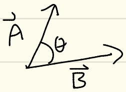
Vector algebra Review

Vector ≡ qty. w/ direction & magnitude (e.g., velocity, position, force, ...)

2 types of products

1.) Dot (Scalar):

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$

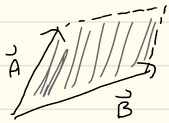


Verify:

$$\vec{A}\cdot\vec{A} = A^2$$
$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$$
$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$$

2.) Vector (cross) Product:

$$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$$



$AB\sin\theta$ = shaded area

* Unit vector $\hat{n}$ from Right Hand Rule

1.) Fingers of RH along $\vec{A}$
2.) Curl RH fingers from $\vec{A}$ to $\vec{B}$
3.) RH thumb = direction of $\hat{n}$ (into the page here)

Verify:

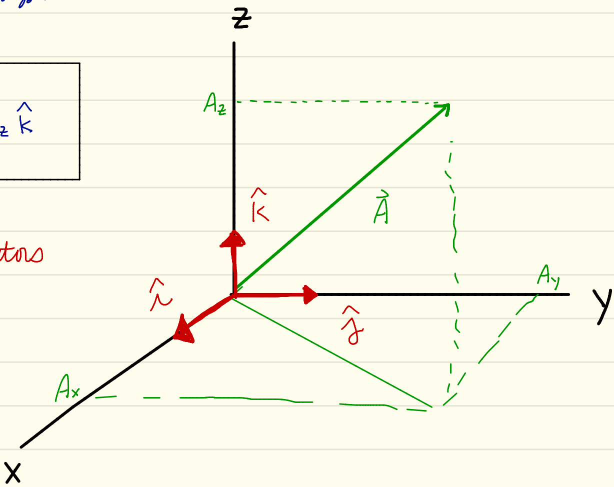
$$\vec{A}\times\vec{A} = 0$$
$$\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$
$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$$

Vector Algebra in Component Form

* Previous defs. were purely geometric (no reference to a coordinate system). Let's now commit to a specific cartesian coordinate system

$$\boxed{\vec{A} = A_x \hat{\imath} + A_y \hat{\jmath} + A_z \hat{k}}$$

$\{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ = unit Vectors

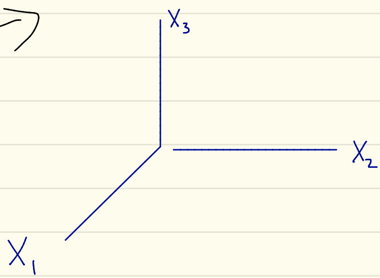


* Use $\hat{\imath} \cdot \hat{\jmath} = \hat{\imath} \cdot \hat{k} = \hat{\jmath} \cdot \hat{k} = 0$ to find

$$\boxed{\begin{aligned} A_x &= \vec{A} \cdot \hat{\imath} \\ A_y &= \vec{A} \cdot \hat{\jmath} \\ A_z &= \vec{A} \cdot \hat{k} \end{aligned}}$$

* Be aware of other notations:

$$\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 \equiv \sum_{i=1}^{3} A_i \hat{e}_i$$



This is the one I'll mostly use.

Vector Algebra in Component Form

1.) $\vec{A}+\vec{B} = \sum_i (A_i + B_i)\,\hat{e}_i$

2) $\vec{A}\cdot\vec{B} = \sum_i \sum_j A_i B_j (\hat{e}_i \cdot \hat{e}_j)$ * but $\hat{e}_i\cdot\hat{e}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \equiv \delta_{ij}$

"Kronecker delta"

$$\boxed{\therefore \quad \vec{A}\cdot\vec{B} = \sum_i A_i B_i}$$

If this is not "obvious" to you, write out all 9 terms in the double sum $\sum_i \sum_j A_i B_j \delta_{ij}$ + use that $\delta_{11} = \delta_{22} = \delta_{33} = 1$, while all other terms vanish.

3) $\vec{A}\times\vec{B} = \sum_i \sum_j A_i B_j (\hat{e}_i \times \hat{e}_j)$ * but $\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$, $\hat{e}_2 \times \hat{e}_3 = \hat{e}_1$, etc.

$$\Rightarrow \quad \boxed{\vec{A}\times\vec{B} = (A_2B_3 - A_3B_2)\hat{e}_1 + (A_3B_1 - A_1B_3)\hat{e}_2 + (A_1B_2 - A_2B_1)\hat{e}_3}$$

* In determinant notation: $\vec{A}\times\vec{B} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$

Levi-Civita Symbol

Disclaimer: The rest of the lecture is not crucial to the bulk of this course. Don't panic if this is too abstract or unfamiliar. I (+ the book) cover it simply because

1) You'll probably see it in more advanced classes you might take
2) Once you get the hang of it, it DRAMATICALLY simplifies the derivation of complicated equations w/ multiple cross products

$$\boxed{\begin{aligned} \varepsilon_{ijk} &= +1 \quad \text{if } (ijk) = (123), (312), (231) \quad \text{"cyclic" or "even"} \\ &= -1 \quad \text{if } (ijk) = (213), (321), (132) \quad \text{"anti-cyclic" or "odd"} \\ &= 0 \quad \text{else} \end{aligned}}$$

Ex: $\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki} = -\varepsilon_{jik} = -\varepsilon_{kji} = -\varepsilon_{ikj}$

$\varepsilon_{iij} = \varepsilon_{ikk} = 0$, etc.

Claim: $$\boxed{(\vec{A} \times \vec{B})_i = \sum_j \sum_k \varepsilon_{ijk} A_j B_k}$$

check it: $(\vec{A} \times \vec{B})_1 = \sum_j \sum_k \varepsilon_{1jk} A_j B_k = \cancelto{+1}{\varepsilon_{123}} A_2 B_3 + \cancelto{-1}{\varepsilon_{132}} A_3 B_2$

$= A_2 B_3 - A_3 B_2$ ✓

* Some Useful identities I state w/out proof

1) $\sum_{k} \varepsilon_{ijk} \varepsilon_{\ell mk} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$

2) $\sum_{jk} \varepsilon_{ijk} \varepsilon_{\ell jk} = 2\delta_{i\ell}$

3) $\sum_{ijk} \varepsilon_{ijk} \varepsilon_{ijk} = 6$

* Here's a couple examples how these ε_{ijk} tricks allow us to derive complicated vector identities in just a few lines

Ex 1: Prove $\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B})$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \sum_{i} A_i(\vec{B}\times\vec{C})_i = \sum_{ijk} \varepsilon_{ijk} A_i B_j C_k = \sum_{ijk} \varepsilon_{kij} C_k A_i B_j$$

$$= \sum_{k} C_k(\vec{A}\times\vec{B})_k = \vec{C}\cdot(\vec{A}\times\vec{B}) \checkmark$$

Ex 2: Prove $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

$$[\vec{A}\times(\vec{B}\times\vec{C})]_i = \sum_{jk} \varepsilon_{ijk} A_j(\vec{B}\times\vec{C})_k = \sum_{jk}\sum_{\ell m} \varepsilon_{ijk} \varepsilon_{k\ell m} A_j B_\ell C_m$$

$$= \sum_{jk}\sum_{\ell m} \varepsilon_{ijk}\varepsilon_{\ell mk} A_j B_\ell C_m = \sum_{j}\sum_{\ell m} (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) A_j B_\ell C_m$$

$$= \sum_{j} B_i A_j C_j - C_i A_j B_j = B_i(\vec{A}\cdot\vec{C}) - C_i(\vec{A}\cdot\vec{B})$$

$$\Rightarrow \vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) \checkmark$$