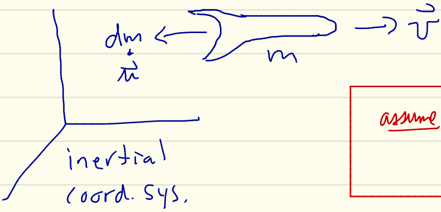
Rocket Motion in Free Space (i.e., no gravity)



assume: $\vec{u}$ = exhaust vel. wrt rocket = const

$dm' > 0$ = exhaust expelled in dt

Momentum conservation: assume all motion along x-axis (1D motion ⇒ drop vector symbols)

$$p(t) = p(t+dt)$$

$$mv = \underbrace{(m-dm')(v+dv)}_{\text{Rocket}} + \underbrace{dm'(v-u)}_{\text{exhaust}}$$

Note: exhaust speed $= -u$ wrt rocket $\therefore (v-u)$ = exhaust speed wrt inertial system

$$\cancel{mv} = \cancel{mv} + m\,dv - \cancel{dm'}v - \cancel{dm'dv}^{\approx 0} + \cancel{dm'}v - dm'u$$

$$\Rightarrow m\,dv = u\,dm'$$

$$dv = u\frac{dm'}{m}$$

* but $dm_{rocket} \equiv dm = -dm'$

$$\therefore dv = -u\frac{dm}{m}$$

* Integrating both sides:

$$\int_{v_0}^{v} dv = -u \int_{m_0}^{m} \frac{dm}{m}$$

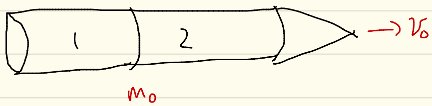
$$v - v_0 = u \ln\left(\frac{m_0}{m}\right)$$

$$\boxed{\Rightarrow v = v_0 + u \ln\left(\frac{m_0}{m}\right)}$$

* How to maximize ship's speed v?

* need to maximize $\frac{m_0}{m}$, but smallest m set by structural issues

Solution: Multistage rockets!



m_0, v_0 = initial mass + speed of 2-stage rocket

m_1 = rocket mass after stage 1 reaches burnout $= m_a + m_b$

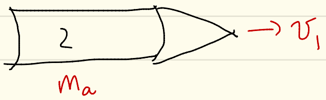
m_a = 1st stage payload (i.e, full tank 2 + capsule)

m_b = mass of empty tank 1

v_1 = speed at stage 1 burnout

$$\boxed{v_1 = v_0 + u \ln\left(\frac{m_0}{m_1}\right)}$$

* Now jettison tank 1 ($m_1 \to m_a$) + ignite phase 2.



m_a = initial mass @ start of phase 2

m_2 = mass at tank 2 burnout $= m_c + m_d$

m_c = 2nd stage payload (capsule mass)

m_d = empty tank 2 mass

v_2 = speed @ phase 2 burnout

$$v_2 = v_1 + u\ln\left(\frac{m_a}{m_2}\right)$$

$$= v_0 + u\ln\left(\frac{m_0}{m_1}\right) + u\ln\left(\frac{m_a}{m_2}\right)$$

$$= v_0 + u\ln\left(\underbrace{\frac{m_0}{m_1}\frac{m_a}{m_2}}_{\text{product of 2 \#'s} > 1}\right)$$

$(\ln A + \ln B = \ln(AB))$

Boxcar problem (HW 4) w/ N people jumping off all at once versus 1-by-1 illustrates this idea.

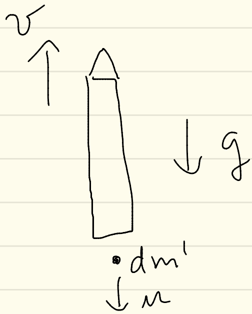
What's the "thrust" of a rocket?

$$dv = -u\frac{dm}{m}$$

$$\Downarrow$$

$$m\frac{dv}{dt} = \underbrace{-u\frac{dm}{dt}}_{\text{"Thrust" (units of Newtons)}} > 0 \quad \left(\text{since } \frac{dm}{dt} < 0\right)$$

* Rocket Motion in presence of Gravity



* gravity acts as an external force. We have

$$F_{ext} = -mg = \frac{dp}{dt}$$

$$\Rightarrow (-mg)\,dt = dp = p(t+dt) - p(t)$$

$$= m\,dv + u\,dm \quad \text{(from before)}$$

$$\boxed{\therefore -mg = m\frac{dv}{dt} + u\frac{dm}{dt}}$$

* Since u = constant (by assumption), $\frac{dm}{dt} = -\alpha$ $\quad \alpha > 0$ $\quad$ "Burn rate"

$$\therefore -mg = m\frac{dv}{dt} - u\alpha$$

$$\Downarrow$$

$$\boxed{dv = \left(-g + \frac{\alpha}{m}u\right) dt} \quad \Rightarrow \quad \boxed{dv = \left(\frac{g}{\alpha} - \frac{u}{m}\right) dm} \; *$$

* Use $\frac{dm}{dt} = -\alpha \Rightarrow dt = -\frac{dm}{\alpha}$

$$\Rightarrow \int_0^{v} dv = \int_{m_0}^{m} \left(\frac{g}{\alpha} - \frac{u}{m}\right) dm$$

$$\boxed{v = -\frac{g}{\alpha}(m_0 - m) + u\ln\left(\frac{m_0}{m}\right)}$$

* Can put time back into the eqn via

$$\frac{dm}{dt} = -\alpha$$

$$\Rightarrow \int dm = -\alpha \int_0^{t} dt \quad \Rightarrow \quad m - m_0 = -\alpha t$$

$$\therefore v = -gt + u\ln\left(\frac{m_0}{m}\right)$$

$$= -gt + u\ln\left(\frac{m_0}{m_0 - \alpha t}\right)$$

$$= -gt - u\ln\left(\frac{m_0 - \alpha t}{m_0}\right)$$

$$\boxed{v(t) = -gt - u\ln\left(1 - \frac{\alpha t}{m_0}\right)}$$

Often written in "mixed" form (m & t-dep)

$$\boxed{v(t) = -gt + u\ln\left(\frac{m_0}{m}\right)}$$

What is height at burnout?

$\boxed{\text{Problem 9.58}}$

$$v(t) = -gt - u\ln\left(1 - \frac{\alpha t}{m_0}\right) = \frac{dy}{dt}$$

$$\Rightarrow y_B = \int_0^{t_B} dt\left[-gt - u\ln\left(1 - \frac{\alpha t}{m_0}\right)\right]$$

$$= \frac{-g t_B^2}{2} - u\int_0^{t_B} dt\, \ln\left(1 - \frac{\alpha t}{m_0}\right)$$

* let $\frac{\alpha t}{m_0} = \tau$

$dt = \frac{m_0}{\alpha} d\tau$

$$\Rightarrow \int_0^{t_B} dt\, \ln\left(1 - \frac{\alpha t}{m_0}\right) = \frac{m_0}{\alpha}\int_0^{\alpha t_B/m_0} d\tau\, \ln(1-\tau)$$

$$= \frac{m_0}{\alpha}\left[-\tau - \ln(1-\tau) + \tau\ln(1-\tau)\right]\Big|_0^{\frac{\alpha t_B}{m_0}}$$

$$= \frac{m_0}{\alpha}\left[-\frac{\alpha t_B}{m_0} - \ln\left(1 - \frac{\alpha t_B}{m_0}\right) + \frac{\alpha t_B}{m_0}\ln\left(1 - \frac{\alpha t_B}{m_0}\right)\right]$$

$$= -t_B - \frac{m_0}{\alpha}\ln\left(\frac{m_0 - \alpha t_B}{m_0}\right) + t_B\ln\left(\frac{m_0 - \alpha t_B}{m_0}\right)$$

$$= -t_B + \frac{m_0}{\alpha}\ln\left(\frac{m_0}{m_0 - \alpha t_B}\right) - t_B\ln\left(\frac{m_0}{m_0 - \alpha t_B}\right)$$

$$= -t_B - t_B\ln\left(\frac{m_0}{m_0 - \alpha t_B}\right)\left[1 - \frac{m_0}{\alpha t_B}\right]$$

* but $m_0 - \alpha t_B = m_B$

and

$\frac{m_0}{\alpha t_B} - 1 = \frac{m_B}{\alpha t_B}$

$$= -t_B - t_B\ln\left(\frac{m_0}{m_B}\right)\left[1 - \frac{m_0}{\alpha t_B}\right]$$

$$= -t_B + \frac{m_B}{\alpha}\ln\left(\frac{m_0}{m_B}\right)$$

$$\boxed{\Rightarrow y_B = \frac{-g t_B^2}{2} + u t_B - \frac{u m_B}{\alpha}\ln\left(\frac{m_0}{m_B}\right)}$$

ex: $\boxed{9.57}$ Rocket in free space

$t=0,\ v(0)=0$

$\Downarrow$

Uniform acceleration a

to find speed v

Find total work done by engine

$$W = W_{rocket} + W_{exhaust}$$

$$W_{rocket} = \int F\,dx = \int \frac{dp}{dt}\,dx = \int v\,dp = \int (at)\,dp \qquad (v = at \text{ for uniform } a)$$

$$\boxed{dp = d(mv) = d(mat) = ma\,dt + at\,dm}$$

Now, we can figure out explicit form of $m(t)$ using

$$v = \cancel{v_0}^{\,0} + u\ln\left(\frac{m_0}{m}\right) = at$$

$$\ln\frac{m_0}{m} = \frac{at}{u}$$

$$\frac{m_0}{m} = e^{at/u} \Rightarrow \boxed{m(t) = m_0 e^{-at/u}}$$

$$\therefore\ dm = -\frac{a}{u} m_0 e^{-at/u}\,dt = -\frac{a}{u} m\,dt$$

$$dp = ma\left(1 - \frac{at}{u}\right)dt = m_0 a e^{-at/u}\left[1 - \frac{at}{u}\right]dt$$

$$\boxed{\Rightarrow W_{rocket} = m_0 a^2 \int_0^t \left(t - \frac{at^2}{u}\right) e^{-at/u}\,dt = \frac{m_0 a}{u}\int_0^t at(u - at)\,e^{-at/u}\,dt}$$

$$W_{exhaust} = \int v_{ex}\, dP_{ex} \qquad v_{ex} = v - u$$

$$dP_{ex} = dM_{ex}(v-u) = -dm(v-u) = \frac{m_0 a}{u} e^{-at/u}(v-u)\, dt$$

$$\boxed{W_{ex} = \frac{m_0 a}{u}\int_0^t (v-u)^2 e^{-at/u}\, dt = \frac{m_0 a}{u}\int_0^t (at-u)^2 e^{-at/u}\, dt}$$

* both integrals are elementary. Doing them + using $v = at \Rightarrow t = \frac{v}{a}$

$$\boxed{W_{rocket} + W_{ex} = muv}$$