Self-Consistent Simulations of Beam and Plasma Systems
Homework 1

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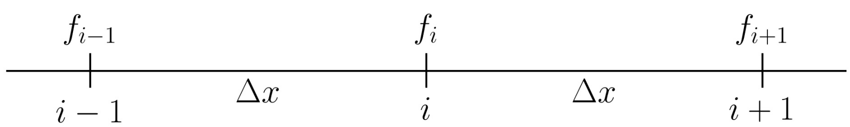
Problem 1 - Derivatives on a non-uniform mesh.

In class, we discretized a first derivative on a uniform grid



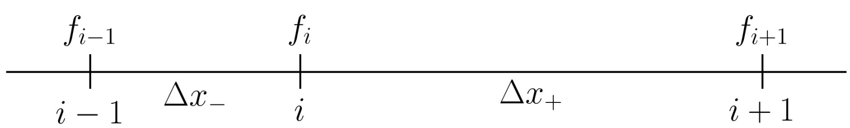
as:

$$\left.\frac{\partial f}{\partial x}\right|_i = \frac{f_{i+1} - f_{i-1}}{2\Delta x} + \mathcal{O}(\Delta x^2)$$

whereas a second derivative is discretized as

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_i = \frac{f_{i+1} - 2f_i + f_{i-1}}{(\Delta x)^2} + \mathcal{O}(\Delta x^2)$$

Similarly, discretize the first and second derivative for a non-uniform grid



i.e. find:

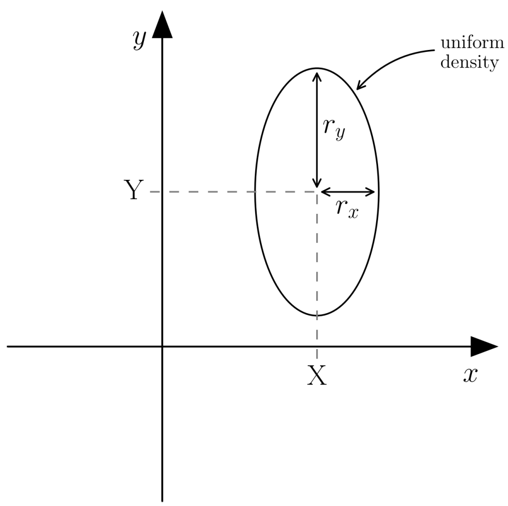
$$\left.\frac{\partial f}{\partial x}\right|_i = \cdots + \mathcal{O}(\Delta x^2_{+,-})$$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_i = \cdots + \mathcal{O}(\Delta x_{+,-})$$

Notice that we find an $\mathcal{O}(\Delta x_{+,-})$ error term for the second derivative instead of an $\mathcal{O}(\Delta x^2)$ term as we did for the uniform grid. Establish the order of error with your answer!

Problem 2 - Moment formulation of the K-V distribution

The equations of motion for a particle evolving within a K-V beam



with

$$\begin{aligned} \tilde{x} &= x - \mathrm{X} \qquad \mathrm{X} = \langle x \rangle_\perp \\ \tilde{y} &= y - \mathrm{Y} \qquad \mathrm{Y} = \langle y \rangle_\perp \end{aligned}$$

are

$$\begin{aligned} \frac{d}{ds}x' + \kappa_x(s)x - \frac{2Q(x-\mathrm{X})}{r_x(r_x+r_y)} &= 0 \\ \frac{d}{ds}y' + \kappa_y(s)y - \frac{2Q(y-\mathrm{Y})}{r_y(r_x+r_y)} &= 0 \end{aligned}$$

when

$$\left(\frac{\tilde{x}}{r_x}\right)^2 + \left(\frac{\tilde{y}}{r_y}\right)^2 \leq 1.$$

Here

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m \gamma_b^3 \beta_b^2 c^2}.$$

a) Show that

$$\begin{aligned} r_x &= 2\langle \tilde{x}^2 \rangle_\perp^{1/2} \\ r_y &= 2\langle \tilde{y}^2 \rangle_\perp^{1/2} \end{aligned}$$

Hint: This is easier if you take

$$\int_{ellipse} d\tilde{x} \int d\tilde{y} \cdots = r_x r_y \int_0^1 d\rho\rho \int_{-\pi}^{\pi} d\theta \cdots$$

$$\begin{aligned}\tilde{x} &= r_x \rho \cos\theta \\ \tilde{y} &= r_y \rho \sin\theta\end{aligned}$$

with

$$\rho \in [0,1], \quad \theta \in [-\pi,\pi], \quad \textit{and} \quad d\tilde{x}d\tilde{y} = r_x r_y \rho d\rho d\theta.$$

b) Derive the equations of motion summarized in class as:

$$\frac{d}{ds}\begin{bmatrix}\langle x\rangle_\perp \\ \langle x'\rangle_\perp \\ \langle y\rangle_\perp \\ \langle y'\rangle_\perp\end{bmatrix} = \begin{bmatrix}\langle x'\rangle_\perp \\ -\kappa_x(s)\langle x\rangle_\perp \\ \langle y'\rangle_\perp \\ -\kappa_y(s)\langle y\rangle_\perp\end{bmatrix}$$

$$\frac{d}{ds}\begin{bmatrix}\langle \tilde{x}^2\rangle_\perp \\ \langle \tilde{x}\tilde{x}'\rangle_\perp \\ \langle \tilde{x}'^2\rangle_\perp \\ \langle \tilde{y}^2\rangle_\perp \\ \langle \tilde{y}\tilde{y}'\rangle_\perp \\ \langle \tilde{y}'^2\rangle_\perp\end{bmatrix} = \begin{bmatrix}2\langle \tilde{x}\tilde{x}'\rangle_\perp \\ \langle \tilde{x}'^2\rangle_\perp - \kappa_x(s)\langle \tilde{x}^2\rangle_\perp + \frac{Q\langle \tilde{x}^2\rangle_\perp^{1/2}}{2[\langle \tilde{x}^2\rangle_\perp^{1/2}+\langle \tilde{y}^2\rangle_\perp^{1/2}]} \\ -2\kappa_x(s)\langle \tilde{x}\tilde{x}'\rangle_\perp + \frac{Q\langle \tilde{x}\tilde{x}'\rangle_\perp}{\langle \tilde{x}^2\rangle_\perp^{1/2}[\langle \tilde{x}^2\rangle_\perp^{1/2}+\langle \tilde{y}^2\rangle_\perp^{1/2}]} \\ 2\langle \tilde{y}\tilde{y}'\rangle_\perp \\ \langle \tilde{y}'^2\rangle_\perp - \kappa_y(s)\langle \tilde{y}^2\rangle_\perp + \frac{Q\langle \tilde{y}^2\rangle_\perp^{1/2}}{2[\langle \tilde{x}^2\rangle_\perp^{1/2}+\langle \tilde{y}^2\rangle_\perp^{1/2}]} \\ -2\kappa_y(s)\langle \tilde{y}\tilde{y}'\rangle_\perp + \frac{Q\langle \tilde{y}\tilde{y}'\rangle_\perp}{\langle \tilde{y}^2\rangle_\perp^{1/2}[\langle \tilde{x}^2\rangle_\perp^{1/2}+\langle \tilde{y}^2\rangle_\perp^{1/2}]}\end{bmatrix}$$

Moments not shown are zero. $\langle \tilde{x}\tilde{y}\rangle = 0$ is obvious and $\langle \tilde{x}\tilde{y}'\rangle = \langle \tilde{x}'\tilde{y}\rangle = 0$ follows from the kinetic distribution.

c) Show from the results in part b) that

$$\epsilon_x = \text{const.} \qquad \text{with} \qquad \epsilon_x = 4[\langle \tilde{x}^2\rangle_\perp \langle \tilde{x}'^2\rangle_\perp - \langle \tilde{x}\tilde{x}'\rangle_\perp^2]^{1/2}$$

d) Show from the results in part b) and c) that

$$\frac{d^2}{ds^2}r_x + \kappa_x(s) r_x - \frac{2Q}{r_x + r_y} - \frac{\epsilon_x^2}{r_x^3} = 0$$

This is the "standard" form of the K-V envelope equation.

Problem 3 - Moment equations and conservation constraints

The non-relativistic Vlasov-equation is:

$$\left\{\frac{\partial}{\partial t} + \vec{v}\cdot\frac{\partial}{\partial \vec{x}} + \frac{q}{m}\left[\vec{E} + \vec{v}\times\vec{B}\right]\cdot\frac{\partial}{\partial \vec{v}}\right\} f(\vec{x},\vec{v},t) = 0$$

Define a fluid density n and a fluid flow velocity $\vec{\mathrm{V}}$ by

$$\begin{aligned}n(\vec{x},t) &= \int d^3v\, f(\vec{x},\vec{v},t) \\ n(\vec{x},t)\vec{\mathrm{V}}(\vec{x},t) &= \int d^3v\, \vec{v}\, f(\vec{x},\vec{v},t)\end{aligned}$$

a) Operate on the Vlasov equation with

$$\int d^3v \cdots$$

to derive the continuity equation:

$$\frac{\partial}{\partial t} n(\vec{x},t) + \frac{\partial}{\partial \vec{x}} \cdot \left[n(\vec{x},t)\vec{\mathrm{V}}(\vec{x},t) \right] = 0$$

b) Can the continuity equation be solved by itself if you specify the intial density field $n(\vec{x}, t = 0)$? Why?

c) Operate on the Vlasov equation with

$$\int d^3v\, \vec{v} \cdots$$

to derive the fluid force equation.

$$\begin{aligned} \frac{\partial}{\partial t}\left(n\vec{\mathrm{V}}\right) + \nabla \cdot (n\langle \vec{v}\vec{v}\rangle_v) &= \frac{q}{m} n \left(\vec{E} + \vec{\mathrm{V}} \times \vec{B}\right) \\ \text{with } \langle \vec{v}\vec{v}\rangle_v &\equiv \frac{\int d^3v\, \vec{v}\vec{v} f}{\int d^3v\, f} \end{aligned}$$

Defining a pressure tensor as

$$\begin{aligned} \underline{\underline{\mathrm{P}}} &= m \int d^3v\, (\vec{v} - \vec{\mathrm{V}})(\vec{v} - \vec{\mathrm{V}}) f(\vec{x}, \vec{v}, t) \\ &= mn\langle \vec{v}\vec{v}\rangle_v - mn\vec{\mathrm{V}}\vec{\mathrm{V}}, \end{aligned}$$

the fluid force equation can be expressed as

$$\frac{\partial}{\partial t}\vec{\mathrm{V}} + \vec{\mathrm{V}} \cdot \frac{\partial}{\partial \vec{x}}\vec{\mathrm{V}} = \frac{q}{m}\left(\vec{E} + \vec{\mathrm{V}} \times \vec{B}\right) - \frac{1}{mn}\frac{\partial}{\partial \vec{x}} \cdot \underline{\underline{\mathrm{P}}}.$$

This form is often used in fluid/plasma analysis.

d) If the continuity and force equation derived in parts a) and c) are analyzed, can they be solved in principle if you specify the initial density field $n(\vec{x}, t = 0)$ and the velocity field $\vec{\mathrm{V}}(\vec{x}, t = 0)$? Why? Does the answer change if we assume a cold initial beam with $\underline{\underline{\mathrm{P}}} = 0$? Why?

e) Let $G(f)$ be some smooth, differentiable function of f, satisfying $G(f \to 0) = 0$. Show that

$$\int d^3x \int d^3v\, G(f) = \text{const.}$$

This so-called "generalized entropy" measure with G specified can be used to check Vlasov simulations. For example:

$$\begin{aligned} G(f) = f : &\quad \int d^3x \int d^3v\, f = \text{const.} \to \text{charge conservation} \\ G(f) = f^2 : &\quad \int d^3x \int d^3v\, f^2 = \text{const.} \to \text{"enstropy" conservation} \\ G(f) = f \ln f : &\quad \int d^3x \int d^3v\, f \ln f = \text{const.} \to \text{entropy conservation} \end{aligned}$$

Problem 4 - Python program for Runge-Kutta order 2

We wish to integrate the differential equation

$$\frac{d\,x(t)}{dt} = x(t)\cos(t) \qquad x(0) = 1$$

with Euler's method and the Runge-Kutta method (order 2), and compare the results.

a) Download the file `euler.py` from:

   `http://raw.githubusercontent.com/RemiLehe/uspas_exercise/master/euler.py`.

   Copy it to a new name `rungekutta.py` and modify it to implement the Runge-Kutta method, order 2. In particular:

   - Change the name of the class from `EulerSolver` to `RKSolver`
   - Change the name of the `euler_integration` method to `rk_integration`, and implement the Runge-Kutta method, order 2 (see the presentation *Overview of Basic Numerical Methods*)

b) In `ipython`, import classes `EulerSolver` from `euler.py` and `RKSolver` from `runge_kutta.py`. Evaluate the results of both methods for `N=100` ; what is the RMS error in both cases?