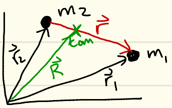
Recap: 2 bodies interacting via Central Forces (ch 8)

$\vec{F}_{12} = -\vec{F}_{21} \equiv \vec{F}$, Central force further obeys $\vec{F} = \hat{r}\, F(r)$



$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M}, \quad \vec{r} = \vec{r}_1 - \vec{r}_2$$

* Writing $\vec{F}_{12} = m_1 \ddot{\vec{r}}_1$ & $\vec{F}_{21} = m_2 \ddot{\vec{r}}_2$ in terms of $(\vec{r}, \vec{R})$

$$\Rightarrow \boxed{\begin{aligned} M\ddot{\vec{R}} &= 0 \\ \mu \ddot{\vec{r}} &= \vec{F}(\vec{r}) \end{aligned}}$$

i.e., $\vec{R}$ & $\vec{r}$ dynamics decouple

Motion of $\vec{R}$ trivial (free particle)

$\Rightarrow$ effective 1-body problem (Work in COM frame $\vec{R} \equiv 0$)

* Assuming $\vec{F}(\vec{r}) = -\vec{\nabla} U(r)$ and $\vec{F} = \hat{r}\, F(r)$ (i.e., conservative central force)

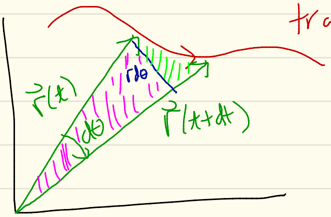
$$\Rightarrow \boxed{\begin{aligned} E &= \tfrac{1}{2}\mu \dot{\vec{r}}^{\,2} + U(r) = \text{constant (ie, conserved)} \\ \vec{L} &= \vec{r} \times \vec{p} \quad (\vec{p} = \mu \dot{\vec{r}}) = \text{constant} \end{aligned}}$$

$\frac{d\vec{L}}{dt} = 0 \Rightarrow$ motion lies in plane $\perp$ to constant $\vec{L}$ (i.e., $\vec{r}, \vec{p} \perp \vec{L}$)

$\therefore$ effective 2D prob. (Polar Coords)

$$\boxed{\begin{aligned} |\vec{L}| &\equiv \ell = \mu r^2 \dot{\theta} \\ E &= \tfrac{1}{2}\mu \dot{r}^2 + \tfrac{1}{2}\mu r^2 \dot{\theta}^2 + U(r) = \tfrac{1}{2}\mu \dot{r}^2 + \tfrac{1}{2}\frac{\ell^2}{\mu r^2} + U(r) \end{aligned}}$$

Ex: Kepler's 2^{nd} Law (Constant "areal velocity")



dA = area swept out by $\vec{r}$ in $(t, t+dt)$ = pink triangle area + $\cancel{\text{green triangle area}}$ $\mathcal{O}(dr\,d\theta)$

$$= \frac{1}{2} r^2 d\theta$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} \qquad \underline{\text{use}}\ \dot{\theta} = \frac{\ell}{\mu r^2}$$

$$\Rightarrow \boxed{\frac{dA}{dt} = \frac{1}{2}\frac{\ell}{\mu} = \text{constant}}$$

i.e., equal area swept out by $\vec{r}(t)$ in equal time intervals

From E, l eqns, we can formally (i.e., not always useful in a practical sense due to nasty integrals) solve for r, θ

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}\left[E - U - \frac{1}{2}\frac{l^2}{\mu r^2}\right]} \Rightarrow \boxed{t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{\mu}\left[E-U-\frac{1}{2}\frac{l^2}{\mu r^2}\right]}}} \quad (1)$$

(i.e., gives $t = t(r)$). Can be (formally) inverted to give $r = r(t)$.

*turning to θ:

$$d\theta = \frac{d\theta}{dt}\frac{dt}{dr}\,dr = \frac{\dot{\theta}}{\dot{r}}\,dr \qquad \text{use } \dot{\theta} = \frac{l}{\mu r^2}$$

$$\Rightarrow d\theta = \frac{l/\mu r^2}{\sqrt{\frac{2}{\mu}\left[E-U-\frac{1}{2}\frac{l^2}{\mu r^2}\right]}}\,dr$$

$$\Rightarrow \boxed{\theta(r) = \int_{r_0}^{r} \frac{\frac{l}{r^2}\,dr}{\sqrt{2\mu\left(E-U-\frac{1}{2}\frac{l^2}{\mu r^2}\right)}}} \quad (2)$$

* (1) + (2) formally solve the problem, but not always useful if $U(r)$ too complicated + integral can't be done.

- turns out for $F(r) \propto r^n$ (hence $U(r) \propto r^{n+1}$), $n = 5, 3, 0, -4, -5, -7$ can be expressed as "Elliptic Integrals" that can be looked up in tables

- for $n = 1, -2, -3$, can be expressed in terms of trig functions

Alternative way to solve the problem directly from Newton's EOM

* rather than solving for $r(t)$, sometimes it's useful to find $r = r(\theta)$. Here, it's best done directly via the $\vec{F} = \mu\vec{a}$ eqns.

$$\mu\ddot{\vec{r}} = \hat{r}F(r) \qquad \ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

$$\Rightarrow \mu(\ddot{r} - r\dot{\theta}^2) = F(r) = -\frac{\partial U}{\partial r} \quad (1)$$

$$\mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \quad (2)$$

NOTE: (2) can be written as $\frac{d}{dt}(\underbrace{\mu r^2\dot{\theta}}_{\ell}) = 0$ $\qquad \dot{\theta} = \frac{\ell}{\mu r^2}$

* let $u = \frac{1}{r}$: $\quad \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\frac{\dot{r}}{\dot{\theta}} = -\frac{\mu}{\ell}\dot{r} \quad$ (a)

$$\frac{d^2u}{d\theta^2} = -\frac{\mu}{\ell}\frac{d}{d\theta}(\dot{r}) = -\frac{\mu}{\ell}\frac{dt}{d\theta}\frac{d}{dt}(\dot{r}) = -\frac{\mu}{\ell}\frac{1}{\dot{\theta}}\ddot{r} = -\frac{\mu^2}{\ell^2}r^2\ddot{r} \quad \text{(b)}$$

(b) $\Rightarrow$ $\boxed{\ddot{r} = -\frac{\ell^2}{\mu^2}u^2\frac{d^2u}{d\theta^2}}$

* $\boxed{r\dot{\theta}^2 = r\frac{\ell^2}{\mu^2 r^4} = \frac{\ell^2}{\mu^2 r^3} = \frac{\ell^2}{\mu^2}u^3}$

plug into eqn (1) + simplify a bit.

$$\Rightarrow \quad \frac{d^2 u}{d\theta^2} + u = -\frac{\mu}{\ell^2}\frac{1}{u^2}F\left(\frac{1}{u}\right)$$

or $u = \frac{1}{r}$

$$\boxed{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{\ell^2}F(r)} \quad \underline{**}$$

$\underline{\text{Ex}}$: $F(r) = -\frac{K}{r^2} = -Ku^2$ $\qquad u'' + u = C \qquad C = \frac{K\mu}{\ell^2}$

$$u(\theta) = u_{HG} + u_{IHG}$$

$$u''_{HG} + u_{HG} = 0 \Rightarrow u_{HG} = A\cos(\theta - \delta) \qquad A, \delta \text{ from IC's}$$

$$u_{IHG} = C$$

$$\Rightarrow u(\theta) = A\cos(\theta - \delta) + C$$

or $\boxed{\frac{1}{r} = A\cos(\theta-\delta) + C}$

* Centrifugal energy + the Effective Potential

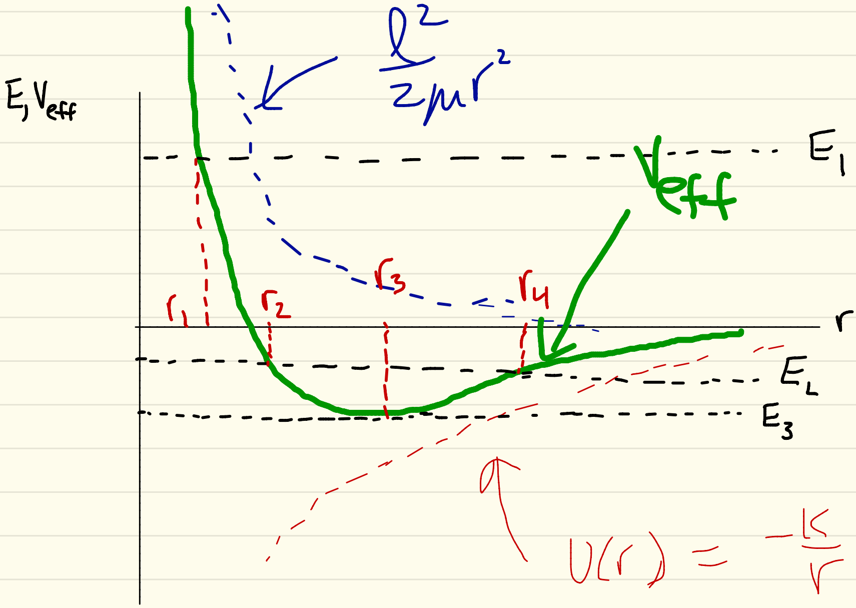
$$E = \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{\ell^2}{2\mu r^2} + U(r)}_{\equiv\, V_{eff}(r)}$$

Quasi 1-D problem like in ch. 2,3

$$\mu r\dot{\theta}^2 = \frac{\mu v_\theta^2}{r} \quad \text{"Centrifugal force"}$$

i.e., $F_{eff}(r) = -\frac{\partial V_{eff}}{\partial r} = \frac{\ell^2}{\mu r^3} + F(r)$

*Just like we did in ch 2,3, where we studied 1d motion with $E = \frac{1}{2}m\dot{x}^2 + U(x)$, we can make qualitative statements about the possible motions by invoking energy arguments



① $E_1 > 0$: Unbound motion

② $V_{eff}^{min} < E_2 < 0$ $\quad r_2 \leq r \leq r_4$ bounded "apsidal distances"

③ $E_3 = V_{eff}^{min}$ $\quad r = r_3$ (circular motion)

$$F_{eff} = -\left.\frac{\partial V_{eff}}{\partial r}\right|_{r_3} = 0 = \frac{\mu v_\theta^2}{r} - \frac{K}{r^2}$$

* Note for the present $U(r) = -\frac{k}{r}$ example,

$$V_{eff}^{min} = -\frac{\mu k^2}{2\ell^2}$$

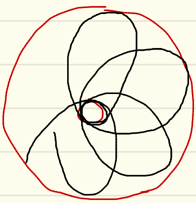
$\Rightarrow$ impossible to have $E < V_{eff}^{min}$

as

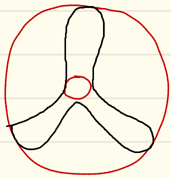
$$E - V_{eff} = \frac{1}{2}\mu\dot{r}^2$$

$$\Rightarrow \dot{r}^2 < 0 \quad \text{impossible.}$$

* Back to bounded motion (E_2 case) between r_2, r_4



open orbit, never repeats



closed orbit repeats after finite # oscillation