# Transverse Particle Resonances with Application to Circular Accelerators*

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# Transverse Particle Resonances: Detailed Outline

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# Transverse Particle Resonances: Detailed Outline - 2

# S1: Overview

In our treatment of transverse single particle orbits of lattices with s-varying focusing, we found that Hill's Equation describes the orbits to leading-order approximation:

$$\boxed{\begin{aligned} x''(s) + \kappa_x(s)x(s) &= 0 \\ y''(s) + \kappa_y(s)y(s) &= 0 \end{aligned}}$$

where $\kappa_x(s), \quad \kappa_y(s)$ are functions that describe linear applied focusing forces of the lattice

- Focusing functions can also incorporate linear space-charge forces
  - Self-consistent for special case of a KV distribution

In analyzing Hill's equations we employed phase-amplitude methods

- See: S.M. Lund lectures on Transverse Particle Dynamics, S8, on the betatron form of the solution

$$x(s) = A_{xi}\sqrt{\beta_x(s)}\cos\psi_x(s) \qquad A_{xi} = \text{const}$$

$$\frac{1}{2}\beta_x(s)\beta_x''(s) - \frac{1}{4}\beta_x'^2(s) + \kappa_x(s)\beta_x^2(s) = 1 \qquad \psi_x(s) = \psi_{xi} + \int_{s_i}^{s}\frac{d\bar{s}}{\beta_x(\bar{s})}$$

$$\beta_x(s+L_p) = \beta_x(s) \qquad \beta_x(s) > 0$$

This formulation simplified identification of the Courant-Snyder invariant:

$$\left(\frac{x}{w_x}\right)^2 + (w_x x' - w_x' x)^2 = A_x^2 = \text{const}$$

$$\frac{1+\beta_x'^2/4}{\beta_x} x^2 - \beta_x \beta_x' x x' + \beta_x x'^2 = A_x^2 = \epsilon_x$$

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 =$$

$$w_x = \sqrt{\beta_x}$$

which helped to interpret the dynamics.

We will now exploit this formulation to better (analytically!) understand resonant instabilities in periodic focusing lattices. This is done by choosing coordinates such that *stable* unperturbed orbits described by Hill's equation:

$$x''(s) + \kappa_x(s) x(s) = 0$$

are mapped to a continuous oscillator

$$\tilde{x}''(\tilde{s}) + \tilde{k}_{\beta 0}^2 \tilde{x}(\tilde{s}) = 0$$

$$\tilde{k}_{\beta 0}^2 = \text{const} > 0$$

$$\cdot\tilde{\cdot}\cdot = \text{Transformed Coordinate}$$

- Because the linear lattice is designed for single particle stability this transformation can be effected for any practical machine operating point

These transforms will help us more simply understand the action of perturbations (from applied field nonlinearities, ....) acting on the particle orbits:

$$x''(s) + \kappa_x(s)x(s) = \mathcal{P}_x(s; \mathbf{x}_\perp, \mathbf{x}'_\perp, \vec{\delta})$$
$$y''(s) + \kappa_y(s)y(s) = \mathcal{P}_y(s; \mathbf{x}_\perp, \mathbf{x}'_\perp, \vec{\delta})$$
$$\mathcal{P}_x,\ \mathcal{P}_y = \text{Perturbations}$$
$$\vec{\delta} = \text{Extra Coupling Variables}$$

For simplicity, we restrict analysis to:

| | |
|---|---|
| $\gamma_b\beta_b = \text{const}$ | No Acceleration |
| $\delta = 0$ | No Axial Momentum Spread |
| $\phi = 0$ | Neglect Space-Charge |

- Acceleration can be incorporated using transformations (see Transverse Particle Dynamics, S10)
- Comments on space-charge effects will be made in S7

We also take the applied focusing lattice to be periodic with:

$$\kappa_x(s + L_p) = \kappa_x(s) \qquad L_p = \text{Lattice Period}$$
$$\kappa_y(s + L_p) = \kappa_y(s)$$

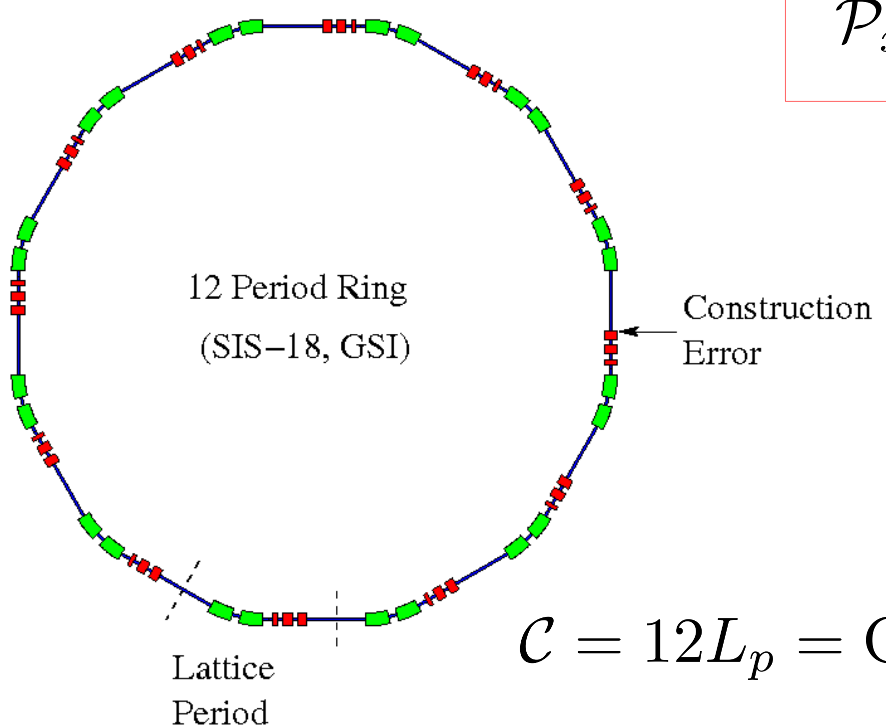
For a ring we also always have the superperiodicity condition:

$$\mathcal{P}_x(s+\mathcal{C};\mathbf{x}_\perp,\mathbf{x}'_\perp,\vec{\delta}) = \mathcal{P}_x(s;\mathbf{x}_\perp,\mathbf{x}'_\perp,\vec{\delta})$$
$$\mathcal{P}_y(s+\mathcal{C};\mathbf{x}_\perp,\mathbf{x}'_\perp,\vec{\delta}) = \mathcal{P}_y(s;\mathbf{x}_\perp,\mathbf{x}'_\perp,\vec{\delta})$$
$$\mathcal{C} = \mathcal{N}L_p = \text{Circumference Ring}$$
$$\mathcal{N} \equiv \text{Superperiodicity}$$

Perturbations can be Random and/or Systematic:

Random Errors in a ring will be felt once per particle lap in the ring rather than every lattice period

$$\mathcal{P}_{x,y}(\cdots,s+\mathcal{N}L_p) = \mathcal{P}_{x,y}(\cdots,s)$$

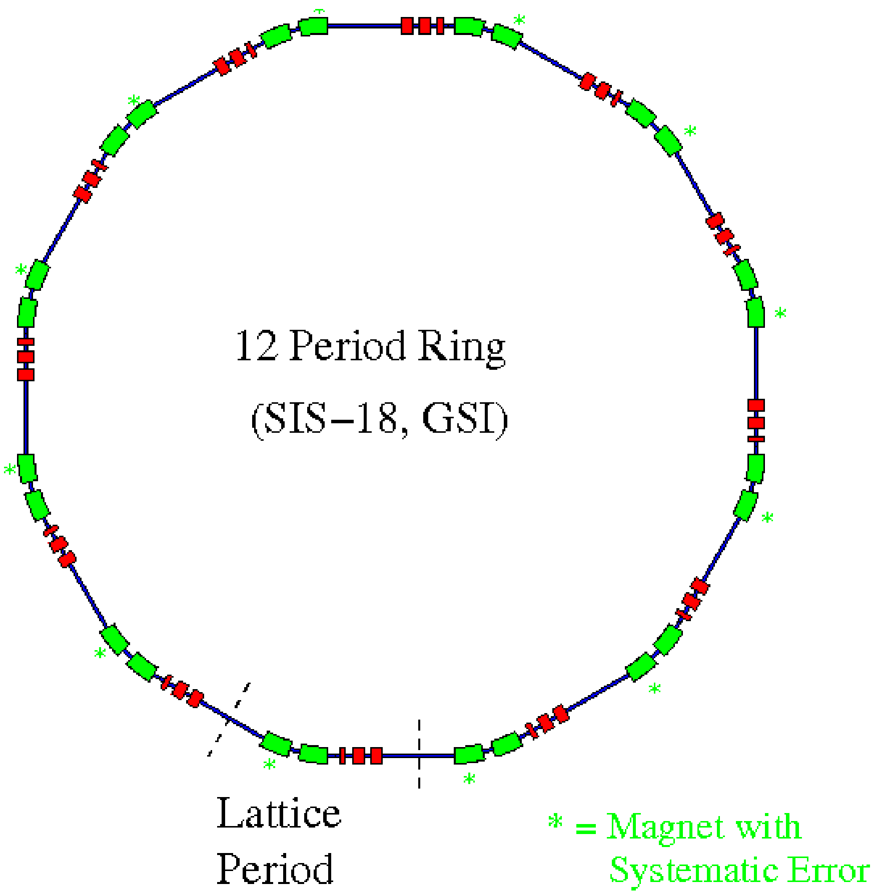



Random Error Sources:
- Fabrication
- Assembly/Construction
- Material Defects
- ....

$$\mathcal{C} = 12L_p = \text{Circumference of Ring}$$

Systematic Errors can occur in both linear machines and rings and effect *every* lattice period in the same manner.

Example: FODO Lattice with the same error in each dipole of pair




$$\mathcal{P}_{x,y}(\cdots, s + L_p) = \mathcal{P}_{x,y}(\cdots, s)$$

Systematic Error Sources:

- Design Idealization (e.g., truncated pole)
- Repeated Construction or Material Error
- ....

We will find that perturbations arising from both random and systematic error can drive resonance phenomena that destabilize particle orbits and limit machine performance

# S2: Floquet Coordinates and Hill's Equation

Define for a *stable* solution to Hill's Equation

- Drop *x* subscripts and only analyze *x*-orbit for now to simplify analysis
- Later will summarize results from coupled *x*-*y* orbit analysis

"Radial" Coordinate: $$u \equiv \frac{x}{\sqrt{\beta}}$$

"Angle" Coordinate: $$\varphi \equiv \frac{1}{\nu_0}\int_{s_i}^{s} \frac{d\bar{s}}{\beta(\bar{s})} \equiv \frac{\Delta\psi(s)}{\nu_0}$$
(dimensionless, normalized)

where: $$\varphi(s = s_i) = 0 \quad \text{reference choice}$$

$$\beta = w^2 = \text{Betatron Amplitude Function}$$

$$\nu_0 \equiv \frac{\Delta\psi(\mathcal{N}L_p)}{2\pi} = \frac{\mathcal{N}\sigma_0}{2\pi} =$$ Number underpressed *x*-betatron oscillations in ring

$$\psi = \text{Phase of } x\text{-orbit}$$

$$\Delta\psi(s) = \psi(s) - \psi(s_i)$$

Can also take $\mathcal{N} = 1$ and then $\nu_0$ is the number (usually fraction thereof) of undepressed particle oscillations in *one* lattice period

## Comment:

$\varphi$ can be interpreted as a normalized angle measured in the particle betatron phase advance:

**Ring:** ($\mathcal{N}$ = Superperiod #) $\Longrightarrow$ $\varphi$ advances by $2\pi$ on one transit around ring for analysis of Random Errors

**Linac or Ring:** ($\mathcal{N} = 1$) $\Longrightarrow$ $\varphi$ advances by $2\pi$ on transit through one lattice period for analysis of Systematic Errors in a ring *or* linac

Take $\varphi$ as the independent coordinate:

$$\boxed{u = u(\varphi)}$$

and define a new "momentum" phase-space coordinate

$$\boxed{\dot{u} \equiv \frac{du}{d\varphi}} \qquad \boxed{\dot{} \equiv \frac{d}{d\varphi}}$$

These new variables will be applied to express the unpreturbed Hill's equation in a simpler (continuously focused oscillator) form

## /// Aside: Comment on use of $\varphi$ as an independent coordinate

To actually use the formulation explicitly, locations of perturbations need to be cast in terms of $\varphi$ rather than the reference particle axial coordinate $s$:

- Will find we do not need to explicitly carry this out to identify parameters leading to resonances
- To analyze resonant growth characteristics or particular orbit phases it is necessary to calculate $s(\varphi)$ to explicitly specify driving perturbation terms

The needed transform is obtained by integration and (in principle) inversion

- Will, in most cases with non-continuous focusing lattices, need to be carried out numerically

$$\varphi(s) = \frac{1}{\nu_0} \int_{s_i}^{s} \frac{d\bar{s}}{\beta(\bar{s})}$$

$$\varphi(s) \qquad \Longrightarrow \qquad s(\varphi)$$

$$\varphi(s) \equiv \frac{1}{\nu_0}\int_{s_i}^{s} \frac{d\bar{s}}{\beta(\bar{s})} \quad \Longrightarrow \quad \frac{d\varphi}{ds} = \frac{1}{\nu_0 \beta}$$

Rate of change in s not constant
except for continuous focusing lattices

Continuous Focusing: Simplest case

$$\kappa_x = k_{\beta 0}^2 = \text{const}$$

$$\frac{d\varphi}{ds} = \frac{2\pi}{\mathcal{C}} = \text{const} \qquad \Longrightarrow \quad \varphi(s) = \frac{2\pi}{\mathcal{C}}(s - s_i)$$

Periodic Focusing: Simple FODO lattice to illustrate

Add numerical example/plot in future version of notes.

///

From the definition

$$u \equiv \frac{x}{\sqrt{\beta}}$$

Rearranging this and using the chain rule:

$$x = \sqrt{\beta}u$$

$$x' = \frac{\beta'}{2\sqrt{\beta}}u + \sqrt{\beta}\frac{du}{d\varphi}\frac{d\varphi}{ds} \qquad\qquad \frac{d}{ds} = \frac{d\varphi}{ds}\frac{d}{d\varphi}$$

From:

$$\varphi \equiv \frac{1}{\nu_0}\int_{s_i}^{s}\frac{d\bar{s}}{\beta(\bar{s})} \qquad \Longrightarrow \qquad \boxed{\frac{d\varphi}{ds} = \frac{1}{\nu_0\beta}}$$

we obtain

$$x' = \frac{\beta'}{2\sqrt{\beta}}u + \frac{1}{\nu_0\sqrt{\beta}}\dot{u}$$

cancels

$$x'' = \frac{d}{ds}x' = \frac{\beta''}{2\sqrt{\beta}}u - \frac{\beta'^2}{4\beta^{3/2}}u + \cancel{\frac{\beta'}{2\nu_0\beta^{3/2}}\dot{u}} - \cancel{\frac{\beta'}{2\nu_0\beta^{3/2}}\dot{u}} + \frac{1}{\nu_0^2\beta^{3/2}}\ddot{u}$$

Summary:

$$\boxed{\begin{aligned} x &= \sqrt{\beta} u \\ x' &= \frac{\beta'}{2\sqrt{\beta}} u + \frac{1}{\nu_0 \sqrt{\beta}} \dot{u} \\ x'' &= \frac{\beta''}{2\sqrt{\beta}} u - \frac{\beta'^2}{4\beta^{3/2}} u + \frac{1}{\nu_0^2 \beta^{3/2}} \ddot{u} \end{aligned}}$$

Insert these results in the perturbed Hill's equation:

$$x''(s) + \kappa_x(s) x(s) = \mathcal{P} \qquad \qquad \mathcal{P} \equiv \mathcal{P}_x$$

giving

$$\frac{\ddot{u}}{\nu_0^2 \beta^{3/2}} + \frac{\beta'' u}{2\sqrt{\beta}} - \frac{\beta'^2 u}{4\beta^{3/2}} + \kappa\beta^2 = \mathcal{P}$$

$$\Longrightarrow \quad \ddot{u} + \nu_0^2 \left[ \frac{\beta\beta''}{2} - \frac{\beta'^2}{4} + \kappa\beta^2 \right] u = \nu_0^2 \beta^{3/2} \mathcal{P}$$

But the betatron amplitude equation satisfies:

$$\frac{\beta\beta''}{2} - \frac{\beta'^2}{4} + \kappa\beta^2 = 1 \qquad \qquad \beta(s + L_p) = \beta(s)$$

So the terms in [...] = 1 and the perturbed Hills equation reduces to

$$\boxed{\ddot{u} + \nu_0^2 u = \nu_0^2 \beta^{3/2} \mathcal{P}}$$

Hill's equation reduces to simple harmonic oscillator form when unperturbed $\mathcal{P} = 0$ :

$$\text{Unperturbed:} \qquad \ddot{u} + \nu_0^2 u = 0$$

$$\text{Perturbed:} \qquad \ddot{u} + \nu_0^2 u = \nu_0^2 \beta^{3/2} \mathcal{P}$$

$$\nu_0^2 = \text{const} > 0$$

In the absence of perturbations, the Floquet transform has mapped a stable, time dependent solution to Hill's equation to a simple harmonic oscillator!

The general solution to the unperturbed simple harmonic oscillator equation can be expressed as:

$$u(\varphi) = u_i \cos(\nu_0 \varphi) + \frac{\dot{u}_i}{\nu_0} \sin(\nu_0 \varphi)$$

$$\dot{u}(\varphi) = -u_i \nu_0 \sin(\nu_0 \varphi) + \dot{u}_i \cos(\nu_0 \varphi)$$

$$u(\varphi = 0) = u_i = \text{const}$$

$$\dot{u}(\varphi = 0) = \dot{u}_i = \text{const}$$

$u_i$ and $\dot{u}_i$ set by $x$, $x'$ initial conditions at $s = s_i$ (phase choice $\varphi = 0$ at $s = s_i$)

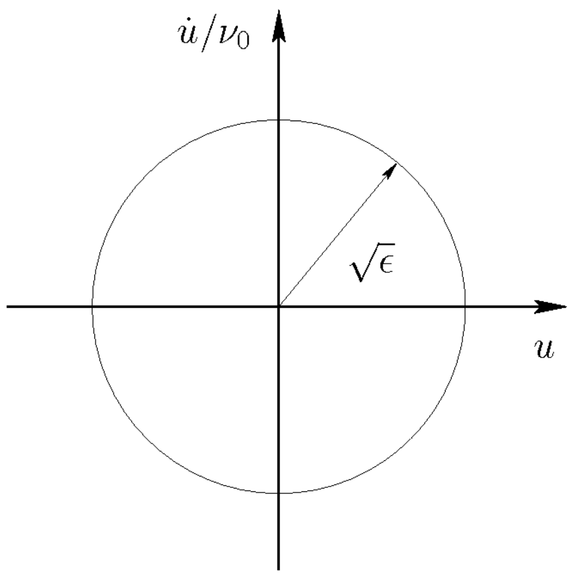
The Floquet representation simplifies the interpretation of the Courant-Snyder invariant:

$$u^2 + \left(\frac{\dot{u}}{\nu_0}\right)^2 = u_i^2 \left[\sin^2(\nu_0\varphi) + \cos^2(\nu_0\varphi)\right] + \left(\frac{\dot{u}_i}{\nu_0}\right)^2 \left[\sin^2(\nu_0\varphi) + \cos^2(\nu_0\varphi)\right]$$

$$\boxed{u^2 + \left(\frac{\dot{u}}{\nu_0}\right)^2 = u_i^2 + \left(\frac{\dot{u}_i}{\nu_0}\right)^2 \equiv \epsilon = \text{const}}$$

- Unperturbed phase-space in $u - \dot{u}/\nu_0$ is a circle of area $\pi\epsilon$ !
- Relate this area to *x*-*x*' phase-space area shortly
    - Preview: areas are equal due to the transform being symplectic
    - Same symbols used as in Transverse Particle Dynamics is on purpose

Unperturbed phase-space ellipse:




This simple structure will also allow more simple visualization of perturbations as distortions on a unit circle, thereby clarifying symmetries:

(Picture to be replaced ... had poor schematic example)

The $u - \dot{u}/\nu_0$ variables also preserve phase-space area

- Feature of the transform being symplectic (Hamiltonian Dynamics)

From previous results:

$$x = \sqrt{\beta} u$$

$$x' = \frac{\beta'}{2\sqrt{\beta}} u + \sqrt{\beta} \frac{d\varphi}{ds} \dot{u} = \frac{\beta'}{2\sqrt{\beta}} u + \frac{1}{\nu_0 \sqrt{\beta}} \dot{u}$$

$$\frac{d\varphi}{ds} = \frac{1}{\nu_0 \beta}$$

Transform area elements by calculating the Jacobian:

$$dx \otimes dx' = |J| du \otimes d\dot{u}$$

$$J = \det \left| \begin{array}{cc} \frac{\partial x}{\partial u}, & \frac{\partial x}{\partial \dot{u}}, \\ \frac{\partial x'}{\partial u} & \frac{\partial x'}{\partial \dot{u}} \end{array} \right] = \det \left| \begin{array}{cc} \sqrt{\beta} & 0 \\ \frac{\beta'}{2\sqrt{\beta}} & \frac{1}{\nu_0 \sqrt{\beta}} \end{array} \right] = \frac{1}{\nu_0}$$

$$\boxed{dx \otimes dx' = du \otimes \frac{d\dot{u}}{\nu_0}}$$

Thus the Courant-Snyder invariant $\epsilon$ is the usual single particle emittance in *x*-*x*' phase-space; see lectures on Transverse Dynamics, S7

# S3: Perturbed Hill's Equation in Floquet Coordinates

Return to the perturbed Hill's equation in S1:

$$x''(s) + \kappa_x(s)x(s) = \mathcal{P}_x(s; \mathbf{x}_\perp, \mathbf{x}'_\perp, \vec{\delta})$$

$$y''(s) + \kappa_y(s)y(s) = \mathcal{P}_y(s; \mathbf{x}_\perp, \mathbf{x}'_\perp, \vec{\delta})$$

$$\mathcal{P}_x,\ \mathcal{P}_y = \text{Perturbations}$$

$$\vec{\delta} = \text{Extra Coupling Variables}$$

Drop the extra coupling variables and apply the Floquet transform in S2 and consider only transverse multipole magnetic field perturbations

- Examine only *x*-equation, *y*-equation analogous
- From S4 in Transverse Particle Dynamics terms $B_x,\ B_y$ only have variation in *x*,*y*. If solenoid magnetic field errors are put in, terms with $x',\ y'$ dependence will also be needed
- Drop *x*-subscript in $\mathcal{P}_x$ to simplify notation

$$\ddot{u} + \nu_0^2 u = \nu_0^2 \beta^{3/2} \mathcal{P}$$

$$\mathcal{P} = \mathcal{P}(s(\varphi), \sqrt{\beta}u, y, \vec{\delta})$$

Transform *y* similarly to *x*
If analyzing general orbit with *x* and *y* motion

Expand the perturbation in a power series:

- Can be done for *all* physical applied field perturbations
- Multipole symmetries can be applied to restrict the form of the perturbations
  - See: S4 in these notes and S3 in Transverse Particle Dynamics
- Perturbations can be random (once per lap; in ring) or systematic (every lattice period; in ring or in linac)

$$\boxed{\begin{aligned}\mathcal{P}(x,y,s) &= \mathcal{P}_0(y,s) + \mathcal{P}_1(y,s)x + \mathcal{P}_2(y,s)x^2 + \cdots \\ &= \sum_{n=0}^{\infty} \mathcal{P}_n(y,s)x^n\end{aligned}}$$

Take:

$$x = \sqrt{\beta}u$$

to obtain:

$$\boxed{\ddot{u} + \nu_0^2 u = \nu_0^2 \sum_{n=0}^{\infty} \beta^{\frac{n+3}{2}} \mathcal{P}_n(y,s) u^n}$$

A similar equation applies in the *y*-plane.

# S4: Sources of and Forms of Perturbation Terms

Within a 2D transverse model it was shown that transverse applied magnetic field components entering the equations of motion can be expanded as:

- See: S3, Transverse Particle Dynamics: 2D components axial integral 3D components
- Applied electric fields can be analogously expanded

$$\underline{B}^*(\underline{z}) = B_x^a(x,y) - iB_y^a(x,y) = \sum_{n=1}^{\infty} \underline{b}_n \left(\frac{\underline{z}}{r_p}\right)^{n-1}$$

$$\underline{b}_n = \text{const (complex)} \equiv \mathcal{A}_n - i\mathcal{B}_n \qquad \underline{z} = x + iy \qquad i = \sqrt{-1}$$

$$n = \text{Multipole Index} \qquad r_p = \text{Aperture "Pipe" Radius}$$

$$\mathcal{B}_n \Longrightarrow \text{"Normal" Multipoles}$$

$$\mathcal{A}_n \Longrightarrow \text{"Skew" Multipoles}$$

Cartesian projections: $\overline{B_x} - i\overline{B_y} = (\mathcal{A}_n - i\mathcal{B}_n)(x+iy)^{n-1}/r_p^{n-1}$

| Index $n$ | Name | Normal ($\mathcal{A}_n = 0$) $B_x r_p^{n-1}/\mathcal{B}_n$ | $B_y r_p^{n-1}/\mathcal{B}_n$ | Skew ($\mathcal{B}_n = 0$) $B_x r_p^{n-1}/\mathcal{A}_n$ | $B_y r_p^{n-1}/\mathcal{A}_n$ |
|---|---|---|---|---|---|
| 1 | Dipole | $0$ | $1$ | $1$ | |
| 2 | Quadrupole | $y$ | $x$ | $x$ | $-y$ |
| 3 | Sextupole | $2xy$ | $x^2 - y^2$ | $x^2 - y^2$ | $-2xy$ |
| 4 | Octupole | $3x^2y - y^3$ | $x^3 - 3xy^2$ | $x^3 - 3xy^2$ | $-3x^2y + y^3$ |
| 5 | Decapole | $4x^3y - 4xy^3$ | $x^4 - 6x^2y^2 + y^4$ | $x^4 - 6x^2y^2 + y^4$ | $-4x^3y + 4xy^3$ |

Trace back how the applied magnetic field terms enter the *x*-plane equation of motion:

- See: S2, Transverse Particle Dynamics
- Apply equation in S2 with: $\beta_b = \text{const}, \ \phi \simeq \text{const}, \ E_x^a \simeq 0, \ B_z^a \simeq 0$
- To include axial $(B_z^a \neq 0)$ field errors, follow a similar pattern to generalize

$$\boxed{x'' = -\frac{q}{m\gamma_b\beta_b c} B_y^a}$$

Express this equation as:

$$x'' + \kappa_x(s)x = -\frac{q}{m\gamma_b\beta_b c}\left[B_y^a(x,y,s) - B_y^a(x,y,s)\Big|_{\text{lin } x\text{-foc}}\right]$$

Nonlinear focusing terms only in []

- "Normal" part of linear applied magnetic field contained in focus func $\kappa_x$

Compare to the form of the perturbed Hill's equation:

$$\boxed{x'' + \kappa_x x = \mathcal{P}_x = \sum_{n=0}^{\infty} \mathcal{P}_n(y,s)x^n}$$

Gives:

$$\Longrightarrow \quad \boxed{\mathcal{P}_x = -\frac{q}{m\gamma_b\beta_b c}\left[B_y^a - B_y^a\big|_{\text{lin }x\text{-foc}}\right]}$$

where the $y$-field components can be obtained from the multipole expansion as:

$$B_y^a = -\text{Im}[\underline{B}^*]$$

$$B_y^a\big|_{\text{lin }x\text{-focus}} = -\text{Im}[\underline{B}^*|_{n=1\text{ term}}]$$

$$\underline{B}^* = \sum_{n=1}^{\infty} \underline{b}_n \left(\frac{x+iy}{r_p}\right)^{n-1}$$

- Use multipole field components of magnets to obtain explicit form of field component perturbations consistent with the Maxwell equations
- Need to subtract off design component of linear filed from $\mathcal{P}_x$ perturbation term since it is included in $\kappa_x$
- Similar steps employed to identify $y$-plane perturbation terms, perturbations from solenoidal field components, and perturbations for applied electric field components

Caution: Multipole index $n$ and power series index $n$ in $\mathcal{P}_x$ expansion not the same (notational overuse: wanted analogous symbol)

- Multipole Expansion for $B_x^a, \; B_y^a$ :

| | | | |
|---|---|---|---|
| $n = 1$ | Dipole | $n = 3$ | Sextupole |
| $n = 2$ | Quadrupole | $n = ....$ | |

- Power Series Expansion for $\mathcal{P}_x$ :

| *x*-plane Motion (*y=0*) | | *x*-*y* plane motion |
|---|---|---|
| $n = 0$ | Dipole | Depends on form of y-coupling |
| $n = 1$ | Quadrupole | |
| $n = 2$ | Sextupole | |
| .... | | |

# S5: Solution of the Perturbed Hill's Equation: Resonances

Analyze the solution of the perturbed orbit equation:

$$\ddot{u} + \nu_0^2 u = \nu_0^2 \sum_{n=0}^{\infty} \beta^{\frac{n+3}{2}} \mathcal{P}_n(y, s) u^n$$

derived in S4.

To more simply illustrate resonances, we analyze motion in the *x*-plane with:

$$y(s) \equiv 0$$

- Essential character of general analysis illustrated most simply in one plane
- Can generalize by expanding $\mathcal{P}_n(y, s)$ in a power series in *y* and generalizing notation to distinguish between Floquet coordinates in the *x*- and *y*-planes
  - Results in coupled *x*- and *y*-equations of motion

Each $n$-labeled perturbation expansion coefficient is periodic with period of the ring circumference (random perturbations) or lattice period (systematic):

$$L_p = \text{Lattice Period}$$

$$\mathcal{C} = \mathcal{N} L_p = \text{Ring Circumference}$$

$$\beta(s + L_p) = \beta(s)$$

$$\beta(s + \mathcal{N} L_p) = \beta(s)$$

Random Perturbation:

$$\mathcal{P}_n(y, s + \mathcal{N} L_p) = \mathcal{P}_n(y, s)$$

$$\Longrightarrow \boxed{\beta^{\frac{n+3}{2}}(s + \mathcal{N} L_p)\mathcal{P}_n(y, s + \mathcal{N} L_p) = \beta^{\frac{n+3}{2}}(s)\mathcal{P}_n(y, s)}$$

Systematic Perturbation:

$$\mathcal{P}_n(y, s + L_p) = \mathcal{P}_n(y, s)$$

$$\Longrightarrow \boxed{\beta^{\frac{n+3}{2}}(s + L_p)\mathcal{P}_n(y, s + L_p) = \beta^{\frac{n+3}{2}}(s)\mathcal{P}_n(y, s)}$$

Expand each *n*-labeled perturbation expansion coefficient in a Fourier series as:

$$\beta^{\frac{n+3}{2}}\mathcal{P}_n(y=0,s) = \sum_{k=-\infty}^{k=\infty} C_{n,k} e^{ikp\varphi}$$

$$i \equiv \sqrt{-1} \qquad p \equiv \begin{cases} 1, & \text{Random perturbation} \\ & \text{(once per lap in ring)} \\ \mathcal{N}, & \text{Systematic perturbation} \\ & \text{(every lattice period)} \end{cases}$$

$$C_{n,k} = \int_{-\pi/p}^{\pi/p} \frac{d\varphi}{2\pi/p}\, e^{-ikp\varphi}\beta^{\frac{n+3}{2}}(s)\mathcal{P}_n(y=0,s) \quad = \quad \text{const}$$

(complex-valued)

$$s = s(\varphi) \qquad \varphi = \int_{s_0}^{s} \frac{1}{\nu_0}\frac{d\tilde{s}}{\beta(\tilde{s})}$$

- Can apply to Rings for random perturbations (with $p=1$) or systematic perturbations (with $p = \mathcal{N}$ )
- Can apply to linacs for periodic perturbations (every lattice period) with $p=1$
- Does not apply to random perturbations in a linac
  - In linac random perturbations will vary every lattice period and drive random walk type effects but not resonances

The perturbed equation of motion becomes:

$$\boxed{\ddot{u} + \nu_0^2 u = \nu_0^2 \sum_{n=0}^{\infty} \sum_{k=-\infty}^{k=\infty} C_{n,k} e^{ikp\varphi} u^n}$$

Expand the solution as:

$$\boxed{u = u_0 + \delta u}$$

$$u_0 = \text{unperturbed solution}$$

$$\delta u = \text{perturbation due to errors}$$

where $u_0$ is the solution to the *simple harmonic oscillator* equation in the absence of perturbations:

$$\boxed{\ddot{u}_0 + \nu_0^2 u_0 = 0}$$

Unperturbed equation of motion

Assume small-amplitude perturbations so that

$$|u_0| \gg |\delta u|$$

Then to leading order, the equation of motion for $\delta u$ is:

$$\boxed{\ddot{\delta u} + \nu_0^2 \delta u \simeq \nu_0^2 \sum_{n=0}^{\infty} \sum_{k=-\infty}^{k=\infty} C_{n,k} e^{ipk\varphi} u_0^n}$$

Perturbed equation of motion

To obtain the perturbed equation of motion, the unperturbed solution $u_0$ is inserted on the RHS terms

- Gives simple harmonic oscillator equation with driving terms

Solution of the unperturbed orbit is simply expressed as:

$$u_0 = u_{0i} \cos(\nu_0 \varphi + \varphi_i) = u_{0i} \frac{e^{i(\nu_0\varphi + \varphi_i)} - ie^{-i(\nu_0\varphi+\varphi_i)}}{2}$$

$$\left.\begin{aligned} u_{0i} &= \text{const} \\ \varphi_i &= \text{const} \end{aligned}\right\} \quad \begin{array}{l} \text{Set by particle initial conditions:} \\ x(s_i) = x_i, \quad x'(s_i) = x'_i \end{array}$$

Then binomial expand:

$$\begin{aligned} u_0^n &= u_{0i}^n \left( \frac{e^{i(\nu_0\varphi+\varphi_i)} + e^{-i(\nu_0\varphi+\varphi_i)}}{2} \right)^n \\ &= \frac{u_{oi}^n}{2^n} \sum_{m=0}^{n} \binom{n}{m} e^{i(n-m)(\nu_0\varphi+\varphi_i)} e^{-im(\nu_0\varphi+\varphi_i)} \\ &= \frac{u_{oi}^n}{2^n} \sum_{m=0}^{n} \binom{n}{m} e^{i(n-2m)\nu_0\varphi} e^{i(n-2m)\varphi_i} \end{aligned}$$

where $\displaystyle \binom{n}{m} \equiv \frac{n!}{m!(n-m)!}$ is a binomial coefficient

Using this expansion the linearized perturbed equation of motion becomes:

$$\delta\ddot{u} + \nu_0^2 \delta u \simeq \nu_0^2 \sum_{n=0}^{\infty} \sum_{k=-\infty}^{k=\infty} \sum_{m=0}^{n} \binom{n}{m} \frac{C_{n,k}}{2^n} e^{i[(n-2m)\nu_0 + pk]\varphi} e^{i(n-2m)\varphi_i}$$

The solution for $\delta u$ can be expanded as:

$$\delta u = \delta u_h + \delta u_p$$

$$\delta u_h = \text{homogenous solution}$$

General solution to: $\delta\ddot{u}_h + \nu_0^2 \delta u_h = 0$

$$\delta u_p = \text{particular solution}$$

Any solution with: $\delta u \rightarrow \delta u_p$

- Can drop homogeneous solution because it can be absorbed in unperturbed solution $u_0$
  - Exception: some classes of linear amplitude errors in adjusting magnets
- Only a particular solution need be found, take:

$$\delta u = \delta u_p$$

$$\boxed{\delta\ddot{u} + \nu_0^2 \delta u \simeq \nu_0^2 \sum_{n=0}^{\infty} \sum_{k=-\infty}^{k=\infty} \sum_{m=0}^{n} \binom{n}{m} \frac{C_{n,k}}{2^n} e^{i[(n-2m)\nu_0 + pk]\varphi} e^{i(n-2m)\varphi_i}}$$

Equation describes a driven simple harmonic oscillator with periodic driving terms on the RHS:

- Homework problem reviews that solution of such an equation will be unstable when the driving term has a frequency component equal to the restoring term
  - Resonant exchange and amplitude grows *linearly* (not exponential!) in $\varphi$
  - Parameters meeting resonance condition will lead to instabilities with particle oscillation amplitude growing in $\varphi(s)$

Resonances occur when:

$$\boxed{(n-2m)\nu_0 + pk = \pm\nu_0}$$

is satisfied for the operating tune $\nu_0$ and some values of:

$$n = 0,\ 1,\ 2,\ \cdots \qquad m = 0,\ 1,\ 2,\ \cdots,\ n$$

$$k = -\infty,\ \cdots,\ -1,\ 0,\ 1,\ \cdots,\ \infty$$

$$p \equiv \begin{cases} 1, & \text{Random perturbation} \\ \mathcal{N}, & \text{Systematic perturbation} \end{cases}$$

If growth rate is sufficiently large, machine operating points satisfying the resonance condition will be problematic since particles will be lost (scraped) by the machine aperture due to increasing oscillation amplitude:

- Machine operating tune ( $\nu_0$ ) can be adjusted to avoid
- Perturbation can be actively corrected to reduce amplitude of driving term

Low order resonance terms with smaller *n*, *k*, *m* magnitudes are expected to be more dangerous because:

- Less likely to be washed out by effects not included in model
- Amplitude coefficients expected to be stronger

More detailed theories consider coherence length, finite amplitude, and nonlinear term effects. Such treatments and numerical analysis concretely motivate importance/strength of terms. A standard reference on analytic theory is:

- Kolomenskii and Lebedev, *Theory of Circular Accelerators*, North-Holland (1966)

We only consider lowest order effects in these notes.

In the next section we will examine how resonances restrict possible machine operating parameters.

- After establishing clear picture of effect on single particle orbits we will then add space-charge

# S6: Machine Operating Points: Tune Restrictions Resulting from Resonances

Examine situations where the *x*-plane motion resonance condition:

$$\boxed{(n-2m)\nu_0 + pk = \pm\nu_0}$$

is satisfied for the operating tune $\nu_0$ and some values of:

Multipole Order Index:

$$n = 0,\ 1,\ 2,\ \cdots$$

Particle Binomial Expansion Index:

$$m = 0,\ 1,\ 2,\ \cdots\ ,\ n$$

Periodicity Fourier Series Expansion Index:

$$k = -\infty,\ \cdots\ ,\ -1,\ 0,\ 1,\ \cdots\ ,\ \infty$$

Perturbation Symmetry Factor:

$$p \equiv \begin{cases} 1, & \text{Random perturbation} \\ \mathcal{N}, & \text{Systematic perturbation} \end{cases}$$

Resonances can be analyzed one at a time using linear superposition

- Analysis valid for small-amplitudes

Analyze resonance possibilities starting with index *n* <==> Multipole Order

## $n = 0$, Dipole Perturbations:

$$n = 0, \quad \Longrightarrow \quad m = 0$$

and the resonance condition gives a single constraint:

$$\nu_0 = \pm pk \qquad pk = \text{integer} \qquad k = -\infty, \cdots, -1, 0, 1, \cdots, \infty$$

$$p = \begin{cases} 1, & \text{Random perturbation} \\ \mathcal{N}, & \text{Systematic perturbation} \end{cases}$$

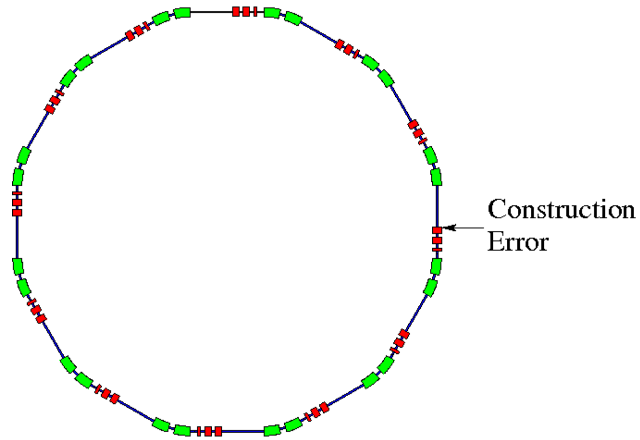
Therefore, to avoid dipole resonances *integer tunes* operating points not allowed:

| | | |
|---|---|---|
| $p = 1$ | Random Perturbation | $\nu_0 \neq 1, 2, 3, \cdots$ |
| $p = \mathcal{N}$ | Systematic Perturbation | $\nu_0 \neq \mathcal{N}, 2\mathcal{N}, 3\mathcal{N}, \cdots$ |

- Systematic errors are less restrictive on machine operating points
- Multiply random perturbation tune restrictions by $\mathcal{N}$ to obtain the systematic perturbation case

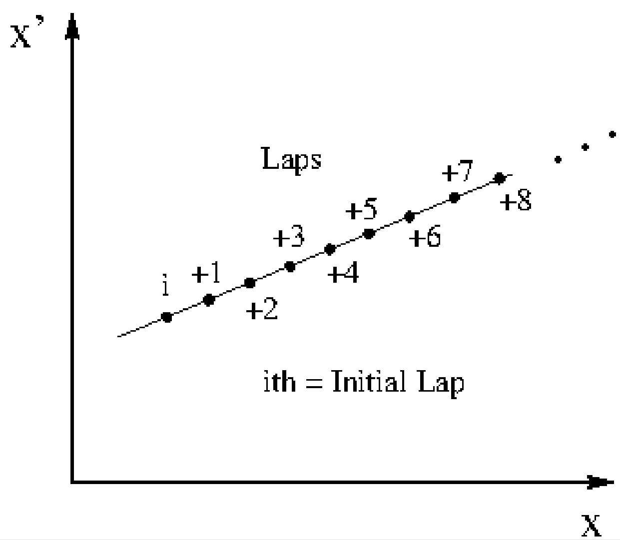
## Interpretation of result:

Consider a ring with a single (random) dipole error along the reference path of the ring:




If the particle is oscillating with integer tune, then the particle experiences the dipole error on each lap in the same oscillation phase and the trajectory will "walk-off" on a lap-to-lap basis in phase-space:

- With finite machine aperture the particle will be scraped/lost



## $n = 1$, Quadrupole Perturbations:

$$n = 1, \implies m = 0,\ 1$$

and the resonance conditions give:

$$n = 1,\ m = 0: \qquad \nu_0 + pk = \pm\nu_0$$

$$n = 1,\ m = 1: \qquad -\nu_0 + pk = \pm\nu_0$$

Give two cases:

$$\implies pk = 0,\ \nu_0 = \pm\frac{pk}{2}$$

Implications of two cases:

$$1)\ pk = 0 \ \Rightarrow k = 0$$

Can be treated by "renormalizing" oscillator focusing strength: need not be considered

$$\ddot{u} + \nu_0^2 u = \nu_0^2 C_{1,0} u$$

$$2)\ \nu_0 = \pm\frac{pk}{2} \ \Rightarrow \nu_0 = \frac{|pk|}{2}$$

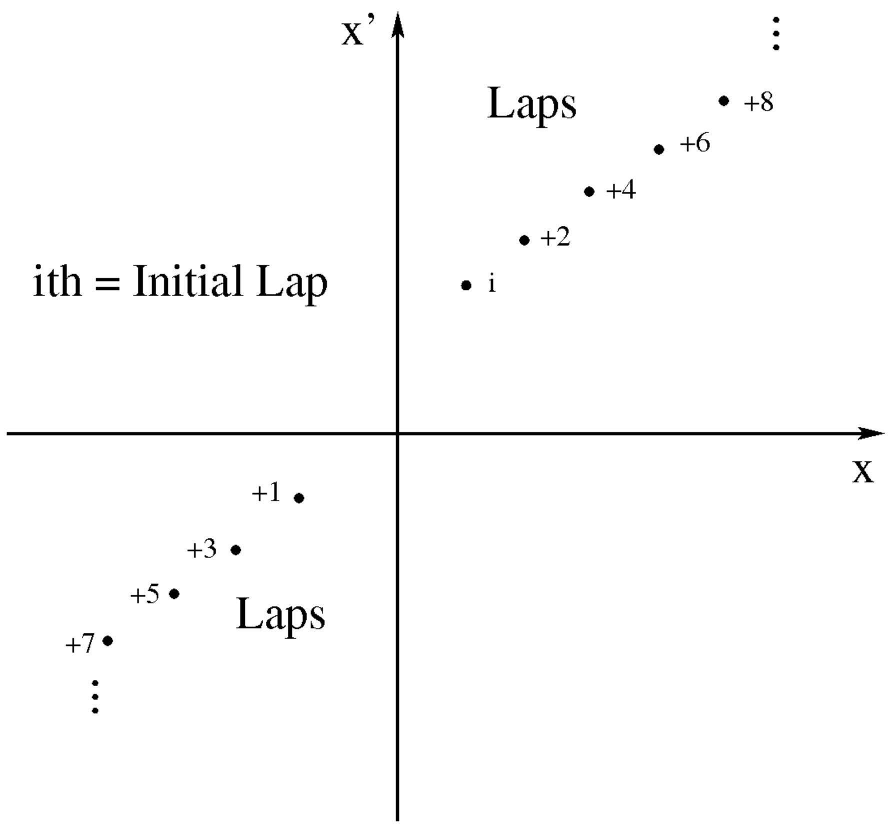
Therefore, to avoid quadrupole resonances, the following tune operating points are not allowed:

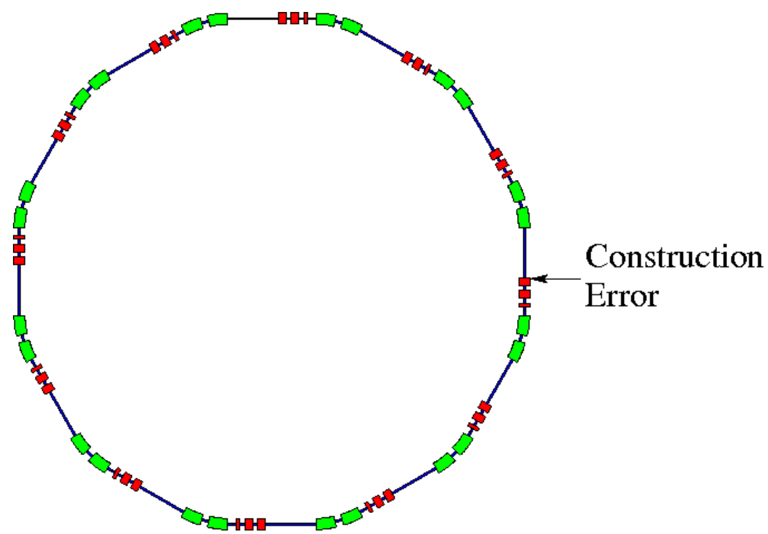
$$\boxed{\nu_0 \neq \frac{|pk|}{2} \qquad p = \begin{cases} 1, & \text{Random perturbation} \\ \mathcal{N}, & \text{Systematic perturbation} \end{cases} \qquad k = -\infty, \cdots, -1,\ 0,\ 1, \cdots, \infty}$$

- New restriction: tunes cannot be half-integer values
- Integers also restricted for $p = 1$ random, but redundant with dipole case
- Some large integers restricted for $p = \mathcal{N}$ systematic perturbations

Interpretation of result (new restrictions):

For a single (random) quadrupole error along the azimuth of a ring, a similar qualitative argument as presented in the dipole resonance case leads one to conclude that if a particle oscillates with ½ integer tune, then the orbit can “walk-off” on a lap-to-lap basis in phase-space:



## $n = 2$, Sextupole Perturbations:

$$n = 2, \implies m = 0,\ 1,\ 2$$

and the resonance conditions give the three constraints below:

$$n = 2,\ m = 0: \qquad 2\nu_0 + pk = \pm\nu_0$$

$$n = 2,\ m = 1: \qquad pk = \pm\nu_0$$

$$n = 2,\ m = 2: \qquad -2\nu_0 + pk = \pm\nu_0$$

Therefore, to avoid sextupole resonances, the following tunes are not allowed:

$$\nu_0 \neq \begin{cases} |pk| & \text{integer} \\ |pk|/2 & \text{half-integer} \\ |pk|/3 & \text{third-integer} \end{cases} \qquad p = \begin{cases} 1, & \text{Random perturbation} \\ \mathcal{N}, & \text{Systematic perturbation} \end{cases}$$

$$k = -\infty, \cdots, -1,\ 0,\ 1, \cdots, \infty$$

- Integer and ½-integer restrictions already obtained for dipole and quadrupole perturbations
- 1/3-integer restriction new

Higher-order ($n > 2$) cases analyzed analogously

- Produce more constraints but expected to be weaker as order increases

# General form of resonance condition

The general resonance condition (all *n*-values) for *x*-plane motion can be summarized as:

$$M\nu_0 = N \qquad \begin{aligned} M,\ N &= \text{Integers of same sign} \\ |M| &= \text{"Order" of resonance} \end{aligned}$$

- Higher order numbers *M* are typically less dangerous
  - Longer coherence length for validity of theory: effects not included can "wash-out" the resonance
  - Coefficients generally smaller

Particle motion is not (measure zero) really restricted to the *x*-plane, and a more complete analysis taking into account coupled *x*- and *y*-plane motion shows that the generalized resonance condition is:

- Place unperturbed y-orbit in rhs perturtation term, then leading-order expand analogously to x-case to obtain additional driving terms

$$M_x\nu_{0x} + M_y\nu_{0y} = N \qquad \begin{aligned} M_x,\ M_y,\ N &= \text{Integers of same sign} \\ |M_x| + |M_y| &= \text{"Order" of resonance} \end{aligned}$$

$$\nu_{0x} = x\text{-plane tune}$$

$$\nu_{0y} = y\text{-plane tune}$$

- Lower order resonances are more dangerous analogously to *x*-case

# Restrictions on machine operating points

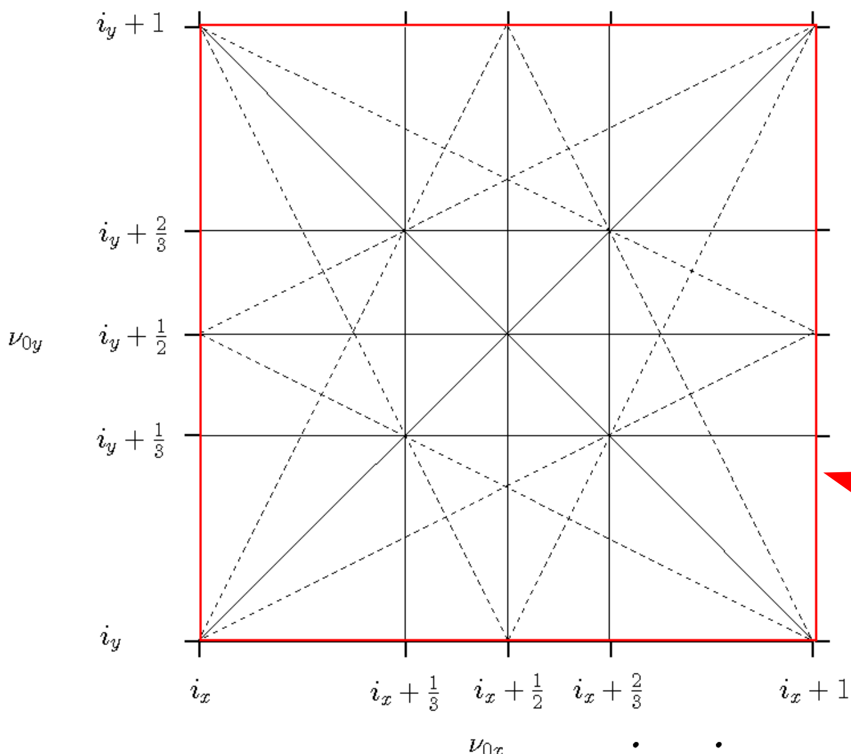
Tune restrictions are typically plotted in $\nu_{0x} - \nu_{0y}$ space order-by-order up to a max order value to find allowed tunes where the machine can safely operate

- Often 3rd order is chosen as a maximum to avoid
- Cases for random ($p = 1$) and systematic ( $p = \mathcal{N}$ ) perturbations considered

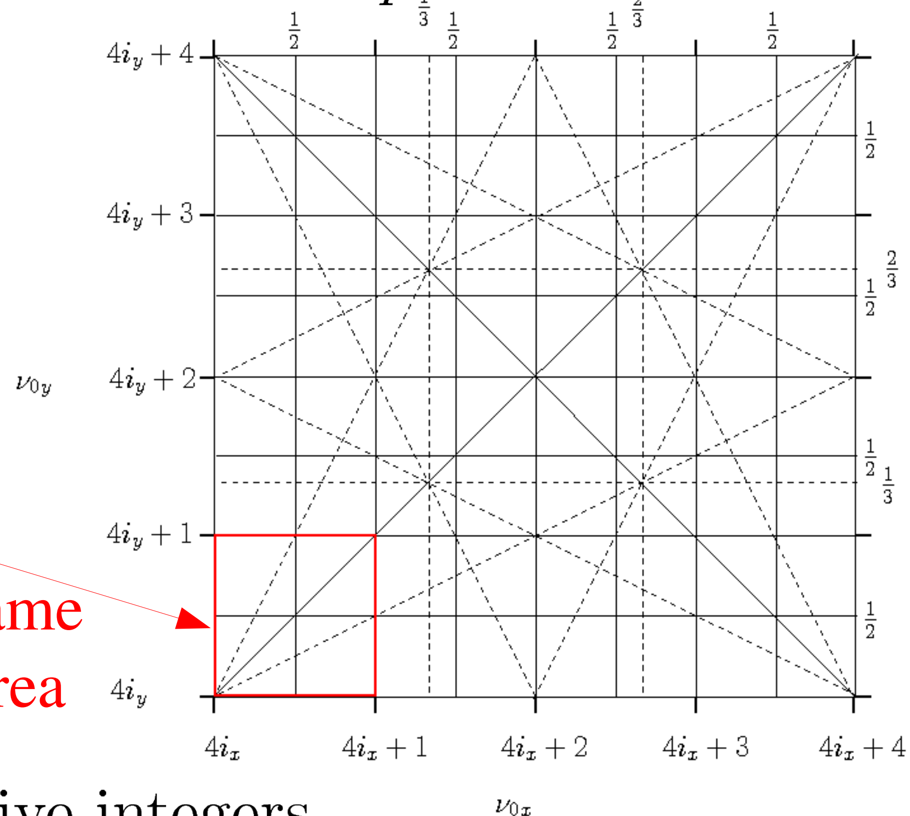
Machine operating points chosen as far as possible from low order resonance lines

<u>Random Perturbations</u> | <u>Systematic Perturbations</u>

$p = 1$ Adapted from Wiedemann $p = \mathcal{N} = 4$




Same Area




$i_x, i_y =$ postive integers

# Discussion: Restrictions on machine operating points

Random Errors:

- Errors always present and give low-order resonances
- Usually have weak amplitude coefficients
  - Can be corrected/compensated to reduce effects

Systematic Errors:

- Lead to higher-order resonances for large $N$ and a lower density of resonance lines (see plots on previous slide comparing the equal boxed red areas)
  - Large symmetric rings with high $N$ values have less operating restrictions from systematic errors
  - Practical issues such as construction cost and getting the beam into and out of the ring can lead to smaller $N$ values (racetrack lattice)
- BUT systematic error Amplitude coefficients can be large
  - Systematic effects accumulate in amplitude period by period

Resonances beyond 3rd order rarely need be considered

- Effects outside of model assumed tend to wash-out higher order resonances

More detailed treatments calculate amplitudes/strengths of resonant terms

- See accelerator physics references:

  Further info: Wiedemann, *Particle Accelerator Physics* (2007)

  Amplitudes/Strengths: Kolomenskii and Lebedev, *Theory of Circular Accel* (1966)

# S7: Space-Charge Effects on Particle Resonances
## S7A: Introduction

Ring operating points are chosen to be far from low-order particle resonance lines in *x*-*y* tune space. Processes that act to shift particle resonances closer towards the low-order lines can prove problematic:

- Oscillation amplitudes increase (spoiling beam quality and control)
- Particles can be lost

Tune shift limits of machine operation are often named "Laslett Limits" in honor of Jackson Laslett who first calculated tune shift limits for various processes:

- Image charges
- Image currents
- Internal beam self-fields
- ...

Processes shifting resonances can be grouped into two broad categories:

| | |
|---|---|
| Coherent | Same for every particle in distribution<br>- Usually most dangerous: full beam resonant |
| Incoherent | Different for particles in separate parts of the distribution<br>- Usually less dangerous: only effects part of beam |

Here we will analyze Laslett tune shift limits induced by coherent space-charge taking a KV distribution model for linear space-charge

- In KV model, space-charge forces interior to the beam are coherent because all particles have the same depressed tune

We will not analyze Laslett limits for other processes in these lectures. But the logical procedure is similar to the space-charge case.

# S7B: Laslett Space-Charge Limit

Laslett first obtained a space-charge limit for rings by assuming that the beam space-charge is uniformly distributed as in a KV model and thereby acts as a coherent shift to previously derived resonance conditions. Denote:

$$\nu_{0x} \equiv x\text{-tune (bare) in absence of space-charge}$$

$$\nu_x \equiv x\text{-tune (depressed) with uniform density beam}$$

$$\Delta\nu_x \equiv \nu_{0x} - \nu_x = \text{Space-charge tune shift} \qquad \Delta\nu_x \geq 0$$

Assume that dipole (integer) and quadrupole (half-integer) tunes only need be excluded when space-charge effects are included.

- Space-charge likely induces more washing-out of higher-order resonances

If the bare tune operating point is chosen as far as possible from ½ -integer resonance lines, the maximum space-charge induced tune shift allowed is ¼-integer, giving:

$$\Delta\nu_x|_{\max} = \frac{1}{4} \implies$$ Establishes maximum current (use KV results in lectures on Transverse Equilibrium Distributions)

- Analogous equation applies in the $y$-plane
  - Identical restriction in lattices with equal $x$- and $y$-focusing strengths

Consider a symmetric ring (not race track for simple arguments) with:

$$\mathcal{N} = \text{Number Lattice Periods}$$

$$L_p = \text{Lattice Period}$$

$$\sigma_0 = \text{Phase advance in } x\text{- or } y\text{-directions}$$

Gives bare (undepressed) tunes:

$$\Longrightarrow \qquad \boxed{\nu_{0x} = \nu_{0y} \equiv \nu_0 = \mathcal{N}\frac{\sigma_0}{2\pi}}$$

Defining the depressed tune in the presence of KV model space-charge analogously with:

$$\boxed{\nu_x = \nu_y \equiv \nu = \mathcal{N}\frac{\sigma}{2\pi}}$$

Then the allowed space-charge depression $\sigma/\sigma_0$ for $\delta\nu = \nu_0 - \nu = 1/4$ is:

$$\boxed{\sigma/\sigma_0|_{\min} = 1 - \frac{\pi/2}{\mathcal{N}\sigma_0}}$$

## Estimate of Maximum Perveance Allowed by Laslett Limit: Simple Continuous Focusing Estimate

Model the focusing as continuous and assume an unbunched, transverse matched KV distribution with:

$$k_{\beta 0} = \frac{\sigma_0}{L_p} = \frac{2\pi\nu_0}{\mathcal{N} L_p}$$

$$\kappa_x = \kappa_y = k_{\beta 0}^2 = \text{const}$$ Focusing

$$\varepsilon_x = \varepsilon_y \equiv \varepsilon = \text{const}$$ Emittance

$$Q = \frac{q\lambda}{2\pi\epsilon_0 m \gamma_b^3 \beta_b^2 c^2} = \text{const}$$ Perveance

The matched envelope equation gives:

$$r_x = r_y = r_b = \text{const}$$

$$\cancelto{0}{r_b''} + k_{\beta 0}^2 r_b - \frac{Q}{r_b} - \frac{\varepsilon^2}{r_b^3} = 0$$

$$\Longrightarrow \quad \boxed{r_b^2 = \frac{Q + \sqrt{4k_{\beta 0}^2 \varepsilon^2 + Q^2}}{2k_{\beta 0}^2}}$$

Depressed phase advance per lattice period can then be calculated from formulas in lectures on Transverse Equilibrium Distributions as:

$$\sigma = \sqrt{k_{\beta 0}^2 - \frac{Q}{r_b^2}} L_p = \varepsilon \int_{s_i}^{s_i + L_p} \frac{ds}{r_b^2}$$

Two forms equivalent from envelope equation

using

$$\nu = \mathcal{N} \frac{\sigma}{2\pi} \qquad k_\beta = \sqrt{k_{\beta 0}^2 - \frac{Q}{r_b^2}} \qquad \sigma = k_\beta L_p$$

and previous formulas gives:

$$\boxed{\nu = \nu_0 \sqrt{1 - \frac{2Q}{Q + \sqrt{\frac{16\pi^2 \nu_0^2 \varepsilon^2}{\mathcal{N}^2 L_p^2} + Q^2}}}}$$

Setting the phase shift to the Laslett current limit value

$$\boxed{\nu|_{Q = Q_{\max}} = \nu_0 - \frac{1}{4}}$$

gives a constraint for the maximum value of $Q = Q_{\rm max}$ to avoid 1/2-integer resonances:

$$\frac{2Q_{\rm max}}{Q_{\rm max} + \sqrt{\frac{16\pi^2\nu_0^2\varepsilon^2}{\mathcal{N}^2 L_p^2} + Q_{\rm max}^2}} = 1 - \left(\frac{\nu_0 - 1/4}{\nu_0}\right)^2 = \frac{1}{2\nu_0^2}(\nu_0 - 1/8)$$

This can be arraigned into a quadratic equation for $Q_{\rm max}$ and solved to show that the Laslett "current" limit expressed in terms of the maximum transportable perveance:

$$\boxed{\begin{aligned} Q < Q_{\rm max} &= \frac{\pi\varepsilon}{\mathcal{N}L_p}\left(\frac{\nu_0 - 1/8}{\nu_0}\right)\frac{1}{\sqrt{1 - \frac{1}{2\nu_0}\left(\frac{\nu_0-1/8}{\nu_0}\right)}} \\ &\simeq \frac{\pi\varepsilon}{\mathcal{N}L_p}\left(1 + \frac{1}{8\nu_0} + \text{Order}(1/\nu_0^2)\right)\end{aligned}}$$

$$\begin{aligned} Q &= \frac{q\lambda}{2\pi\epsilon_0 m\gamma_b^3\beta_b^2c^2} \\ &= \frac{qI}{2\pi\epsilon_0 m\gamma_b^3\beta_b c}\end{aligned}$$

$I = \text{Beam Current}$

// Example: Take (typical synchrotron numbers, represents peak charge in rf bunch)

$\mathcal{N}L_p = \mathcal{C} = \text{Ring Circumfrance} \sim 300\ \text{m}$

$\varepsilon \sim 50$ mm-mrad

Neglect $1/\nu_0$ term

$$\Longrightarrow \quad Q < Q_{\rm max} \simeq \frac{\pi\varepsilon}{\mathcal{C}} \simeq 5\times 10^{-7}$$

Not a lot of charge .... //

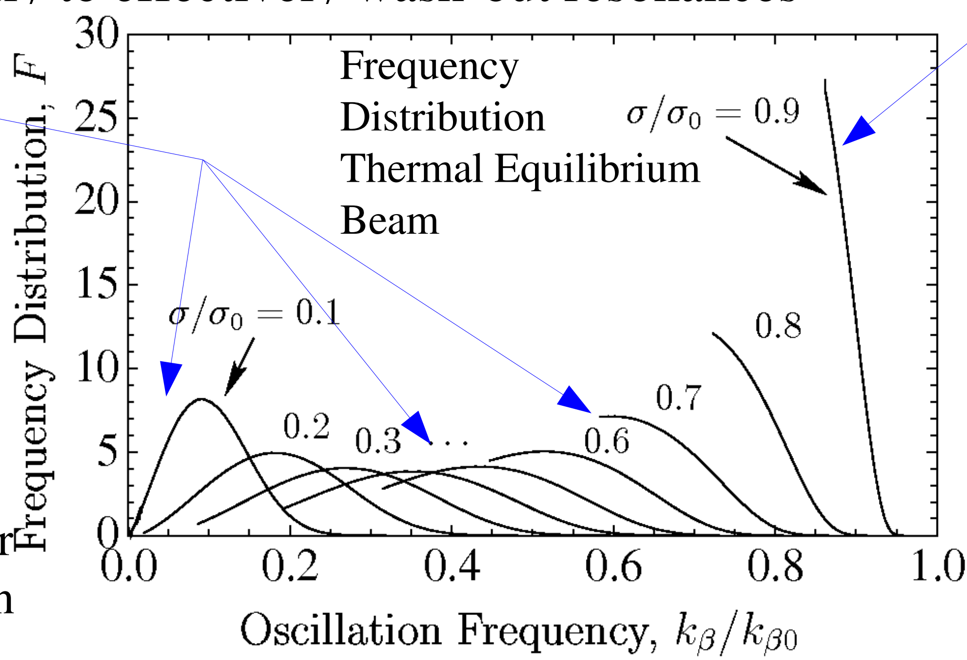
## Discussion:

Laslett limit may be overly restrictive:

- KV model assumes all particles in beam have the same tune
  - Significant spectrum of particle tunes likely in real beam
    Particularly if space-charge strong: see Transverse Equilibrium Dists, S7
  - No equilibrium beam: core oscillates and space-charge may act incoherently to effectively wash-out resonances

For strong space-charge:

- Frequency spread large and KV approx bad
- Does not work in spite of beam density being near uniform density for smooth distribution



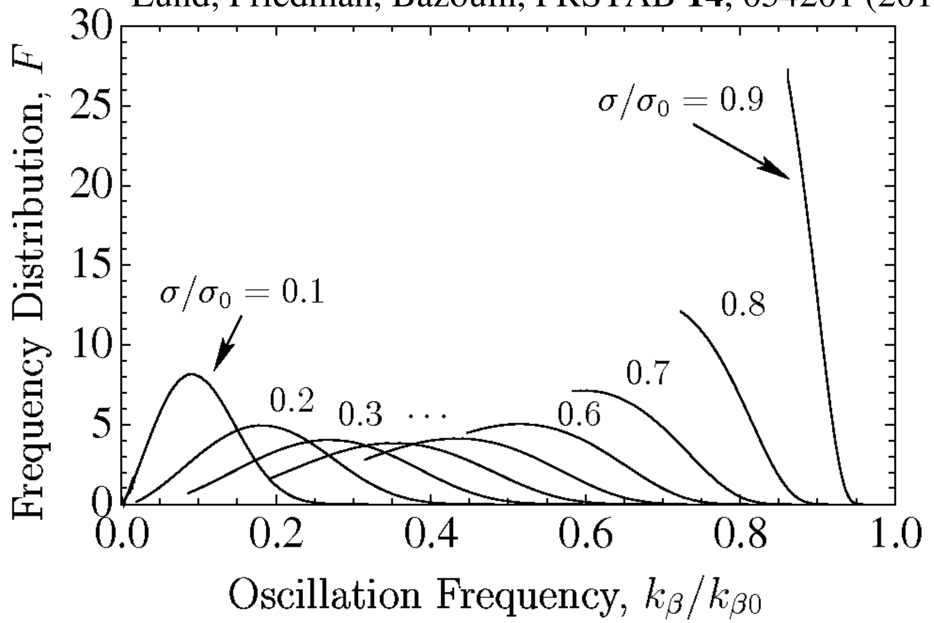

For weak space-charge:

- Frequency spread small and KV approx good
- Works in spite of beam density being far from uniform density for smooth distribution

- Simulations suggest Laslett limit poses little issues over 10s – 100s of laps in rings (Small Recirculator, LLNL) and in fast bunch compressions in rings
  - Longer simulations very difficult to resolve: see Simulation Techniques
- Future experiments can hopefully address this issue
  - University of Maryland electron ring will have strong space-charge

Frequency distribution for an idealized 1D thermal equilibrium beam suggest significant deviations from KV coherent picture when space-charge intensity is high.

Lund, Friedman, Bazouin, PRSTAB **14**, 054201 (2011)




| | | Statistical Measures | | | |
|---|---|---|---|---|---|
| | | Mean: | RMS: $\sigma_F =$ | Width: | Relative Width: |
| $\sigma/\sigma_0$ | $\Delta$ | $\mu_F = \overline{k_\beta}/k_{\beta 0}$ | $\sqrt{\overline{k_\beta^2} - \overline{k_\beta}^2}/k_{\beta 0}$ | $F_w = 2\sqrt{3}\sigma_F$ | $F_w/\mu_F$ |
| 0.9 | 2.879 | 0.886 | 0.0176 | 0.0610 | 0.0689 |
| 0.8 | 1.093 | 0.774 | 0.0354 | 0.123 | 0.159 |
| 0.7 | 0.5181 | 0.663 | 0.0531 | 0.184 | 0.277 |
| 0.6 | 0.2500 | 0.557 | 0.0696 | 0.241 | 0.433 |
| 0.5 | 0.1097 | 0.456 | 0.0833 | 0.289 | 0.634 |
| 0.4 | $3.780 \times 10^{-2}$ | 0.361 | 0.0915 | 0.317 | 0.878 |
| 0.3 | $7.562 \times 10^{-3}$ | 0.274 | 0.0898 | 0.311 | 1.14 |
| 0.2 | $3.649 \times 10^{-4}$ | 0.190 | 0.0750 | 0.260 | 1.37 |
| 0.1 | $5.522 \times 10^{-8}$ | 0.102 | 0.0465 | 0.161 | 1.58 |

## Discussion Continued:

- Even if internal resonances in the core of the beam are washed out due to nonlinear space-charge at high intensity, centroid resonances may still behave more as a single particle (see notes on Transverse Centroid and Envelope Descriptions of Beam Evolution) to limit beam control.
  - Steering and correction can mitigate low order centroid instabilities
  - Centroid will also have (likely weak if steering used) image charge correction to the tune
- Caution: Terminology can be very bad/confusing on topic. Some researchers:
  - Call KV Laslett space-charge shift an "incoherent tune shift" limit in spite of it being (KV) coherent
  - Call anything space-charge related "incoherent" regardless of model
  - Call beam transport near the KV Laslett space-charge shift limit a "space charge dominated beam" even though space-charge defocusing likely is only a small fraction of the applied focusing

## More research on this topic is needed!

- Higher intensities can open new applications for energy and material processing
- Many possibilities to extend operating range of existing machines and make new use of developed technology
- Good area for graduate thesis projects!

# Laslett Space Charge Limit for an Elliptical KV Beam

For more basic info, see material in the G. Franchetti lecture from the 2014 CERN Accelerator School in the reference below:

- Will summarize results from here and other sources in future version of notes

http://cas.web.cern.ch/CAS/CzechRepublic2014/Lectures/FranchettiSC.pdf

# S8: Limits Induced by Space-Charge Collective Modes

Add in future edition of notes here or in kinetic theory:

Review simple 1D theory results of Sacherer and implications for rings

# Corrections and suggestions for improvements welcome!

These notes will be corrected and expanded for reference and for use in future editions of US Particle Accelerator School (USPAS) and Michigan State University (MSU) courses. Contact: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Please provide corrections with respect to the present archived version at:

https://people.nscl.msu.edu/~lund/uspas/bpisc_2015

Redistributions of class material welcome. Please do not remove author credits.

# References: For more information see:

These course notes are posted with updates, corrections, and supplemental material at:

https://people.nscl.msu.edu/~lund/uspas/bpisc_2015

Materials associated with previous and related versions of this course are archived at:

JJ Barnard and SM Lund, *Beam Physics with Intense Space-Charge,* USPAS:

http://hifweb.lbl.gov/USPAS_2011 2011 Lecture Notes + Info

https://people.nscl.msu.edu/~lund/uspas/bpisc_2011/

http://uspas.fnal.gov/programs/past-programs.shtml (2008, 2006, 2004)

JJ Barnard and SM Lund, *Interaction of Intense Charged Particle Beams with Electric and Magnetic Fields*, UC Berkeley, Nuclear Engineering NE290H

http://hifweb.lbl.gov/NE290H 2009 Lecture Notes + Info

# References: Continued (2):


E. D. Courant and H. S. Snyder, "Theory of the Alternating Gradient Synchrotron," Annals of Physics **3**, 1 (1958)

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