Problem Set # 1

Barnard/Lund USPAS 2015

Problem 1

Consider a round uniform ion beam with a current of 1 ampere, compose of Au^+ ions (atomic mass A =197), a kinetic energy of 2 MeV, a beam radius of 2 cm and normalized emittance of 1 mm-mrad.

Calculate for these beam parameters (to 1 or 2 significant figures):

a) $\beta = v_0/c$ (assume non-relativistic beam)
b) n = number density of ions in beam
c) kT = transverse temperature (express in eV)
d) λ_D = transverse Debye length
e) Q = generalized perveance
f) Λ = plasma parameter
g) $\Delta\phi$ = potential difference between center and edge of beam.

For reference:

$e = 1.6 \times 10^{-19}$ C [proton charge]
$k_B = 1.38 \times 10^{-23}$ J/K [Boltzmann's constant]
$\varepsilon_0 = 8.854 \times 10^{-12}$ F/m [permittivity of free space]
$c = 3 \times 10^8$ m/s [speed of light in free space]
$m_{amu} = 1.66 \times 10^{-27}$ kg [atomic mass unit]
$m_{amu}c^2 = 931.1 \times 10^6$ eV [atomic mass unit in eV]

Problem 2

Show that:

$$\left\langle r\frac{\partial\phi}{\partial r}\right\rangle = -\frac{\lambda}{4\pi\varepsilon_0}$$

for a charge distribution in which $\rho(r,\theta)=\rho(r)$ only.

Here $\lambda =$ line charge density $= \int_0^\infty 2\pi r\rho(r)dr$

$$\langle g\rangle = \frac{1}{\lambda}\int_0^\infty g(r)2\pi r\rho(r)dr$$

where g is any beam quantity that is a function of r only.

Problem 3

Let the equation of motion for a single particle be:

$$x'' = -\alpha(s)x^n$$

Here x is the usual transverse coordinate and s is the longitudinal coordinate.

Calculate the derivative with respect to s of the square of the emittance:

$$\varepsilon^2 = 16\left(\langle x^2\rangle\langle x'^2\rangle - \langle xx'\rangle^2\right)$$

Express $\dfrac{d\varepsilon^2}{ds}$ in terms of $\langle x^2\rangle, \langle xx'\rangle, \langle x' x^n\rangle$, and $\langle x^{n+1}\rangle$.

For what value of n is $\dfrac{d\varepsilon^2}{ds}$ identically zero?

Problem 4

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JPD Problem 1 – Larmor Frame

For a uniform solenoidal channel:

$$B_z^a(s) = B_0 = \text{const}$$

with no acceleration

$$\gamma_b \beta_b = \text{const}$$

and an axisymmetric $(\partial/\partial\theta = 0)$ beam with

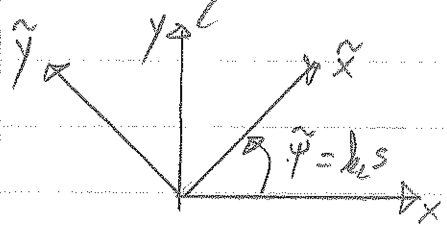
$$\frac{\partial \phi}{\partial \vec{x}_\perp} = \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial \vec{x}_\perp} = \frac{\partial \phi}{\partial r}\frac{\vec{x}_\perp}{r} \qquad r = \sqrt{x^2+y^2}$$

the particle equations of motion reduce to:

$$x'' = \frac{qB_0}{m\gamma_b\beta_b c}\, y' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial r}\frac{x}{r}$$

$$y'' = -\frac{qB_0}{m\gamma_b\beta_b c}\, x' - \frac{q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial \phi}{\partial r}\frac{y}{r}$$

a) Parallel steps taken in the class notes to transform the equations of motion to a co-rotating frame:



$k_L = \text{const} =$ Larmor wavenumber

$$\tilde{x} = x\cos(k_L s) + y\sin(k_L s)$$
$$\tilde{y} = -x\sin(k_L s) + y\cos(k_L s)$$

Find an expression for k_L to reduce the equations of motion to the decoupled form:

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$$\tilde{x}'' + \mathcal{K}\tilde{x} = \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{x}}{r}$$

$$\tilde{y}'' + \mathcal{K}\tilde{y} = \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial r}\frac{\tilde{y}}{r}$$

and identify $\mathcal{K} = \text{const.}$

Hint:
The transformation can be carried out directly. But you may find the algebra simpler using complex coordinates as in the class notes:

$$\underset{\sim}{z} = x + iy \qquad i = \sqrt{-1} \qquad e^{i\theta} = \cos\theta + i\sin\theta$$
$$\tilde{\underset{\sim}{z}} = \tilde{x} + i\tilde{y}$$

b) If the direction of the magnetic field is reversed:

$$B_0 \rightarrow -B_0$$

how will the dynamics be influenced?

c) Neglect space-charge:

$$\phi = 0$$

and sketch a typical orbit in the rotating Larmor frame. Will this orbit appear more complicated in the Laboratory frame? Why?

Bonus: Sketch the orbit taking advantage of simple choices of initial conditions that can always be made through choice of coordinates.

Problem 5

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a) From the Lorentz force equation, show that a static magnetic field $\vec{B}^a$ cannot change the kinetic energy of a particle; $\mathcal{E} = (\gamma - 1) mc^2 = \text{const.}$

$$m \frac{d}{dt}(\gamma \vec{\beta}) = q \vec{\beta} \times \vec{B}^a \qquad \text{Lorentz Force Eqn}$$

$$\gamma = \frac{1}{\sqrt{1 - \vec{\beta}^2}} \qquad \vec{\beta} = \frac{1}{c}\frac{d\vec{x}}{dt}$$

b) In class, it was shown for a solenoid magnet with azimuthal symmetry ($\partial/\partial\theta = 0$), the magnetic field can be expanded in terms of the on-axis field as:

$$B_r^a = \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{\nu!(\nu-1)!} \frac{\partial^{2\nu-1} B_{z0}(z)}{\partial z^{2\nu-1}} \left(\frac{r}{2}\right)^{2\nu-1}$$

$$B_z^a = B_{z0}(z) + \sum_{\nu=1}^{\infty} \frac{(-1)^\nu}{(\nu!)^2} \frac{\partial^{2\nu} B_{z0}(z)}{\partial z^{2\nu}} \left(\frac{r}{2}\right)^{2\nu}$$

$$B_{z0}(z) \equiv B_z^a(r=0, z) \qquad \text{on-axis field}$$

Take $\mathcal{E} \simeq \mathcal{E}_b$ and apply the paraxial equations of motion (see Sec S1G class notes) to show that if nonlinear applied force terms are dropped ($\propto x^2, xy, xy', \text{etc}$), the equations of motion are

$$x'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}x' - \frac{B_{z0}'}{2[B\rho]}y - \frac{B_{z0}}{[B\rho]}y' = \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial x}$$

$$y'' + \frac{(\gamma_b\beta_b)'}{(\gamma_b\beta_b)}y' + \frac{B_{z0}'}{2[B\rho]}x + \frac{B_{z0}}{[B\rho]}x' = \frac{-q}{m\gamma_b^3\beta_b^2c^2}\frac{\partial\phi}{\partial y}$$

$$[B\rho] = \frac{\gamma_b\beta_b mc}{q}$$

c) Qualitative only: If there are no axial acceleration fields, we take $\gamma_b\beta_b = \text{const}$, $[B\rho] = \text{const}$, are the results of part b) inconsistent with part a)? If so, could they still be OK to use?

d) Show if we take $\vec{B}^a = \nabla\times\vec{A}$, we can generate the linear field components

$$B_r^a = -\frac{1}{2}\frac{\partial B_{z0}}{\partial z}r$$

$$B_z^a = B_{z0}$$

from $\vec{A} = \frac{\hat{\theta}}{2}B_{z0}r$

$$\nabla\times\vec{A} = \hat{r}\left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right) + \hat{\theta}\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + \frac{\hat{z}}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)$$

e) Paraxial approximate the canonical angular momentum

$$P_\theta = [\vec{x}\times(\vec{p}+q\vec{A})]\cdot\hat{z}$$

as

$$P_\theta \simeq m\gamma_b\beta_b c(xy' - yx') + q\frac{B_{z0}}{2}(x^2+y^2)$$

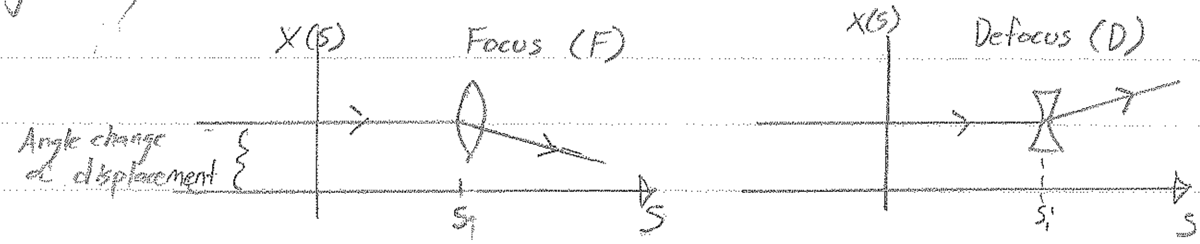
Use eqn of motion in part b) to show that $\frac{d}{dz}P_\theta = 0 \Rightarrow P_\theta = \text{const}$ for a beam with $\phi = \phi(r)$.

Problem 6

TPD Problem 4

S.M. Lund P4/

4/ A thin lense changes the angle of a particle trajectory but not the coordinate:



This action can be specified by transfer matrices applied at $s = s_i$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = M(s|s_i) \begin{pmatrix} x_i \\ x_i' \end{pmatrix}$$

Focusing:

$$M_F = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

$$f > 0$$

Defocusing:

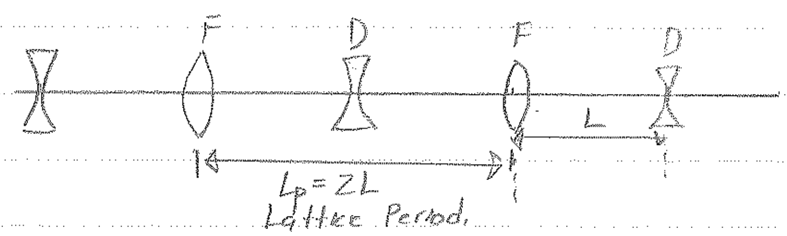
$$M_D = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix}$$

$$f > 0$$

From TPD Problem 3, a free-space drift of length L has a transport matrix:

$$M_O = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

Consider a lattice of period $2L$ made up of equally spaced F and D lenses with equal values of f.



This is the simplist "FODO" alternating gradient lattice!

a) Use the transfer matrix analysis developed in class to find the range of f for which the particle orbit is stable.

b) Calculate $\cos\sigma_0$ where σ_0 is the particle phase advance.

c) For the case of of f chosen to correspond to the stability limit, sketch the motion of a particle with initial condition

$$\lim_{s \to s_i^-} x(s) = x_0$$
$$\lim_{s \to s_i^-} x'(s) = x_0/L$$

where $s = s_i$ is the axial location of a focusing thin lens kick, and $s \to s_i^-$ is just before the kick. Sketch the particle orbit for focusing strength slightly larger than the stability limit. Superimpose the orbit sketch on a diagram of the lattice (see below):