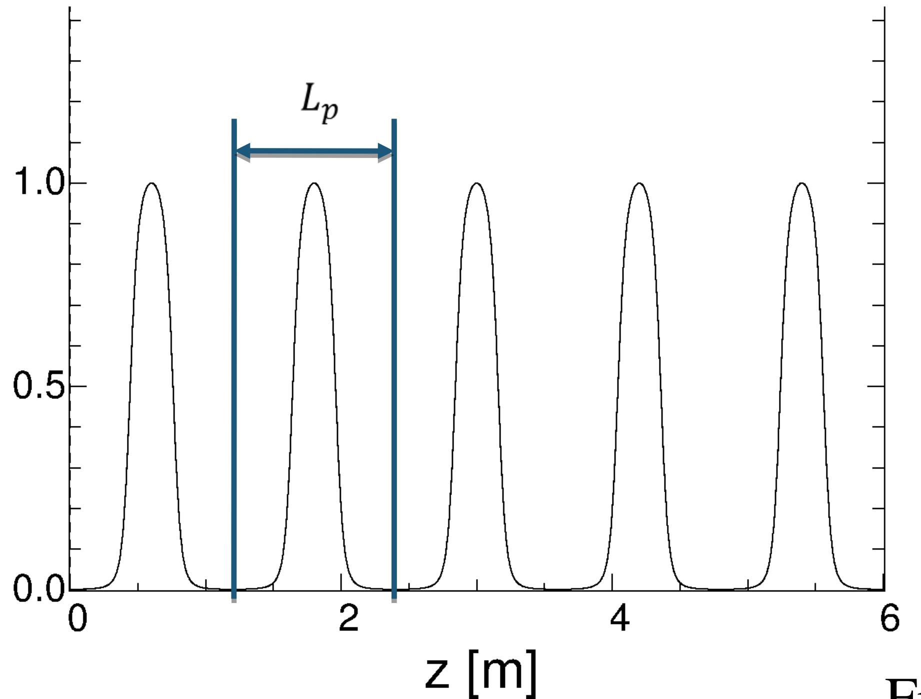
Resolution of Applied Field Structures

Enough steps should be taken to adequately resolve applied field structures

- Characteristic axial variation of fringe field entering/exiting optics should be well resolved by the step size to negligibly small values
  - What constitutes small enough depends on problem
  - Example: found solenoids needed 10e-4 resolution of peak field to respect canonical angular momentum conservation to degree needed

Example: Periodic Solenoid Lattice

- Detailed magnet model
- Scaled B_z on-axis vs z



From C.Y. Wong

$$v_z \Delta_t < \lambda \qquad \lambda = \text{shortest wavelength of variation of applied field}$$

Comment:

- In addition, if there is a local error in the field, one must choose an advance increment consistent with resolution

Resolution of collective oscillations (waves)

- For a leap-frog mover this requires minimally

$$\tau = \text{period of wave component}$$

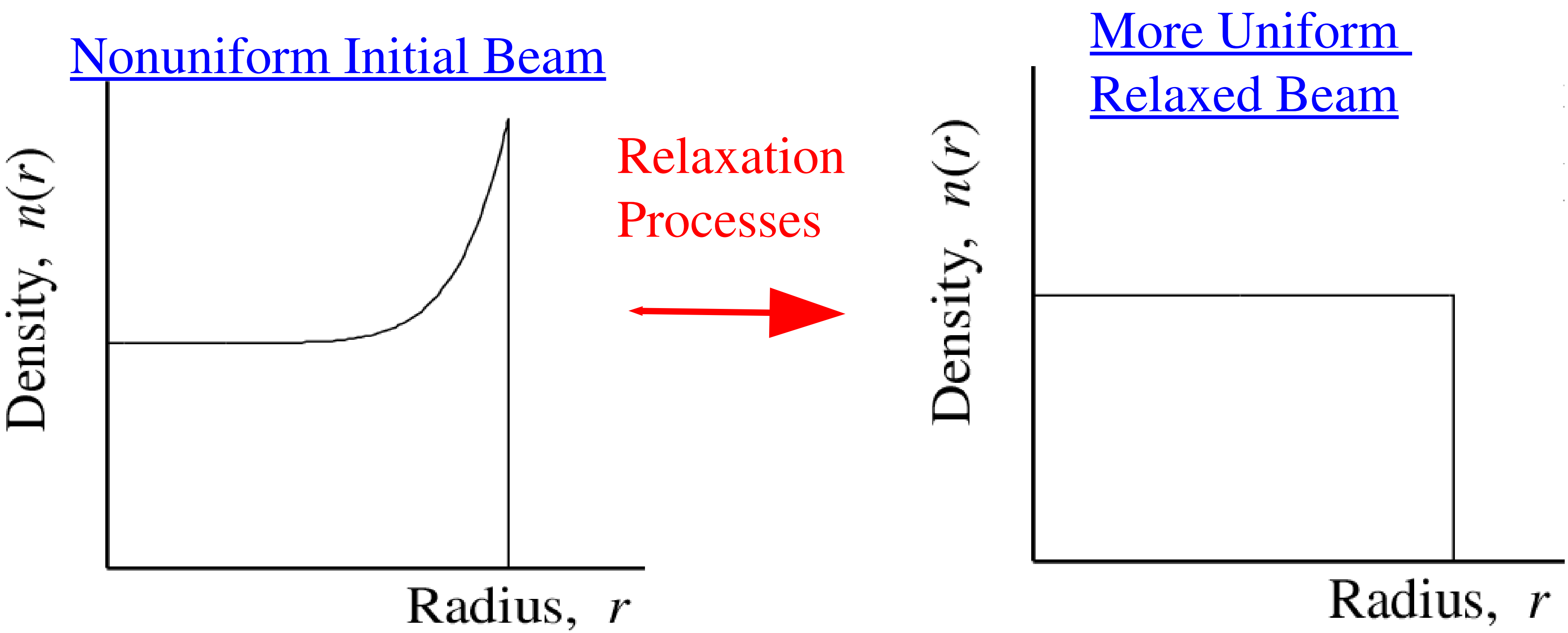
$$\frac{\Delta_t}{\tau} < \frac{1}{2}$$

Collective modes can have high harmonic components which evolve rapidly (at harmonics of the plasma frequency) rending resolution issues difficult. See USPAS notes on *Beam Physics with Intense Space Charge*, Transverse Kinetic Stability

Topic is difficult, but very important. Take a digression on space charge waves to understand demands

Digression: Space charge waves in beams

Strong space charge and Debye screening takes to make the beam density profile flat in a linear focusing channel if the beam starts out nonuiform due to nonlinear errors/aberrations



Reference: High resolution self-consistent PIC simulations shown in class

- Continuous focusing and a more realistic FODO transport lattice
  - Relaxation more complete in FODO lattice due to a richer frequency spectrum
- Relaxations surprisingly rapid: few undepressed betatron wavelengths observed in simulations

Digression: Space charge waves in beams

The space-charge profile of intense beams can be born highly nonuniform out of nonideal (real) injectors or become nonuniform due to a variety of (error) processes. Also, low-order envelope matching of the beam may be incorrect due to focusing and/or distribution errors.

How much emittance growth and changes in other characteristic parameters may be induced by relaxation of characteristic perturbations?

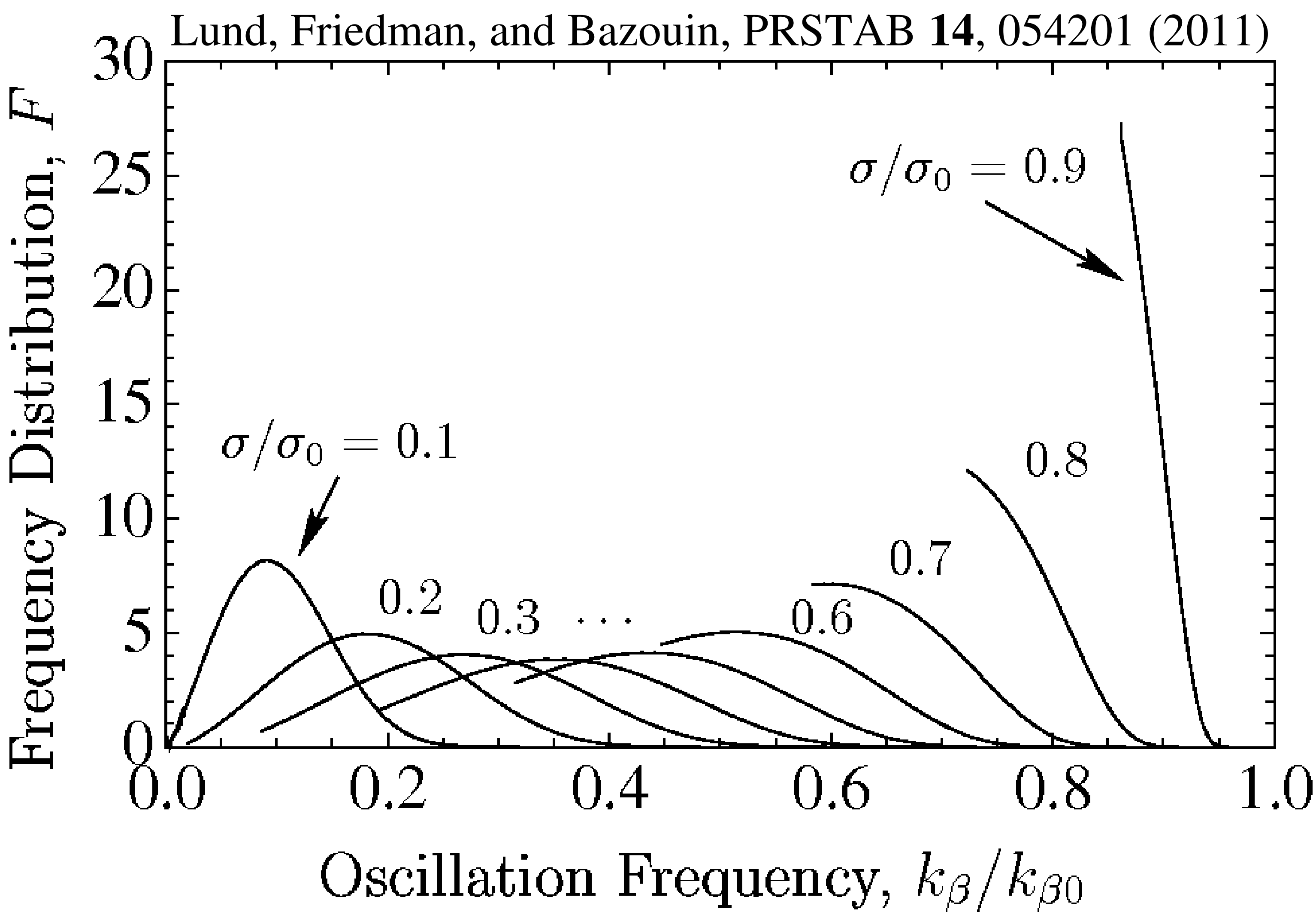
- Employ Global Conservation Constraints of system to bound possible changes
- Assume full relaxation to a final, uniform density state for simplicity

What is the mechanism for the assumed relaxation?

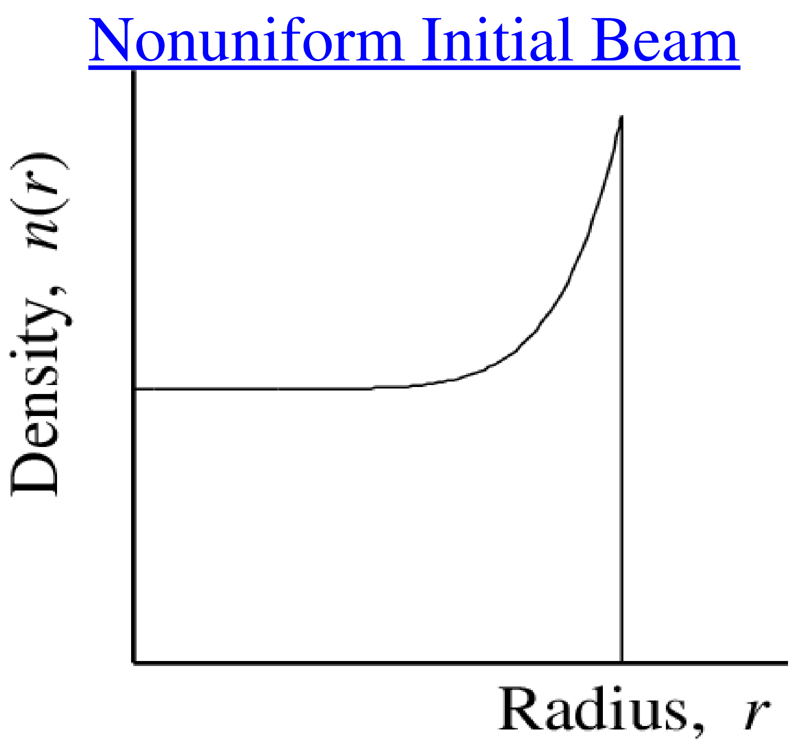
- Collective modes launched by errors will have a broad spectrum
  - Phase mixing can smooth nonuniformities – mode frequencies incommensurate
- Nonlinear interactions, Landau damping, interaction with external errors, ...
- Certain errors more/less likely to relax:
  - Internal wave perturbations expected to relax due to many interactions
  - Envelope mismatch will not (coherent mode) unless amplitudes are very large producing copious halo and nonlinear interactions

Motivation for rapid phase-mixing mechanism for beams with intense space-charge: strong spread in distribution of particle oscillation frequencies in the core of the beam

Distribution of particle oscillation frequences in a smooth beam with nonlinear space-charge forces (sheet beam model for simple curves to illustrate)



Analyze/simulate an initial nonuniform beam to better understand what happens



Initial Nonuniform Beam Parameterization

$$n(r) = \begin{cases} \hat{n}\left[1 + \frac{1-h}{h}\left(\frac{r}{r_e}\right)^p\right], & 0 \leq r \leq r_e \\ 0, & r_e < r \leq r_p \end{cases}$$

$h =$ hollowing parameter $= n(r=0)/n(r=r_e)$

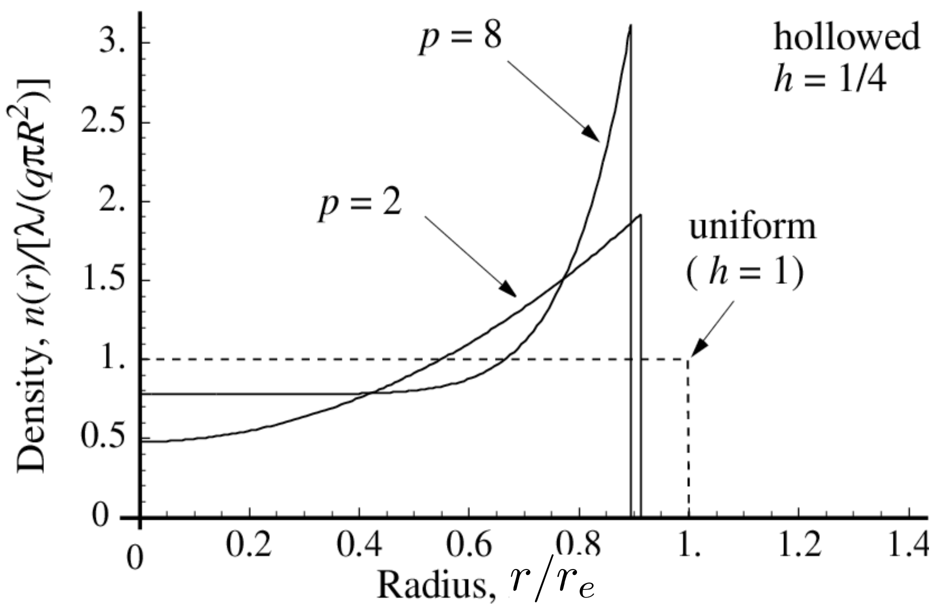
$p =$ radial index

$r_e =$ edge radius

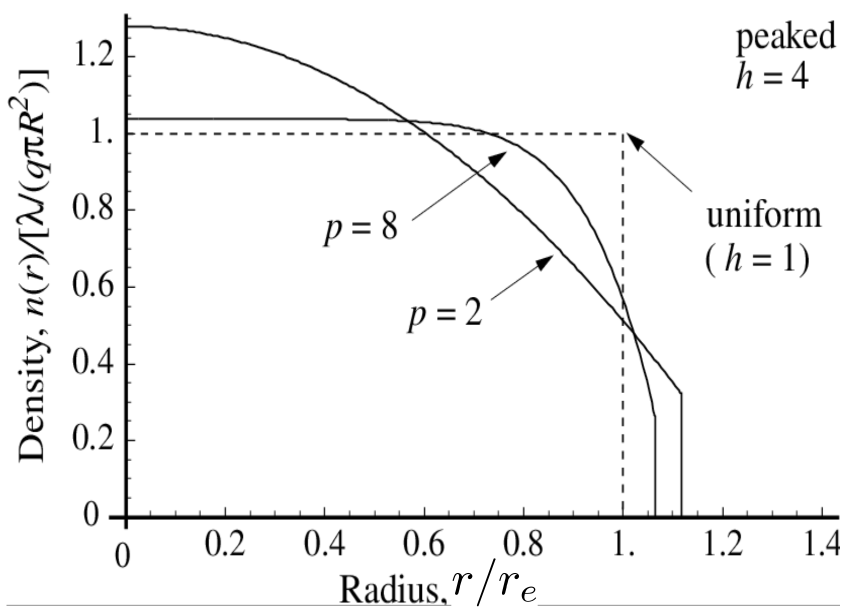
$$\lambda = \int d^2x_\perp \, n = \pi q \hat{n} r_e^2 \left[\frac{(ph+2)}{(p+2)h}\right] \qquad r_b = 2\langle x^2 \rangle_\perp^{1/2} = \sqrt{\frac{(p+2)(ph+4)}{(p+4)(ph+2)}} r_e$$

Normalize profiles to compare common rms radius (r_b) and total charge (λ)

Hollowed Initial Density



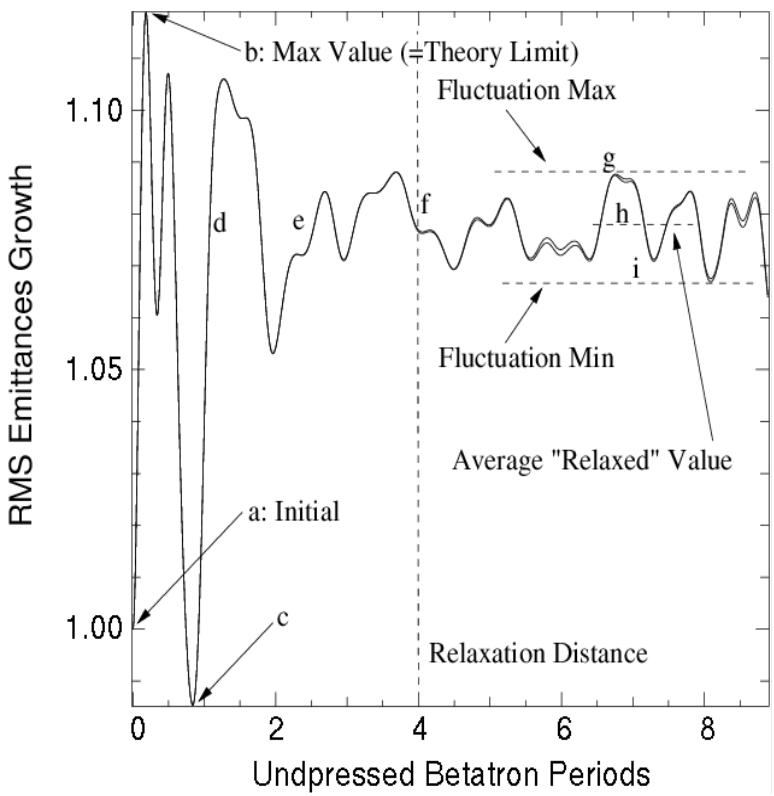
Peaked Initial Density

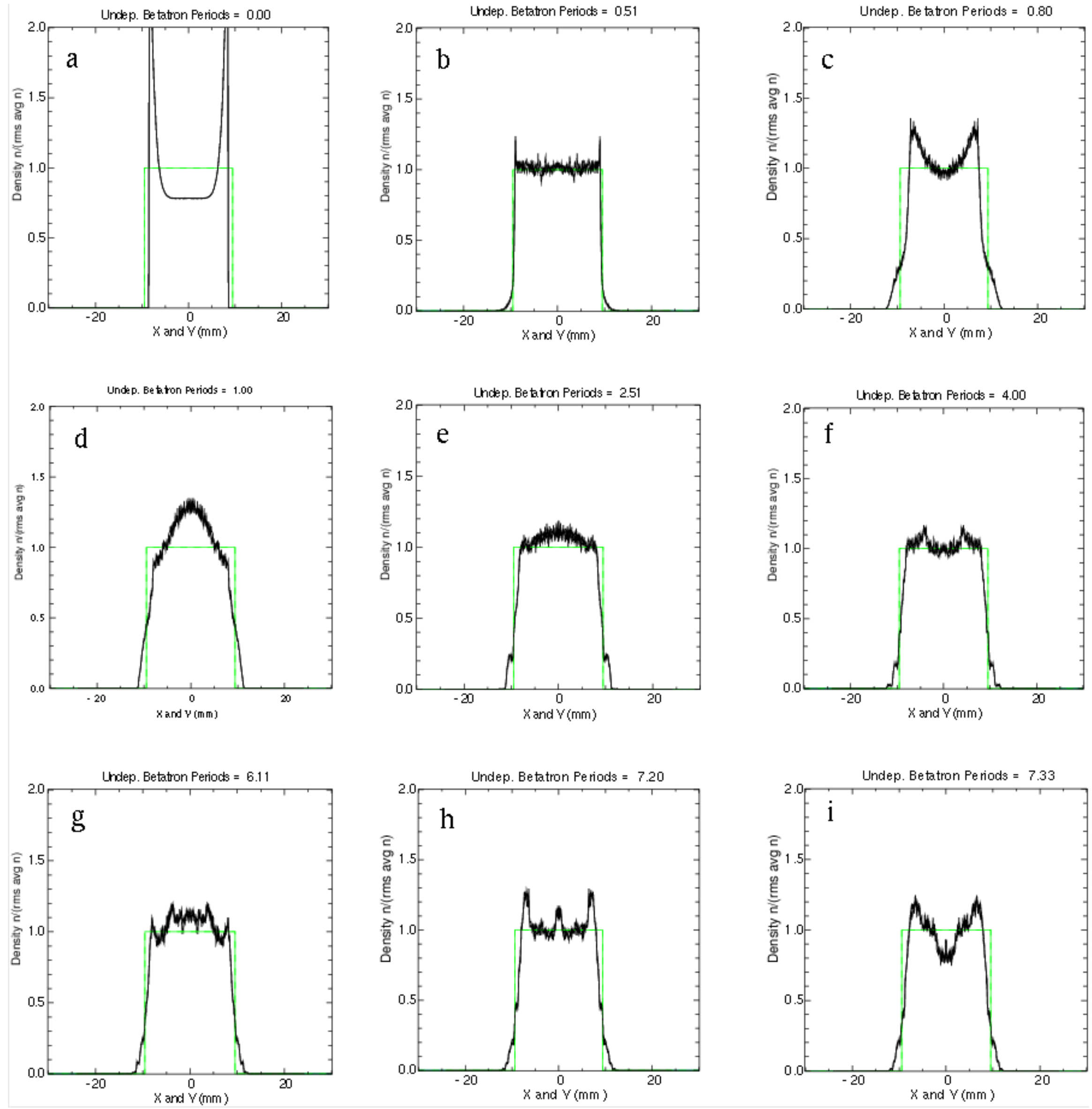


- Analogous definitions are made for the radial temperature profile of the beam

Example Simulation, Initial Nonuniform Beam

$\sigma/\sigma_0 = 0.2$ Initial density: *h*=1/4, *p*=8 Initial Temp: *h* = infinity, *p*=2





Show movie of evolution

[Lund, Grote, and Davidson, Nuc. Instr. Meth. A 544, 472 (2005)]

Hollowed beam simulation/theory results for strong space-charge

- Peaked beam shows very small emittance growth

Initial beam					Relaxed and transient beam		
σ_i/σ_0	Density		Temperature		Emittance growth		Undep. betatron periods to relax
	h	p	h	p	Theory	Simulation	
0.1	0.25	4	1	arb.	1.57	1.42 (1.57, 1.31–1.52)	3.5
			∞	2		1.45 (1.57, 1.38–1.52)	3.0
			0.5			1.41 (1.57, 1.30–1.52)	3.0
	0.25	8	1	arb.	1.43	1.33 (1.43, 1.28–1.38)	3.5
			∞	2		1.35 (1.43, 1.30–1.40)	4.5
			0.5			1.32 (1.43, 1.26–1.38)	4.0
0.20	0.25	4	1	arb.	1.17	1.11 (1.16, 1.09–1.13)	4.5
			∞	2		1.12 (1.16, 1.10–1.13)	3.0
			0.5			1.11 (1.16, 1.09–1.13)	4.0
	0.25	8	1	arb.	1.12	1.08 (1.12, 1.06–1.09)	5.5
			∞	2		1.08 (1.12, 1.07–1.09)	4.0
			0.5			1.08 (1.12, 1.06–1.09)	4.5

Theory results based on conservation of system charge and energy used to calculate the change in rms edge radius between initial (*i*) and final (*f*) matched beam states

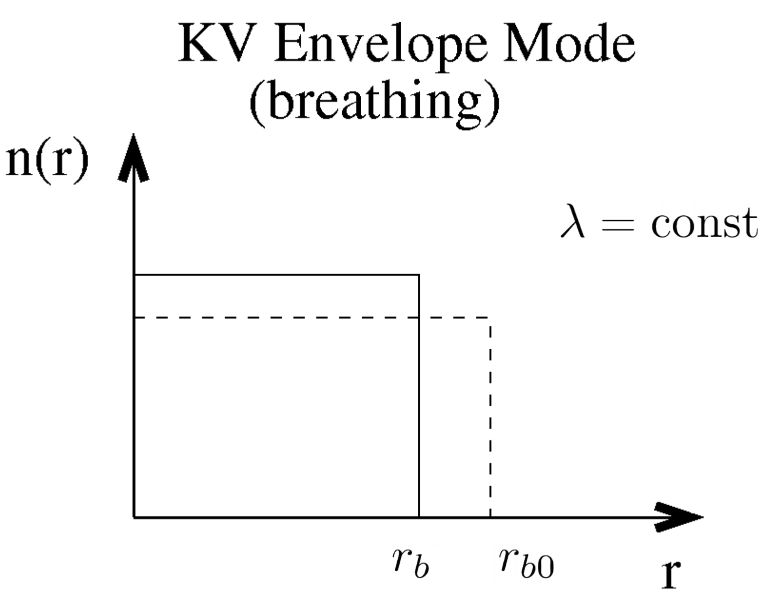
$$\frac{(r_{bf}/r_{bi})^2 - 1}{1-(\sigma_i/\sigma_0)^2} + \frac{p(1-h)[4+p+(3+p)h]}{(p+2)(p+4)(2+ph)^2} - \ln\left[\sqrt{\frac{(p+2)(ph+4)}{(p+4)(ph+2)}}\frac{r_{bf}}{r_{bi}}\right] = 0$$

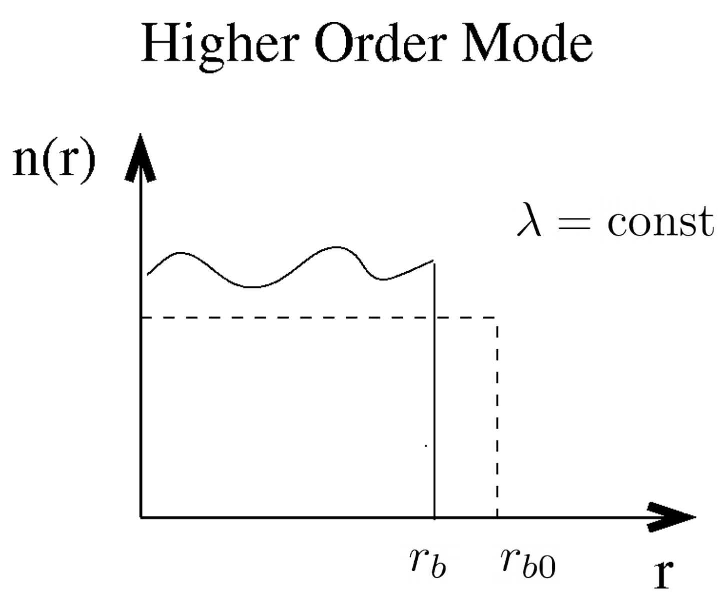
Ratios of final to initial emittance are then obtainable from the matched envelope eqns:

$$\frac{\varepsilon_{xf}}{\varepsilon_{xi}} = \frac{r_{bf}}{r_{bi}}\sqrt{\frac{(r_{bf}/r_{bi})^2 - [1-(\sigma_i/\sigma_0)^2]}{(\sigma_i/\sigma_0)^2}}$$

Higher-order Collective (internal) Mode Stability

- Perturbations will generally drive nonlinear space-charge forces
- Evolution of such perturbations can change the beam rms emittance
- Many possible internal modes of oscillation should be possible
  - Frequencies can differ significantly from envelope modes
  - Creates more possibilities for resonant exchanges with a periodic focusing lattice and various beam characteristic responses opening possibilities for system destabilization



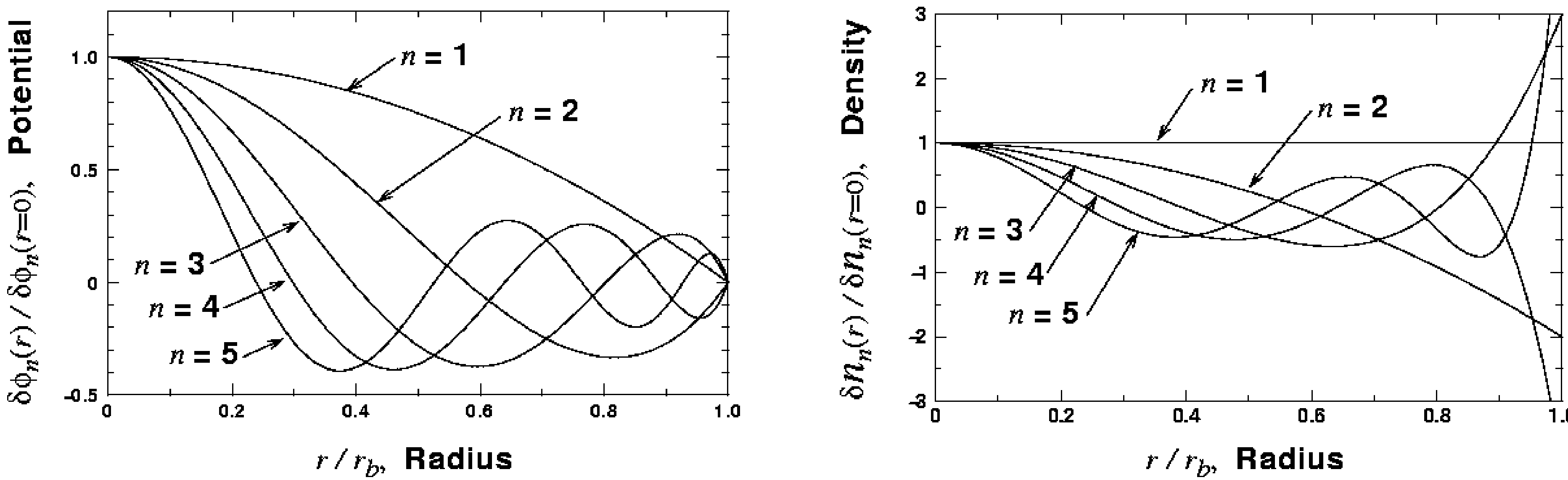


Results of normal mode analysis based on a fully analytic fluid theory:

Mode eigenfunctions: [S. Lund and R. Davidson, Physics of Plasmas **5**, 3028 (1998)]

Exactly the same as derived under kinetic theory!

Potential   Density ($\delta n_n = \epsilon_0 \nabla_\perp^2 \delta\phi_n / q$)



Mode dispersion relation:

$$\frac{k}{k_{\beta 0}} = \sqrt{2 + 2\left(\frac{\sigma}{\sigma_0}\right)^2 (2n^2 - 1)}$$

$$n = 1,\ 2,\ 3,\ \cdots$$

$$k = \frac{2\pi}{\lambda} \qquad k_{\beta 0} = \frac{\sigma_0}{L_p}$$

$\lambda =$ mode wavelength

- Agrees well with the stable high frequency branch in KV kinetic theory without spurious unphysical instabilities of KV kinetic model

Results show should expect many collective modes internal to beam seeded by intiail large perturbations

End of digression on space-charge Waves

Resolution of space-charge collective oscillations (waves)

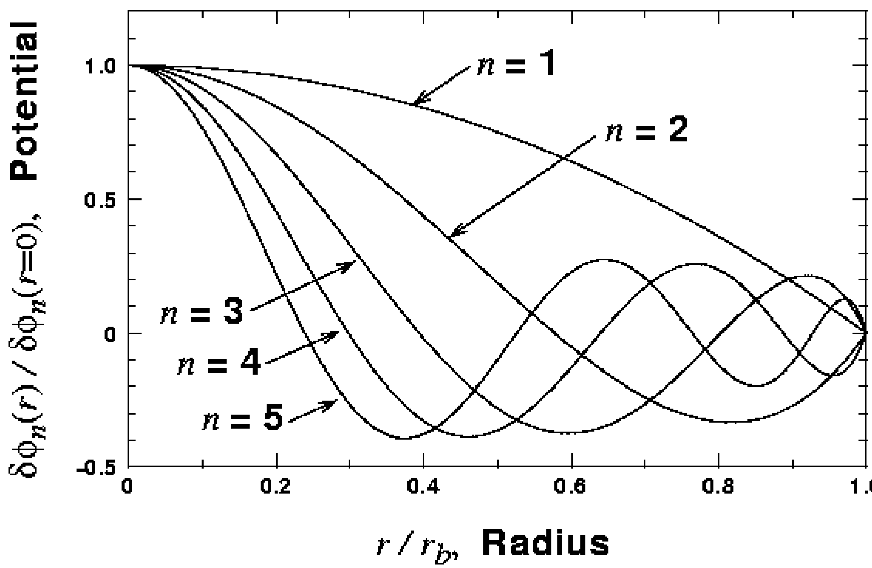
- For a leap-frog mover this requires minimally

$$\tau = \text{period of wave component}$$

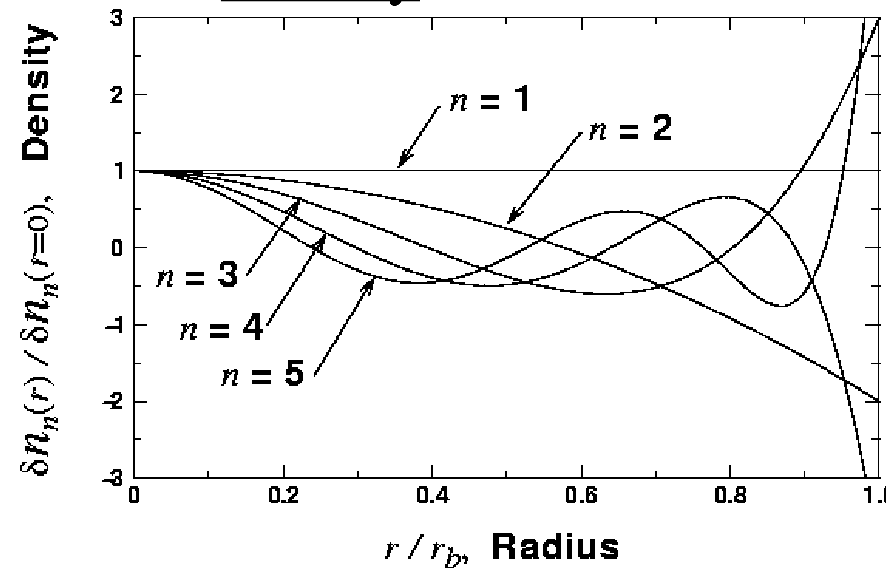
$$\frac{\Delta_t}{\tau} < \frac{1}{2} \qquad \tau = \frac{\lambda}{\beta_b c} \qquad \frac{k}{k_{\beta 0}} = \sqrt{2 + 2\left(\frac{\sigma}{\sigma_0}\right)^2 (2n^2 - 1)}$$

$$n = 1,\ 2,\ 3,\ \cdots$$

Potential



Density ($\delta n_n = \epsilon_0 \nabla_\perp^2 \delta\phi_n / q$)



C: Resolution: Spatial Grid

The spatial grid should resolve both space-charge variations of the beam associated with both the bulk structure of the of the finite radial extent beam and collective waves

Beam Edge

- Screening leads to a flat core in linear focusing for strong space-charge
- Estimate from equilibrium beam properties with the edge falloff being on the scale of the characteristic thermal Debye length.

$$\lambda_D = \frac{v_t}{\omega_p} = \left(\frac{\epsilon_0 T}{q^2 \hat{n}}\right)^{1/2}$$

$v_t = \text{thermal velocity}$

$\omega_p = \text{plasma frequency}$

$T = \text{kinetic temp (energy units)}$

$\hat{n} = \text{characteristic density}$

- Know species, need $\hat{n},\ T$

Estimate from known:

$\lambda = \text{line charge}$

$r_x = 2\langle \tilde{x}^2 \rangle^{1/2} = \text{beam edge}$

$\varepsilon_x = 4\left[\langle \tilde{x}^2 \rangle_\perp \langle \tilde{x}'^2 \rangle_\perp - \langle \tilde{x}\tilde{x}' \rangle_\perp^2\right]^{1/2} = \text{emittance}$

+ species and kinetic energy $\leftrightarrow m, \gamma_b,\ \beta_b$

$\Longrightarrow$

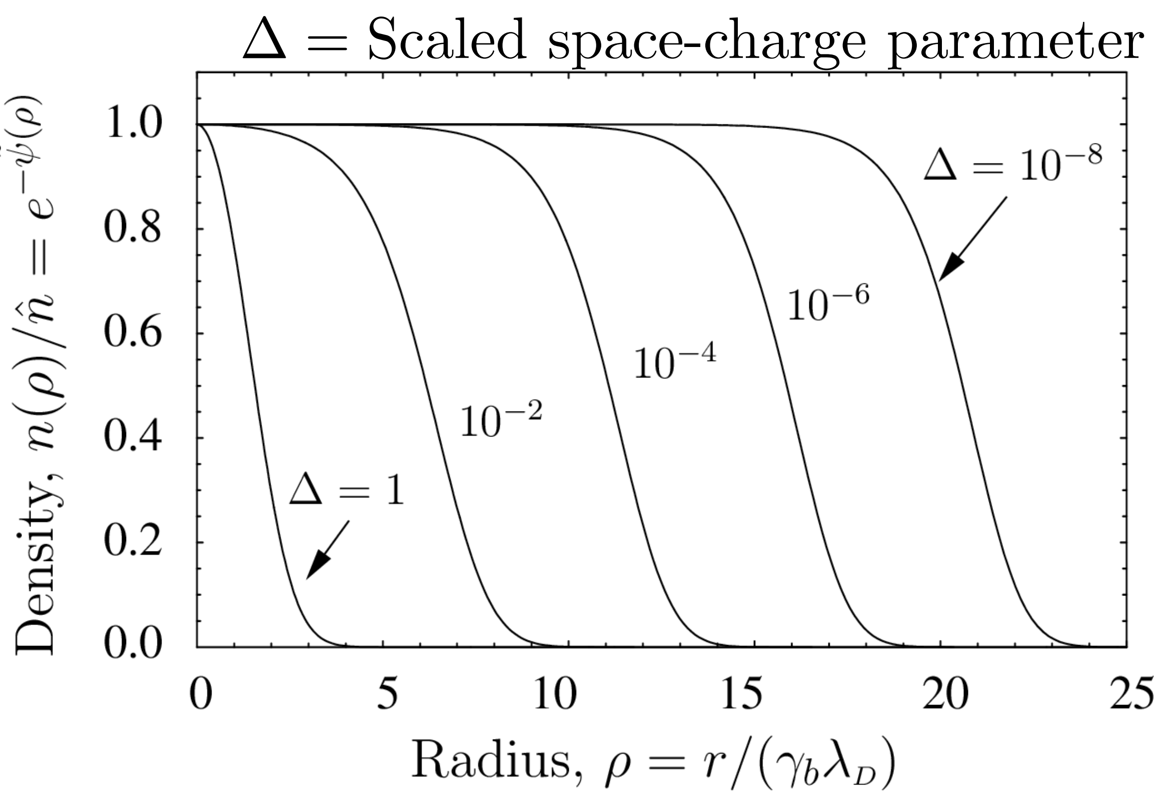
Obtain:

$$\hat{n} \sim \frac{\lambda}{q\pi r_x r_y}$$

$$\frac{T}{\gamma_b m \beta_b^2 c^2} \sim \frac{4\varepsilon_x \varepsilon_y}{r_x r_y}$$

For a smooth, thermal equilbrium core scaled theory shows that

- See USPAS notes, *Beam Physics with Intense Space Charge*



Debye Lengths

Density *always* falls off in a few Debye lengths regardless of huge range of space-charge!

$\ell_{\text{edge}} \sim 5\lambda_D$

Regardless of Δ value characterizing

$f_\perp = \exp(-H_\perp/T)$

This suggests that to reasonably resolve the beam edge that the spatial mesh increments should satisfy

$$\Delta_x, \Delta_y \lesssim \lambda_D$$

- Typical beams with low emittance and space-charge dominated flow have

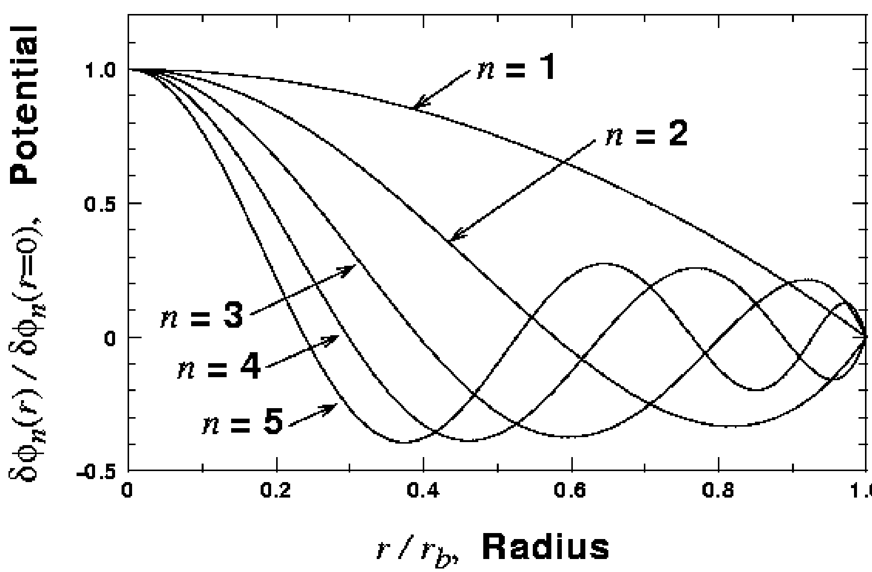
  beam radius $\sim \sqrt{r_x r_y} \sim 50 - 100$'s λ_D
- Resolving this can be demanding: particularly in 3D

Spatial variation of collective space-charge waves

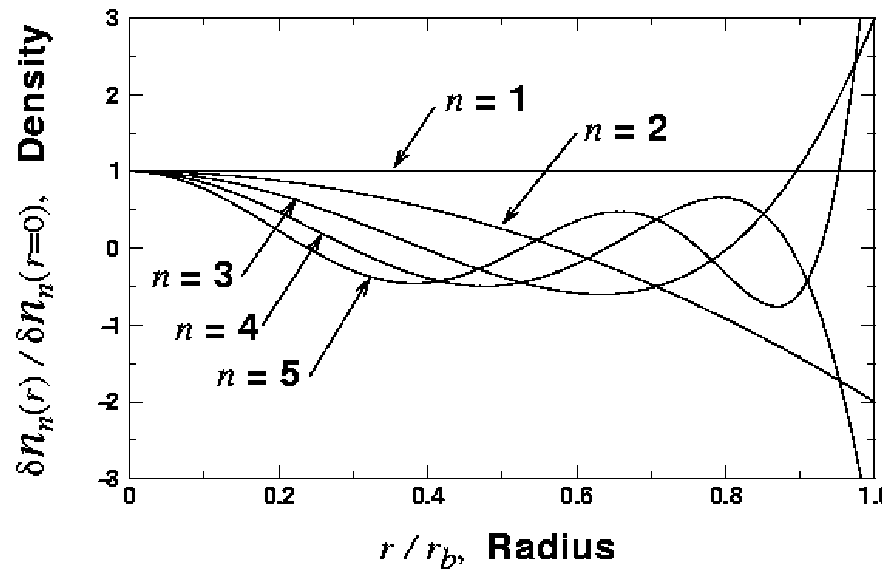
Space charge waves not only constrain the time advance, but they also require radial variations to be resolved

- Usually resolution of beam edge more demanding on choice of $\Delta_x, \; \Delta_y$
  – High orders can be demanding
- Modes usually have more variation near the edge of the beam
  – n'th order mode has n-1 radial nodes

Potential



Density $(\; \delta n_n = \epsilon_0 \nabla_\perp^2 \delta\phi_n / q \;)$



$$\Delta_x, \Delta_y \lesssim \frac{1}{2} \frac{\sqrt{r_x r_y}}{n}$$

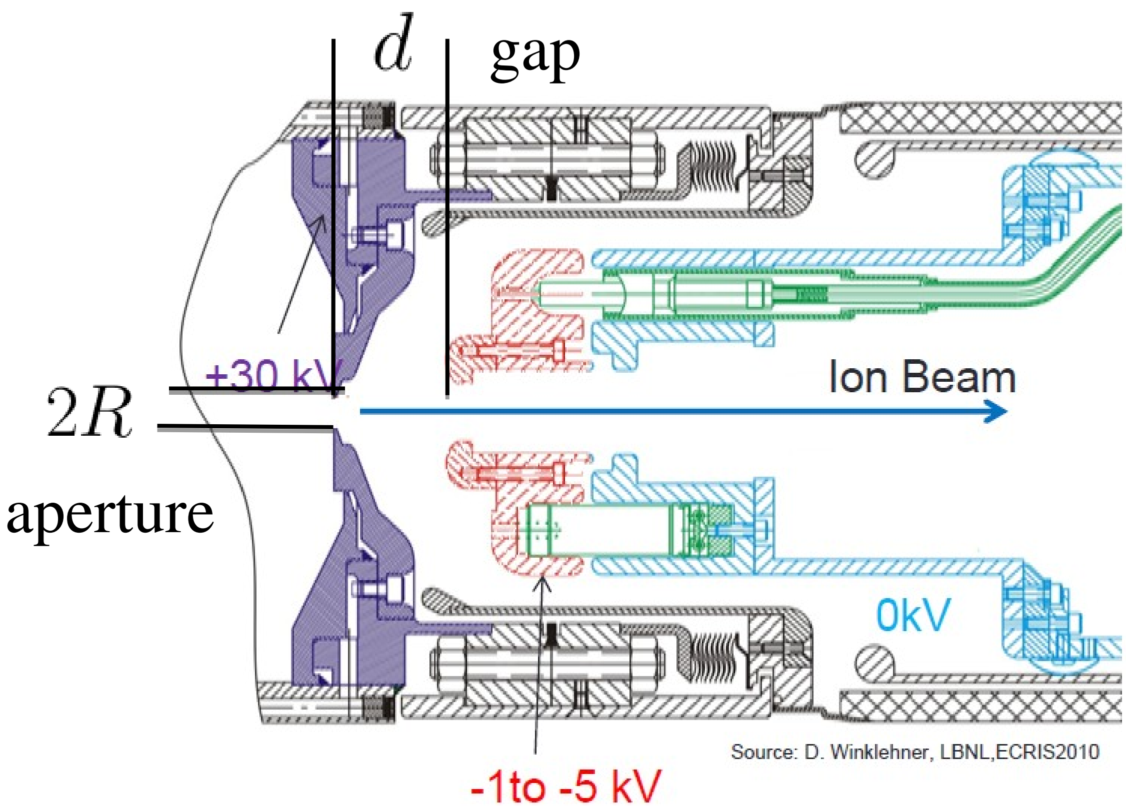
$$\sqrt{r_x r_y} \sim \text{edge radius beam}$$

Electrostatic Structures on Mesh

If applied fields are calculated with biased conductors on the mesh, then the mesh should resolve structures

- Only "features" that impact multipole field components within the aperture where the beam particles are need need be resolved

Example: Puller electrodes from
ECR ion source at Michigan State University



(Source: Daniel Winklehner)

Choose:

$$\Delta_x, \ \Delta_y \lesssim \frac{R}{2}$$

R = *smallest* radius curvature that "matters" to beam

- Mesh refinement can relax overall zones but need to check more carefully

D: Statistics

Collective effects require having a significant number of particles within the "volume" bounded by the characteristic shielding distance

- Shielding distance given by the Debye length:

$$\lambda_D = \frac{v_t}{\omega_p} = \left(\frac{\epsilon_0 k_B T}{q^2 \hat{n}} \right)^{1/2}$$

See following discussion to motivate screening length

- "Volume" bounded by shielding distance will depend on the dimension of the simulation being carried out. For simulations with N macro-particles require number of macro particles in Debye screening length to be large:

$$2\text{D}: \qquad N_D = \sum_i \int_{\text{circle}} d^2x \delta(x - x_i)\delta(y - y_i) \qquad \gg 1$$

$$3\text{D}: \qquad N_D = \sum_i \int_{\text{sphere}} d^2x \delta(x - x_i)\delta(y - y_i)\delta(z - z_i) \qquad \gg 1$$

circle and sphere have radius of Debye length

$x_i = \text{macro particle coordinate}$

Debye screened potential for a test charge inserted in a thermal equilibrium beam essentially the same in 1D, 2D, and 3D

Test Charge:

1D:

Sheet Charge Density: Σ_t

2D:

Line Charge Density: λ_t

3D: (physical case)

Point Charge: q_t

All Cases:

$$\lambda_D = \left(\frac{\epsilon_0 T}{q^2 \hat{n}} \right)^{1/2}$$

Dimension	Distance Measure	Test Charge Density $\rho =$	Screened Potential $\delta\phi \simeq$
1D	$\|x\|$	$\Sigma_t \delta(x)$	$\frac{\gamma_b \lambda_D \Sigma_t}{2\epsilon_0} e^{-\|x\|/(\gamma_b \lambda_D)}$
2D	$r = \sqrt{x^2 + y^2}$	$\lambda_t \frac{\delta(r)}{2\pi r}$	$\frac{\lambda_t}{2\sqrt{2\pi}\epsilon_0} \frac{1}{\sqrt{r/(\gamma_b \lambda_D)}} e^{-r/(\gamma_b \lambda_D)}, \quad r \gg \gamma_b \lambda_D$
3D	$r = \sqrt{x^2 + y^2 + z^2}$	$q_t \delta(x)\delta(y)\delta(z)$	$\frac{q_t}{4\pi\epsilon_0 r} e^{-r/(\gamma_b \lambda_D)}$

References for Calculation:

1D: Lund, Friedman, Bazouin, PRSTAB **14**, 054201 (2011)

2D: USPAS lecture notes on *Beam Physics with Intense Space Charge*

3D: Davidson, *Theory of Nonneutral Plasmas*, Addison-Wesley 1989

In all these cases: 1D, 2D, and 3D, the screened interaction potential $\delta\phi$ has approximately the form:

$$\delta\phi = \phi(r) e^{-r/\lambda_D}$$

r = distance measure in each dimension case

Bare Screen

Comments

- If a lower dimensional models produce the same screened interaction as in physical 3D, then the lower dimensional model can produce essentially the same collective interaction as in 3D. This is why lower dimensional models can give right answers!
  - This is important and seems to be poorly realized by newer generations of scientists who run big codes routinely
  - Parameters can be returned for optimal equivalency with 3D
- In 1D the bare Coulomb interaction is infinite range (sheet charges) but the screened interaction is still the same as in physical 3D
  - Paper [Lund, Friedman, Bazouin, PRSTAB **14**, 054201 (2011)] shows how to exploit this with optimal equivalences to model space-charge effects in beams: sheet beam model simpler to analyze

Comments Continued

- In 2D the screened form is approximately the same as in physical 3D in spite of the radically different Coulomb forces
  – Equivalence, ironically, a little more approximate than for 1D
- It is MUCH easier to get good convergance in statistics in lower dimensional models. This can be exploited to guide setting of numerical parameters in 3D codes.
  – Results sometimes sobering: can be difficult!
  – Recommend strongly testing models in 2D to gain insight

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Statistics:

Collective effects typically require having a significant number of particles N_D within the characteristic screening radius characterized by the Debye length:

2D: $$N_D = \sum_i \int_{\substack{\text{circle} \\ |\vec{x}|<\lambda_D}} d^2x \; \delta^{(2)}(\vec{x}-\vec{x}_i) \gg 1$$

$\vec{x}_i$ = macro-particle coordinate.

3D: $$N_D = \sum_i \int_{\substack{\text{sphere} \\ |\vec{x}|<\lambda_D}} d^3x \; \delta^{(3)}(\vec{x}-\vec{x}_i) \gg 1$$

where:

$$\lambda_D = \frac{v_t}{\omega_p} = \left(\frac{\varepsilon_0 k_B T}{\hat{n} q^2}\right)^{1/2}$$

$$v_t = (k_B T/m)^{1/2}$$

$$\omega_p = \left(\frac{q^2 \hat{n}}{\varepsilon_0 m}\right)^{1/2}$$

$\sum_i \Rightarrow$ sum over all macro particles.

In simulations of higher order collective modes it may also be necessary to have a significant number of particles per cell on a mesh that resolves the relevant spatial variations of mode induced self-field fluctuations.

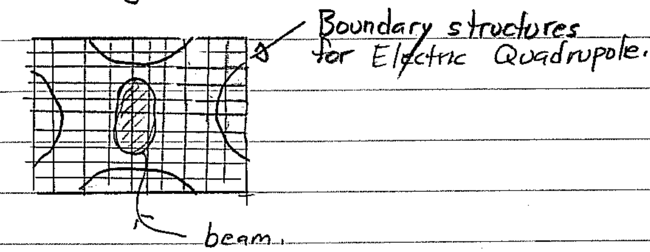
2D: $$N_{cell} = \sum_i \int_{cell} d^2x \; \delta^{(2)}(\vec{x}-\vec{x}_i) \gg 1$$

3D: $$N_{cell} = \sum_i \int_{cell} d^3x \; \delta^{(3)}(\vec{x}-\vec{x}_i) \gg 1$$

- Larger N_{cell} prevents local self-fields from being noise dominated.
- Larger N_{cell} leads to larger N_D, typically $N_D > N_{cell}$ since λ_D must be resolved on the grid.

Good statistics are <u>only</u> needed in the beam core with the possible exception of certain beam-halo problems and near the beam edge.

- Most beams will only occupy a fraction of the full grid.



statistics should be evaluated in the cells that the beam occupies rather than average grid measures.

No comprehensive rules exist for how good the statistics must be. <u>Individual problems must be checked and verified.</u> Some general comments:

- What is adequate will typically depend on what is analyzed
  - Image fields may be resolved with few particles
  - Collective waves may take many particles if low noise (interpetable) diagnostic projections are needed.
- Longer runs generally require increased statistics
- Poor statistics result in unphysical collisionality that is often characterized by a linear rise in beam emittances with simulation time.

Classes of Particle Simulations.

How important is smoothing?

3D Beam: $N \sim 10^{10} - 10^{14}$ particles typical

Simulations: $N \lesssim 10^{8}$ practical (modern parallel computers); typical $10^{3} - 10^{6}$

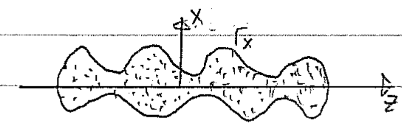
Each simulation particle may represent: $10^{3} \rightarrow 10^{11}$ particles in the real beam for 3D simulations.

- Smoothing involved with particle weightings are key to obtaining physical answers. and limiting collisionality.

Is the situation really this bad?

- Lower dimensional models typically simulated.

3D Model



N point particles with smoothed interactions

Phase Space:

x, y, z } 6D
p_x, p_y, p_z

Physical charge - point charges

$$\rho = \sum_i q\, \delta(x-x_i)\delta(y-y_i)\delta(z-z_i)$$

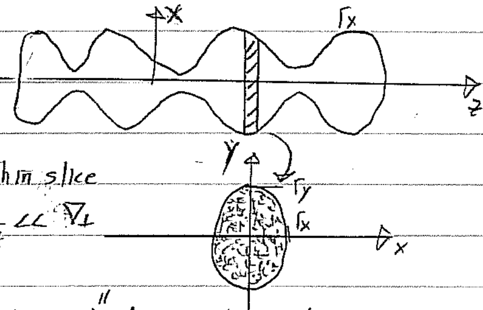
Smoothed charge

$$\rho = \sum_i q_M\, f(x-x_i, y-y_i, z-z_i)$$

q_M: Macro particle charge

f: smoothed shape function

2D ⊥ Thin Slice Model



N line charges with smoothed interactions.

Thin slice

$\frac{\partial}{\partial z} \ll \nabla_\perp$

Phase Space:

x, y } 4D + possible p_z 1D
p_x, p_y }

4D or 5D

"Physical" charge - line charges

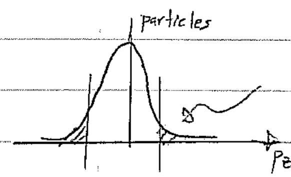
$$\rho = \sum_i \lambda_p \delta(x-x_i)\delta(y-y_i)$$

Smoothed charge - λ_M: Macro particle line charge

$$\rho = \sum_i \lambda_M f(x-x_i, y-y_i) \qquad f: \text{ smoothing function}$$

The slice must be tracked in s with each particle moving the same increment in s with each step so that a slice maps to a slice.

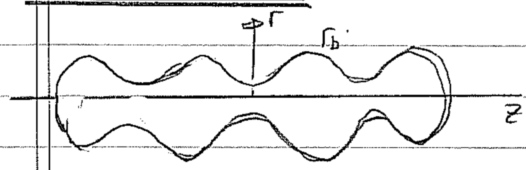
- If p_z is included the velocity distribution must be assumed "frozen in".



parts that would leave assumed replenished by particles from adjacent slices.

- Response to acceleration may be modeled with
- "Thick" slice models also possible with periodic boundary conditions on the "slice" to try to recover some 3D effects of a long pulse in a periodic lattice.

2D r-z Model



N charged Rings with smoothed interactions

$\partial/\partial\theta = 0$ Axisymmetric

Phase Space.

r, z \} 4D + possible 1D
p_r, p_z \} p_θ (angular mom.)

4D or 5D.

physical charge - cylindrical rings

$$\rho = \sum_i \frac{Q_i}{2\pi} \frac{\delta(r-r_i)}{r_i} \delta(z-z_i)$$

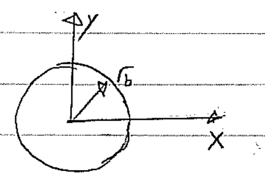
smoothed charge

$$\rho = \sum_i \frac{Q_M}{2\pi} \frac{f(r-r_i, z-z_i)}{r_i}$$

Q_M: Macro particle charge
f: Smoothing function.

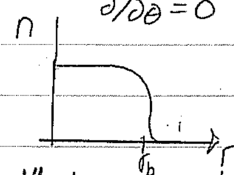
- Used to model solinoidal transport of an initial axisymmetric beam.
- Sometimes used to model AG beams with an approximately equivalent, s-dependant focusing force. $\vec{x}_\perp'' = k_{\beta 0}^2(s)\, \vec{x}_\perp + \ldots\ldots$

1D Axisymmetric Model



N charged cylinders with smoothed interactions

$\partial/\partial\theta = 0$



Phase - Space

r \} 2D + possible
p_r \} p_θ, p_z

2D to 4D

"physical" charge - cylindrical sheets

$$\rho = \sum_i \frac{Q_i}{2\pi} \frac{\delta(r-r_i)}{r_i}$$

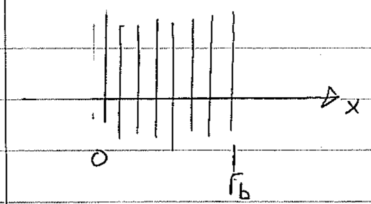
Smoothed charge

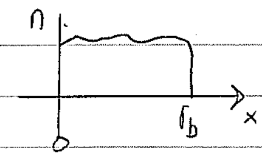
$$\rho = \sum_i \frac{Q_M}{2\pi} \frac{f(r-r_i)}{r_i}$$

Q_M : Macro charge
f : Smoothing function

- Simple model for continuously focused, axisymmetric beams

1D slab Model





N charged slabs with smoothed interactions.

Phase-Space

x, p_x } 2D + possible p_y, p_z

2D to 4D

"physical" charge - sheets

$$\rho = \sum_i \sigma \, \delta(x - x_i)$$

smoothed charge

$$\rho = \sum_i \sigma_M \, f(x - x_i)$$

σ_M : macro particle charge
f : smoothing function

- Most simple model, but slab geometry is least physical.

It is not immediately clear how such different models can in many cases represent qualitatively similar collective interactions since force laws can change form with dimension. For example, in free space, we find that:

Model	Free Space Field due to ith "particle"	
3D	$\vec{E} = \dfrac{q_i (\vec{x} - \vec{x}_i)}{4\pi\varepsilon_0 \lvert \vec{x} - \vec{x}_i \rvert^3}$	
2D	$\vec{E} = \dfrac{\lambda_i (\vec{x} - \vec{x}_i)}{2\pi\varepsilon_0 \lvert \vec{x} - \vec{x}_i \rvert^2}$	λ_i = line-charge "particle"
1D	$E_x = \dfrac{\sigma_i}{2\varepsilon_0} \dfrac{(x - x_i)}{\lvert x - x_i \rvert}$	σ_i = sheet charge "particle"

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The reason these radically different interactions can give similar physics is that the screening associated with collective interactions is found to be similar:

- Debye screening has similar characteristics in each dimension.

Showed 2D form in class. Will show in final that the 3D scaling obtains the same Debye length definition.

$$\Rightarrow \quad \lambda_D = \frac{v_{thermal}}{\omega_p} = \left(\frac{\epsilon_0 k_B T}{\hat{n} q^2}\right)^{1/2} \qquad v_t = (k_B T/m)^{1/2} \qquad \omega_p = \left(\frac{q^2 \hat{n}}{\epsilon_0 m}\right)^{1/2}$$

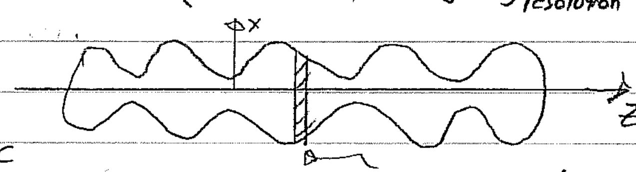
- <u>It is much easier to have a significant number of particles within the characteristic screening distances for lower dimensional problems.</u>

  - Lower dimensional simulations can more easily resolve collective effects! (Sometimes people run 3D simulations for collective modes and present garbage answers due to resolution difficulties)

// Example

$q = Ze$ ions

$e = 1.6 \times 10^{-19}$ C



$\lambda \sim 10^{-13} \rightarrow 10^{-7}$ C/m typical for intense beams

$$\frac{\#\ \text{particles}}{\text{cm}} = \frac{\lambda}{Ze \cdot 100} \sim \frac{10^4}{Z} \rightarrow \frac{10^{10}}{Z}$$

$q \equiv Ze$ (Z: charge state)

- Smoothing still important in lower dimensions and real beam is 3D

//

E: Illustrative Examples with the Warp Code

Sorry ran out of time. Have large series of computer runs to illustrate but need a week of work to distill and make proper summaries. I will try to post an updated version here after the course and extend for future versions.

The intent will be to show examples on the influence of resolution and statistics with the xy Warp transverse slice simulation (x,x',y,y') of an alternating gradient focused beam in a linear hard-edge periodic transport lattice using the script

xy-quad-mag-mg.py

Cases to be covered (common lattice):

- Weak Space-Charge
- Intermediate Space-Charge
- Strong Space-Charge
- Strong Space-Charge with Instability

Corrections and suggestions for improvements welcome!

These notes will be corrected and expanded for reference and for use in future editions of US Particle Accelerator School (USPAS) and Michigan State University (MSU) courses. Contact:

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Please provide corrections with respect to the present archived version at:

https://people.nscl.msu.edu/~lund/uspas/scs_2016

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Review of Initial Distribution Loads

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