Phy 321 HW #1 Solutions

1.) Taylor expansion: $f(x) \approx f(x_0) + (x-x_0)f'(x_0) + \frac{1}{2!}(x-x_0)^2 f''(x_0) + \cdots$

where, $x_0 = \frac{\pi}{4}$ and $f(x) = \sin x$

$f(x) = \sin x \Rightarrow f(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$f'(x) = \cos x \Rightarrow f'(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

$f''(x) = -\sin x \Rightarrow f''(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$

$f'''(x) = -\cos x \Rightarrow f'''(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$

etc...

$$\boxed{\therefore \sin x \simeq \frac{1}{\sqrt{2}}\left(1 + (x-\frac{\pi}{4}) - \frac{1}{2!}(x-\frac{\pi}{4})^2 - \frac{1}{3!}(x-\frac{\pi}{4})^3 + \cdots\right)}$$

2.) $\vec{A} = \hat{i} - \hat{j} + 2\hat{k}$, $\vec{B} = -2\hat{i} + \hat{j} + 3\hat{k}$

a) $\vec{A}-\vec{B} = \underline{3\hat{i} - 2\hat{j} - \hat{k}}$, and $|\vec{A}-\vec{B}| = \sqrt{(3)^2 + (-2)^2 + (-1)^2} = \underline{\sqrt{14} = 3.74}$

b) let $\hat{A} \equiv \frac{\vec{A}}{|\vec{A}|}$. From the def. of the dot product, $\vec{B}\cdot\hat{A}$ gives the component of $\vec{B}$ along $\vec{A}$.

$$\Rightarrow B_A = \frac{\vec{B}\cdot\vec{A}}{A} = \frac{-2-1+6}{\sqrt{1^2+1^2+2^2}} = \underline{\frac{3}{\sqrt{6}} = 1.225}$$

c) Since $\vec{A}\cdot\vec{B} = AB\cos\theta \Rightarrow \cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{3}{\sqrt{6}\cdot\sqrt{14}} = \frac{\sqrt{3}}{\sqrt{2}\cdot\sqrt{14}} \Rightarrow \theta = \cos^{-1}\left(\sqrt{\tfrac{3}{2}}\,\tfrac{1}{\sqrt{14}}\right) = 1.237$ rad $= 70.9°$

d) $\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}$

$= (-3-2)\hat{i} + (-4-3)\hat{j} + (1-2)\hat{k}$

$= \underline{-5\hat{i} - 7\hat{j} - \hat{k}}$

2e) $(\vec{A}+\vec{B}) \times (\vec{A}-\vec{B}) = \cancel{\vec{A}\times\vec{A}}^{\,0} - \vec{A}\times\vec{B} + \vec{B}\times\vec{A} - \cancel{\vec{B}\times\vec{B}}^{\,0}$

$= -2\,\vec{A}\times\vec{B} = 10\hat{i} + 14\hat{j} + 2\hat{k}$

3) $z = h(x,y) = 10 + \frac{1}{2}x + \frac{1}{4}y + \frac{1}{2}xy - \frac{1}{4}x^2 - \frac{1}{2}y^2$

a) Find extremum by setting $\vec{\nabla} h(x,y) = 0$

1) $\frac{\partial h}{\partial x} = \frac{1}{2} + \frac{1}{2}y - \frac{1}{2}x = 0$
2) $\frac{\partial h}{\partial y} = \frac{1}{4} + \frac{1}{2}x - y = 0$

} 2 eqns, 2 unknowns. Solving for x & y:

2) $\Rightarrow$ $y = \frac{1}{4} + \frac{1}{2}x$. plug into 1)

$$0 = \frac{1}{2} + \frac{1}{8} + \frac{1}{4}x - \frac{1}{2}x = \frac{5}{8} - \frac{1}{4}x \Rightarrow \boxed{x = \frac{5}{2}}$$

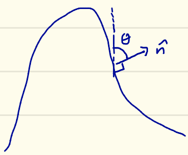
Plugging back into 2) $\Rightarrow$ $y = \frac{1}{4} + \frac{5}{4} = \boxed{\frac{3}{2}}$

Finally, evaluating $h\left(\frac{5}{2}, \frac{3}{2}\right) = 10 + \frac{5}{4} + \frac{3}{8} + \frac{15}{8} - \frac{25}{16} - \frac{9}{8}$

$$= \frac{160 + 20 + 6 + 30 - 25 - 18}{16}$$

$$= \frac{216 - 43}{16} = \boxed{10.8125}$$

b) We want to find the normal vector $\hat{n}$ to the mountain at $(x,y) = (1,1)$



* recall that for a 3d surface $f(x,y,z) = C$ (C = constant), then $\dfrac{\vec{\nabla} f(x,y,z)}{|\vec{\nabla} f(x,y,z)|} = \hat{n}$

* for the present case, we let $f(x,y,z) = z - h(x,y)$

$$\therefore \vec{\nabla} f = -\left(\tfrac{1}{2} + \tfrac{y}{2} - \tfrac{x}{2}\right)\hat{\imath} - \left(\tfrac{1}{4} + \tfrac{x}{2} - y\right)\hat{\jmath} + \hat{k}$$

$$\vec{\nabla} f \Big|_{x,y=1,1} = -\tfrac{1}{2}\hat{\imath} + \tfrac{1}{4}\hat{\jmath} + \hat{k}$$

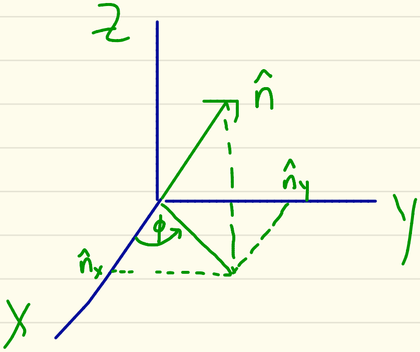
$$\boxed{\therefore \hat{n} = \frac{-\frac{1}{2}\hat{\imath} + \frac{1}{4}\hat{\jmath} + \hat{k}}{\sqrt{\frac{1}{4} + \frac{1}{16} + 1}} = -.436\,\hat{\imath} + .218\,\hat{\jmath} + .873\,\hat{k}}$$

Finally, we use that $\hat{n} \cdot \hat{k} = \cos\theta \Rightarrow$

$$\boxed{\begin{aligned} \theta &= \cos^{-1}[.873] \\ &= .5095 \text{ rad} \\ &= 29.19^\circ \end{aligned}}$$

*

c.) First, let's find the azimuthal angle ϕ (measured from x-axis)



$$\tan\phi = \frac{\hat{n}_y}{\hat{n}_x} \Rightarrow \phi = \tan^{-1}\left(\frac{.218}{-.436}\right) = -.464 \text{ rad}$$

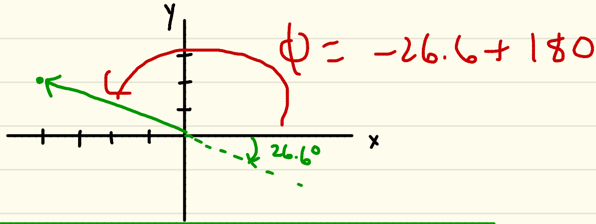
$$\phi = -26.56^\circ + 180^\circ$$

$$\phi = 153.4^\circ$$

Note: If you're rusty w/ your inverse Trig functions + wondering why I took

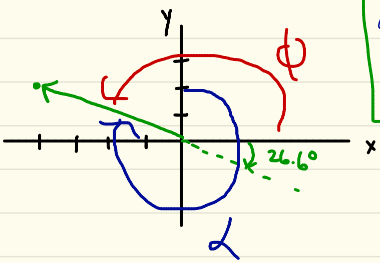
$$\phi = \tan^{-1}\left(\frac{.218}{-.436}\right) = -26.6^\circ$$

and then added 180° to it to get $\phi = 153.4^\circ$, draw a little picture in xy plane of $(n_x\hat{\imath} + n_y\hat{\jmath})$



$\boxed{\text{So, } \phi = 153.4^\circ \text{ measured from x-axis as shown.}}$

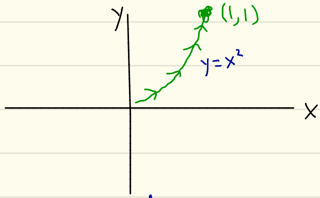
Lastly, we can convert this to a compass ∡, which is measured from the North (Y) axis in the <u>clockwise</u> direction



$$\boxed{\begin{aligned}\text{compass angle } \alpha &= 360 - (153.4 - 90)\\ &= 296.6^\circ\end{aligned}}$$

4) let $\vec{F} = y^2\,\hat{\imath} + \alpha xy\,\hat{\jmath}$ where $\alpha = 1$ (force B), $\alpha = 2$ (force A)

a) $W = \int \vec{F}\cdot d\vec{r}$



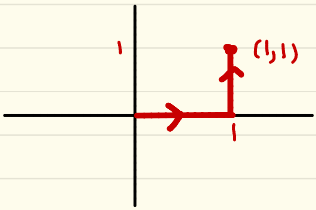
$$W = \int F_x\,dx + \int F_y\,dy = \int_0^1 \left(F_x + F_y \frac{dy}{dx}\right) dx$$

$$= \int_0^1 \left(y^2 + \alpha xy \frac{dy}{dx}\right) dx \qquad \text{but } y = x^2$$

$$= \int_0^1 (x^4 + 2\alpha x^4)\,dx = \left(\tfrac{1}{5}x^5 + \tfrac{2}{5}\alpha x^5\right)\Big|_0^1 = \tfrac{1}{5}(1 + 2\alpha)$$

$$\boxed{\begin{aligned} \therefore\ W_A &= 1 \text{ Joule} \\ W_B &= \tfrac{3}{5} \text{ Joule} \end{aligned}}$$

b) For the path:



$$W = \int_0^1 F_x(x,0)\,dx + \int_0^1 F_y(1,y)\,dy$$

$$= 0 + \alpha \int_0^1 y\,dy = \frac{\alpha}{2}$$

$$\boxed{\begin{aligned} \therefore\ W_A &= 1 \text{ Joule} \\ W_B &= \tfrac{1}{2} \text{ Joule} \end{aligned}}$$

4c) $\boxed{\text{Force A might be conservative since } W_A \text{ is path-indep. for the 2 cases. Force B is not.}}$

To conclusively answer if $\vec{F}_A$ is conservative, we need to see if we can write it as

$$\vec{F}_A \stackrel{?}{=} -\vec{\nabla} U$$

Claim: $U = -xy^2 + C$ does the trick, as you can easily verify.

5) $\vec{R} = \dfrac{\int dx\,dy\,\vec{r}}{\int dx\,dy}$ $\qquad A = \int dx\,dy = \int_0^1 dx \int_0^{2x} dy = 1$

$$\therefore \vec{R} = \int_0^1 dx \int_0^{2x} dy\,(x\hat{\imath} + y\hat{\jmath})$$

$$R_x = X = \int_0^1 dx\,x \int_0^{2x} dy = 2\int_0^1 dx\,x^2 = \frac{2}{3}$$

$$R_y = Y = \int_0^1 dx \int_0^{2x} dy\,y = \int_0^1 dx\,\frac{(2x)^2}{2} = 2\int_0^1 dx\,x^2 = \frac{2}{3}$$

$$\boxed{\therefore \vec{R} = \frac{2}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath}}$$