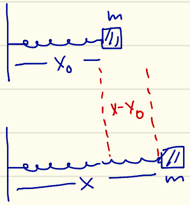
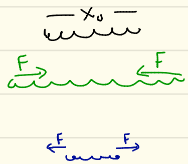
# Recap of Simple Harmonic Oscillator (SHO) from last class




**Hooke's Law:** $F = -k(x - x_0)$

restoring force:




$$m\ddot{x} = -k(x - x_0)$$

$$\ddot{x} + \frac{k}{m}(x - x_0) = 0 \qquad * \text{ let } \omega_0^2 \equiv \frac{k}{m}$$

$$\ddot{x} + \omega_0^2 (x - x_0) = 0$$

let $x - x_0 = y \Rightarrow \left.\begin{matrix}\dot{x} = \dot{y} \\ \ddot{x} = \ddot{y}\end{matrix}\right\} \Rightarrow \boxed{\ddot{y} + \omega_0^2 y = 0}$

$\Downarrow$ showed 3 equiv. ways to write sol'n

1.) $y = A e^{i\omega_0 t} + A^* e^{-i\omega_0 t}$ $\qquad$ $A$ = complex constant

2.) $y = A \cos \omega_0 t + B \sin \omega_0 t$

$A, B$ = real constants

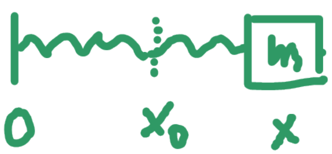
3.) $y = A \cos(\omega_0 t + \theta)$

$A, \theta$ = real constants

You should know how to jump back & forth!

* Fix constants from 2 IC's (usually $y(0)$ & $\dot{y}(0)$)

Mass $m$ is attached to the origin, in one dimension, by a spring of neutral length $x_0$. When pulled, the mass begins to oscillate at frequency $\omega_0$. What frequency would result if the mass were attached to the origin by two springs connected a) one after another, b) in parallel?




$\omega_0$

$$m\ddot{x} = -k(x-x_0) \Rightarrow \ddot{x} = -\frac{k}{m}(x-x_0)$$

new vble $y = x - x_0$ $\quad \ddot{y} = \ddot{x}$

$$\ddot{y} = -\frac{k}{m} y \Rightarrow y = x - x_0 = C\cos(\omega_0 t - \varphi)$$

(a) 

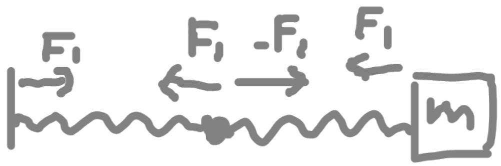

$\omega = ?$

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \text{OR} \quad k = m\omega_0^2$$

(b) 


$\omega = ?$

extension/one spring $= \frac{1}{2}$ of net

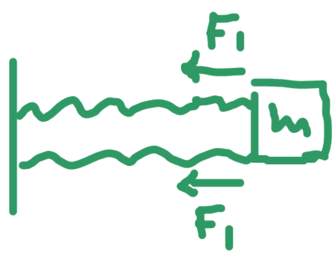
(a) 


$\Delta x = x - 2x_0$ $\qquad (\Delta x)\big|_{1\,SPRING} = \frac{\Delta x}{2} = \frac{x}{2} - x_0$

$|F_{1\,SPRING}| = k\left|\frac{x}{2} - x_0\right|$ $\qquad m\ddot{x} = -k\left(\frac{x}{2} - x_0\right) = -\frac{k}{2}(x - 2x_0)$

$F = F_{1\,SPRING}$

$y = x - 2x_0$ $\qquad y = x - 2x_0 = C\cdot\cos(\omega_0^a t - \varphi)$ $\qquad \omega_0^a = \sqrt{\frac{k}{2m}} = \frac{\omega_0}{\sqrt{2}}$

(b) 


$F_{1\,SPRING} = -k(x - x_0)$ $\qquad F = 2F_{1\,SPRING}$

$m\ddot{x} = -2k(x - x_0)$ $\qquad \ddot{x} = -\frac{2k}{m}(x - x_0)$

$y = x - x_0$ $\qquad y = x - x_0 = C\cos(\omega_0^b t - \varphi)$ $\qquad \omega_0^b = \sqrt{\frac{2k}{m}} = \omega_0\sqrt{2}$

The oscillations are more frequent for springs in parallel and less frequent for the springs in sequence.

Springs in parallel are equivalent to a spring out of thicker wire, that is stiffer. Springs put in sequence make a softer spring.

# Damped Oscillations

* previous ODE for SHO oversimplified. In "real life," there are frictional forces that reduce energy/motion w/time so oscillations ultimately damp out.

Simple model: $f_{friction} = -bv = -b\dot{x}$ $(b>0)$

$$m\ddot{x} = -kx - b\dot{x} \quad \Rightarrow \quad \ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

* let $2\beta \equiv \frac{b}{m}$ "Damping Parameter"

$\omega_0 = \sqrt{\frac{k}{m}}$ "Natural or Characteristic ∢ frequency"

$$\boxed{\Rightarrow \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \text{ is our ODE we must solve}}$$

☆ Reminder - review App. C if these details of solving the ODE are not familiar to you

Guess: $x(t) = Ce^{\alpha t}$

$$\dot{x} = \alpha Ce^{\alpha t} = \alpha x$$

$$\& \quad \ddot{x} = \alpha^2 Ce^{\alpha t} = \alpha^2 x$$

$$\overset{ODE}{\Rightarrow} (\alpha^2 + 2\beta\alpha + \omega_0^2)Ce^{\alpha t} = 0 \quad \Rightarrow \quad \alpha^2 + 2\beta\alpha + \omega_0^2 = 0 \quad //$$

$$\alpha^2 + 2\beta\alpha + \omega_0^2 = 0$$

$$\Downarrow$$

$$\alpha = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega_0^2}}{2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

* General Sol'n = Linear Comb. of the sol'ns corresponding to the 2 roots for $\alpha$

$$\boxed{X(t) = e^{-\beta t}\left[A_1 e^{\sqrt{\beta^2-\omega_0^2}\,t} + A_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right]}$$

* Nature of sol'n controlled by $\sqrt{\beta^2-\omega_0^2}$ term ("the discriminant"). 3 Cases to consider

1) $\omega_0^2 > \beta^2$ "Underdamped"

2) $\omega_0^2 = \beta^2$ "Critical Damping"

3) $\omega_0^2 < \beta^2$ "Overdamped"

Case 1: $\omega_0^2 > \beta^2$

$$\alpha = -\beta \pm i\sqrt{\omega_0^2 - \beta^2}$$

def: $\omega_1^2 = \omega_0^2 - \beta^2$

$$\boxed{\Rightarrow X(t) = e^{-\beta t}\left(A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t}\right)}$$

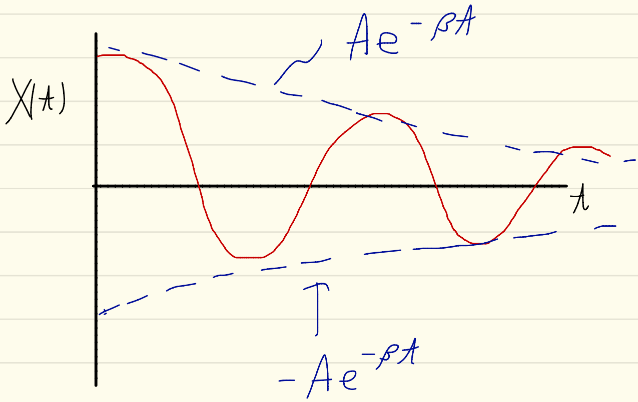
Damped SHO oscillates at a lower freq. $\omega_1 < \omega_0$ !

* Note that $X(t) = X^*(t) \Rightarrow A_1 = A_2^*$

$$X = e^{-\beta t}\left(A_1 e^{i\omega_1 t} + A_1^* e^{-i\omega_1 t}\right)$$

Using $e^{\pm i\omega_1 t} = \cos\omega_1 t \pm i \sin\omega_1 t$ (just like our derivation last lecture)

$$\Rightarrow \boxed{X(t) = e^{-\beta t}\left(A\cos\omega_1 t + B\sin\omega_1 t\right)}$$




$\Rightarrow$ Oscillations die out exponentially w/time.

## * Case 2: Overdamped motion ($\omega_0^2 < \beta^2$)

$$\alpha = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \quad = \text{purely real}$$

* Let $\omega_2 = \sqrt{\beta^2 - \omega_0^2}$

$$\Rightarrow \quad \boxed{X(t) = e^{-\beta t}\left[A_1 e^{w_2 t} + A_2 e^{-w_2 t}\right] \qquad (A_1, A_2 = \text{real})}$$

Note: $w_2 < \beta \Rightarrow$ both terms exp. decay.
(2nd term dies out first)

NO Periodic Motion!

* Suppose you want $X(t) \to 0$ w/out oscillations as fast as possible. Naively, you might think the answer is to simply crank up a large $b$ ($\beta$). THIS IS WRONG. Why?

* look at 1st term since it dominates for large $t$.

$$X(t) \simeq A_1 e^{-t(\beta - w_2)}$$

$$\beta - w_2 = \beta - \sqrt{\beta^2 - w_0^2} = \beta - \beta\sqrt{1 - \frac{w_0^2}{\beta^2}}$$

$$\approx \beta - \beta\left(1 - \frac{w_0^2}{2\beta^2}\right) \qquad \left(\text{ok for } \frac{w_0^2}{\beta^2} \ll 1\right)$$

$$\beta - w_2 \cong \frac{w_0^2}{2\beta} \quad \text{for large } \beta.$$

$$\therefore \; X(t) \simeq A_1 e^{-t\frac{w_0^2}{2\beta}}$$

$\Rightarrow$ Slow decrease of $X(t)$ w/time if $\beta \to \infty$.

## * Case 3: Critical Damping ($\omega_0^2 = \beta^2$)

$$\alpha = -\beta \pm \sqrt{\cancel{\beta^2 - \omega_0^2}}^{\,0}$$

* seems to imply only 1 indep. sol'n $\quad X(t) = Ae^{-\beta t}$

* This would be bad since we'd only have 1 constant $A$ to try to satisfy $X(0)$ + $\dot{X}(0)$ conditions! Luckily, this is not the case.

Claim: $\quad X = Bt\,e^{-\beta t}$ is a 2nd indep. solution of the ODE.

$$\dot{X} = B\left(e^{-\beta t} - \beta t\,e^{-\beta t}\right)$$

$$\ddot{X} = B\left(-\beta e^{-\beta t} - \beta e^{-\beta t} + \beta^2 t\,e^{-\beta t}\right) = B(-2\beta + \beta^2 t)\,e^{-\beta t}$$

Plug into ODE: $\quad \ddot{X} + 2\beta\dot{X} + \beta^2 X = 0$

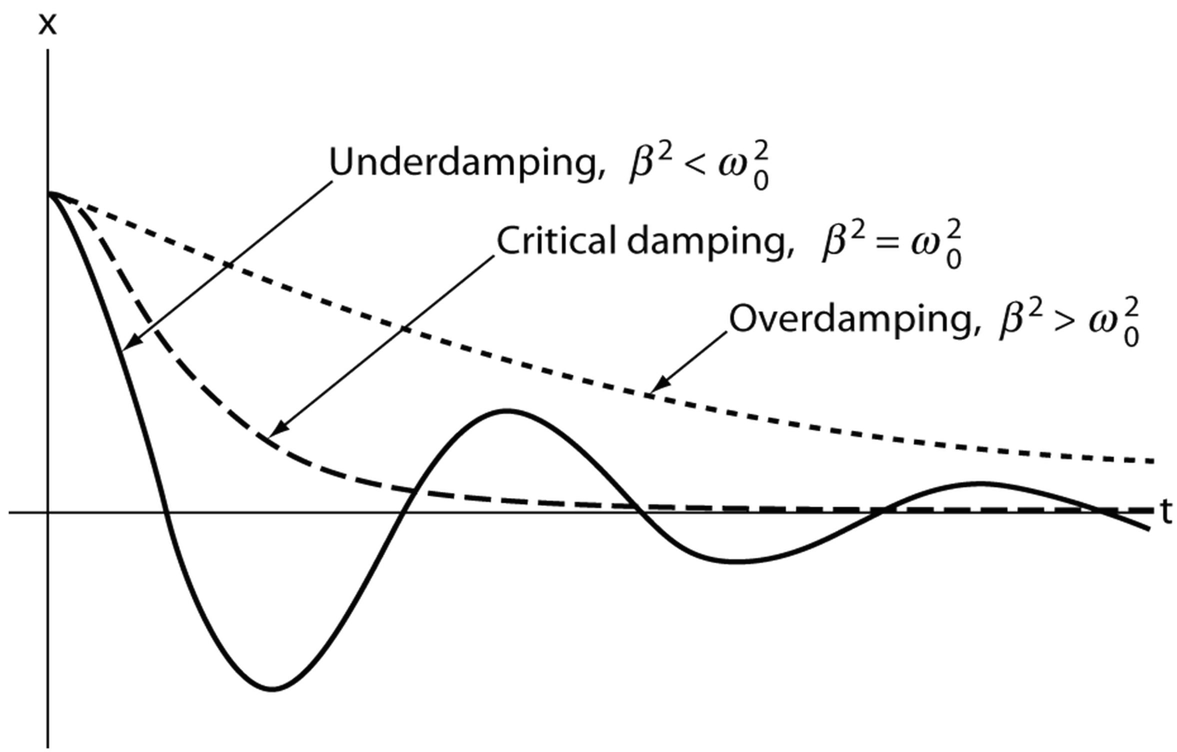
$$\cancel{B}(-2\beta + \beta^2 t)\cancel{e^{-\beta t}} + \cancel{B}(2\beta - 2\beta^2 t)\cancel{e^{-\beta t}} + \beta^2 t\,\cancel{B}\cancel{e^{-\beta t}} \stackrel{?}{=} 0$$

$$-2\beta + \beta^2 t + 2\beta - 2\beta^2 t + \beta^2 t = 0 \qquad \checkmark.$$

$\Rightarrow$ General sol'n for $\omega_0^2 = \beta^2$ case:

$$\boxed{X(t) = e^{-\beta t}(A + Bt)}$$

# SUMMARY OF DAMPING RESPONSES




Note: Critical yields fastest relaxation of the system to the vicinity of neutral position.

In technical design, proximity to critical damping is often desired. Examples: class door, car suspension, structures such as bridges and tall buildings.