Ch. 3 - Small Oscillations

* The Simple Harmonic Oscillator

Consider a mass free to move in 1D about a position of stable equilibrium X_0. I.e., if it moves away from X_0 then it's subject to a restoring force that

i) depends on $X - X_0$

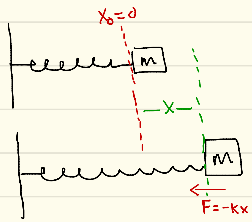
ii) tries to push/pull it back to X_0

Without loss of generality, take $X_0 = 0$.

$$F(x) = -kx \qquad \text{[Hooke's Law]} \qquad \Rightarrow \text{Oscillatory Motion}$$

"spring stiffness const"

*** The classic example is a spring:**



$\left(\nu_0 = \frac{\omega_0}{2\pi} \quad \text{sec}^{-1} \text{ or Hz} \right)$

$\left(T = \frac{1}{\nu_0} = \frac{2\pi}{\omega_0} = \text{period in sec} \right)$

EOM: $\quad m\ddot{x} = -kx$

$$\ddot{x} = -\frac{k}{m}x \equiv -\omega_0^2 x \qquad \omega_0 = \text{"Natural frequency"}$$

NOTE: Now is a good time to read Appendix C if you're a bit rusty on how to solve simple 2^{nd}-order ODEs.

Guess: $X = e^{\alpha t}$ α = const

$$\dot{X} = \alpha e^{\alpha t}$$
$$\ddot{X} = \alpha^2 e^{\alpha t} = \alpha^2 X$$

$$\Downarrow$$

$$\ddot{X} + \omega_0^2 X = 0 \Rightarrow (\alpha^2 + \omega_0^2) X = 0$$

$\therefore$ need $\alpha^2 = -\omega_0^2$ (recall, $\omega_0 > 0$)

$$\underline{\underline{\alpha = \pm i \omega_0}}$$

* Therefore, we have two linearly independent solutions

$$X_1(t) = e^{i\omega_0 t} \quad \text{and} \quad X_2(t) = e^{-i\omega_0 t}$$

* Recall that our ODE is Linear. That is, if $\ddot{X}_1 = -\omega_0^2 X_1$
$\ddot{X}_2 = -\omega_0^2 X_2$

then any linear combination of X_1 & X_2

e.g., $X = A_1 X_1 + A_2 X_2$ also obeys $\ddot{X} = -\omega_0^2 X$

* This is a good thing, as formal (see App. C) ODE theory tells us we can fix A_1 & A_2 to the 2 independent initial conditions needed to uniquely specify the sol'n of a 2^{nd}-order ODE.

* So, our general solution takes the form

$$\boxed{X(t) = A_1 e^{i\omega_0 t} + A_2 e^{-i\omega_0 t}}$$

Note: A_1 + A_2 are arbitrary complex coeffs.

* We can cast this in several equivalent forms that can be more convenient for calculations.

e.g., recall
$$e^{i\omega_0 t} = \cos\omega_0 t + i\sin\omega_0 t$$
$$e^{-i\omega_0 t} = \cos\omega_0 t - i\sin\omega_0 t$$

$$\Rightarrow X(t) = A_1\cos\omega_0 t + iA_1\sin\omega_0 t + A_2\cos\omega_0 t - iA_2\sin\omega_0 t$$

$$= (A_1+A_2)\cos\omega_0 t + i(A_1-A_2)\sin\omega_0 t$$

$$\boxed{\begin{array}{l} X(t) \equiv A\cos\omega_0 t + B\sin\omega_0 t \\ \qquad A, B = \text{arbitrary (real) constants} \end{array}}$$

$$X = X^* \Rightarrow A_1 = A_2^*$$

* Still yet another way to write it:

let $A_1 = a\,e^{-i\delta}$ $\quad A_2 = a\,e^{i\delta}$ $\quad a, \delta$ = real const.

$$X(t) = a\,e^{i(\omega_0 t-\delta)} + a\,e^{-i(\omega_0 t-\delta)} = a\left(\cos(\omega_0 t-\delta) + \cancel{i\sin(\omega_0 t-\delta)} + \cos(\omega_0 t-\delta) - \cancel{i\sin(\omega_0 t-\delta)}\right)$$

$$= 2a\cos(\omega_0 t-\delta)$$

$$\boxed{X(t) \equiv C\cos(\omega_0 t-\delta)}$$

* Will find the explicitly real forms (i.e., using trig functions) are more convenient

Question: How to pin down A+B (or C+δ)?

Use 2 IC's, usually taken at $t=0$ (i.e, $X(0)$ + $\dot{X}(0)$ are assumed to be known)

$$X(0) = A\cancel{\cos 0}^{1} + B\cancel{\sin 0}^{0}$$

$$\dot{X}(0) = -A\omega_0 \cancel{\sin 0}^{0} + B\omega_0 \cancel{\cos 0}^{1}$$

$$\boxed{\begin{aligned} \Rightarrow A &= X(0) \\ B &= \frac{\dot{X}(0)}{\omega_0} \end{aligned}}$$

* Energy for Simple Harmonic Motion

* assume $X(0)=0$

$\Rightarrow A=0$ and $X(t) = B\sin\omega_0 t$

$$\dot{X}(t) = B\omega_0 \cos\omega_0 t$$

* Spring PE $\left(F=-kx = -\frac{dU}{dx} \Rightarrow U = \frac{1}{2}kx^2\right)$

$$\boxed{U = \frac{1}{2}kB^2 \sin^2\omega_0 t}$$

* KE of System (note: we assume the spring is nearly massless, so to good accuracy the only KE is from the oscillating mass)

$$T = \frac{1}{2} m \dot{x}^2 = \frac{1}{2} m B^2 \omega_0^2 \cos^2 \omega_0 t \qquad \text{*but } \omega_0^2 = \frac{k}{m}$$

$$\therefore \underline{\underline{T = \frac{1}{2} k B^2 \cos^2 \omega_0 t}}$$

<u>* Total Energy</u>

$$E = T + U$$

$$= \frac{1}{2} k B^2 \sin^2 \omega_0 t + \frac{1}{2} k B^2 \cos^2 \omega_0 t$$

$$= \frac{1}{2} k B^2 \left(\cancelto{1}{\sin^2 \omega_0 t + \cos^2 \omega_0 t} \right)$$

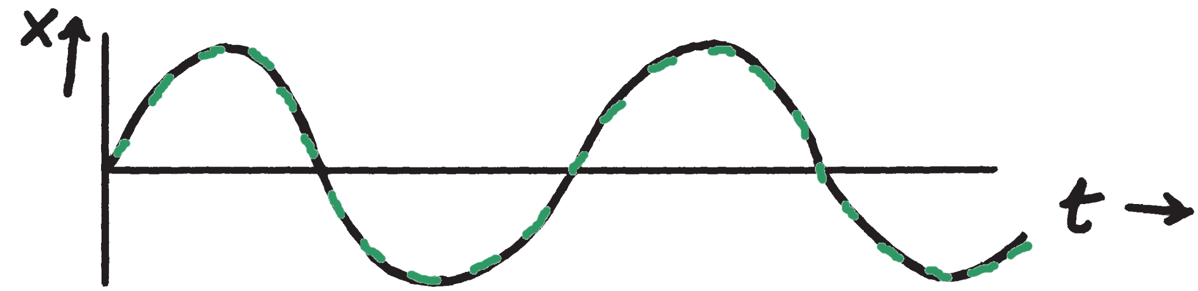
$$\boxed{\therefore E = \frac{1}{2} k B^2 = \text{constant!}}$$

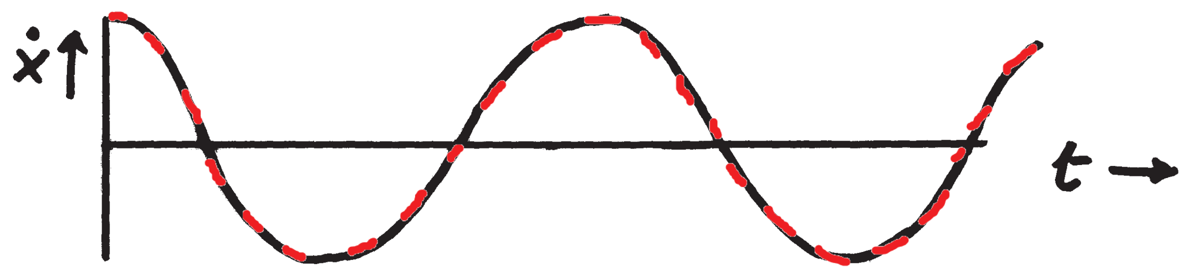
<u>Note</u>:

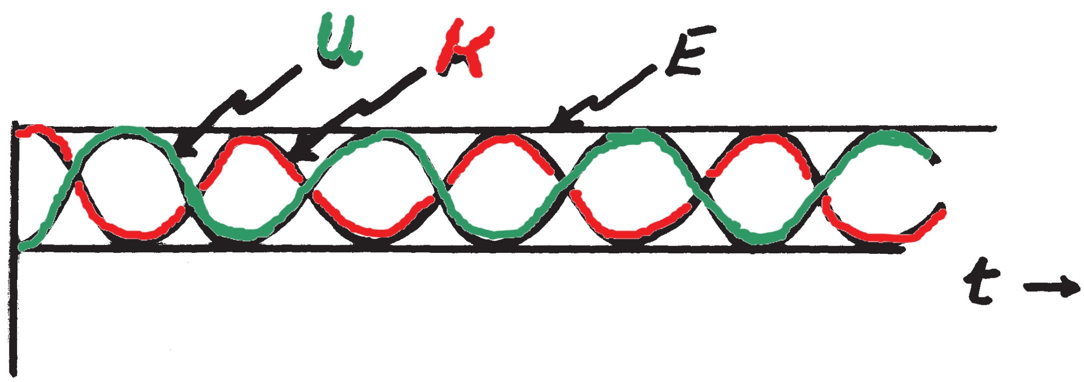
$$T = \frac{1}{2} k B^2 \cos^2 \omega_0 t = \frac{1}{4} k B^2 \left(1 + \cos(2\omega_0 t) \right)$$

$$U = \frac{1}{2} k B^2 \sin^2 \omega_0 t = \frac{1}{4} k B^2 \left(1 - \cos(2\omega_0 t) \right)$$

i.e., T & U each have oscillating pieces (that cancel against eachother) that individually oscillate at <u>twice</u> ($2\omega_0$) the frequency that $x(t)$ & $\dot{x}(t)$ oscillate at.







Energy oscillates back and forth between U and K but with <u>twice</u> the frequency of oscillation.

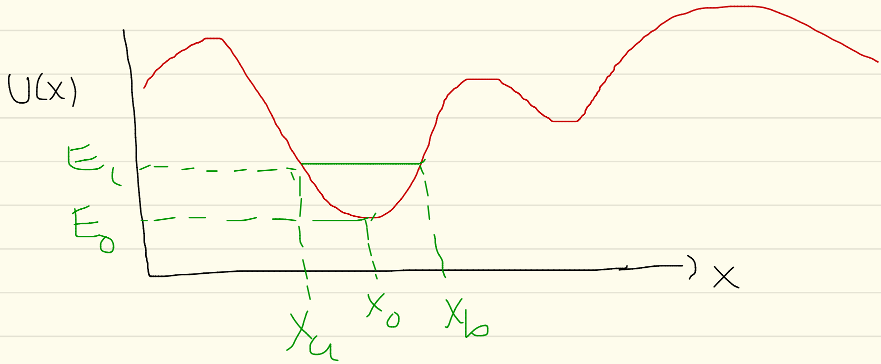
* Simple Harmonic Oscillators are everywhere in Physics (i.e., much more than just masses on springs!)

Ex: 1D motion w/ conservative forces

$$F = -\frac{d}{dx}U(x) \qquad \text{and} \qquad T + U = E = \text{constant}$$

* since $T = \frac{1}{2}mv^2 > 0$, we see physically allowable motion has $E \geq U$

* qualitative insight about the motion is possible from $U(x)$ curve



e.g.: for E_0, particle at x_0 stays there forever (Why? $T(x_0) = 0 \Rightarrow v = 0$ and $F(x_0) = -\frac{dU}{dx}\Big|_{x_0} = 0$)

e.g.: for E_1, particle motion is bounded & oscillatory between x_a & x_b

$v(x_a) = v(x_b) = 0$ (since $T = 0$ there)

$F(x_a) = -\frac{dU}{dx}\Big|_{x_a} => 0 \Rightarrow$ force to the right

$F(x_b) = -\frac{dU}{dx}\Big|_{x_b} < 0 \Rightarrow$ force to the left.

Near x_0:

$$U(x) \approx U(x_0) + \cancel{(x-x_0)\frac{dU}{dx}\Big|_{x_0}}^{\,0} + \frac{1}{2}(x-x_0)^2 \frac{d^2U}{dx^2}\Big|_{x_0} + \cdots$$

just a constant #.
We can ignore it
since PE arbitrary
up to a constant

For x close to x_0, $\quad U(x) \approx \frac{1}{2}(x-x_0)^2\, U''(x_0)$

$U''(x_0) > 0$ for stable equilibrium (i.e., local minimum in U)

$\therefore$ Call $U''(x_0) = k$

Therefore, in the neighborhood of x_0, we have

$$U(x) \simeq \frac{1}{2}k(x-x_0)^2$$

$$F(x) \simeq -k(x-x_0) \qquad \text{(Hooke's Law!)}$$

$\therefore$ near x_0, we have simple harmonic motion

$$m\ddot{x} = -k(x-x_0) \quad \Rightarrow \text{let } y = x - x_0 \quad \Rightarrow \ddot{x} = \ddot{y}$$

$$m\ddot{y} = -ky$$

$$\ddot{y} = -\omega_0^2 y \qquad \text{where } \omega_0^2 = \frac{k}{m} = \frac{U''(x_0)}{m}.$$