Recap on SHO w/ damping (friction) forces

$F_{fric} = -b\dot{x}$

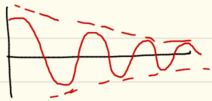
$\Rightarrow \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \qquad 2\beta = \frac{b}{m}, \quad \omega_0 = \sqrt{\frac{k}{m}}$

$\Downarrow$

$$\boxed{x(t) = e^{-\beta t}\left[A_1 e^{\sqrt{\beta^2-\omega_0^2}\,t} + A_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right]}$$

* 3 cases: 1) $\omega_0^2 > \beta^2$ ("Underdamped")

$$\boxed{\begin{gathered} x(t) = e^{-\beta t}\left(A\cos\omega_1 t + B\sin\omega_1 t\right) \\ \omega_1^2 = \omega_0^2 - \beta^2 \end{gathered}}$$



2) $\beta^2 > \omega_0^2$ ("overdamped")

$$\boxed{\begin{gathered} x(t) = e^{-\beta t}\left[A_1 e^{\omega_2 t} + A_2 e^{-\omega_2 t}\right] \\ \omega_2^2 = \beta^2 - \omega_0^2 \end{gathered}}$$

$\swarrow$

Slow decay to equilibrium ($x_{eq} = 0$) for $\beta^2 \gg \omega_0^2$ $\qquad x(t) \sim A_1 e^{-t\frac{\omega_0^2}{2\beta}}$

* Case 3: Critical Damping ($\omega_0^2 = \beta^2$)

$$\alpha = -\beta \pm \sqrt{\cancel{\beta^2 - \omega_0^2}}^{\,0}$$

* seems to imply only 1 indep. sol'n $\quad X(t) = Ae^{-\beta t}$

* This would be bad since we'd only have 1 constant A to try to satisfy $X(0)$ + $\dot{X}(0)$ conditions! Luckily, this is not the case.

Claim: $\quad X = Bt\, e^{-\beta t}$ is a 2nd indep. solution of the ODE.

$$\dot{X} = B\left(e^{-\beta t} - \beta t\, e^{-\beta t}\right)$$

$$\ddot{X} = B\left(-\beta e^{-\beta t} - \beta e^{-\beta t} + \beta^2 t\, e^{-\beta t}\right) = B(-2\beta + \beta^2 t)\, e^{-\beta t}$$

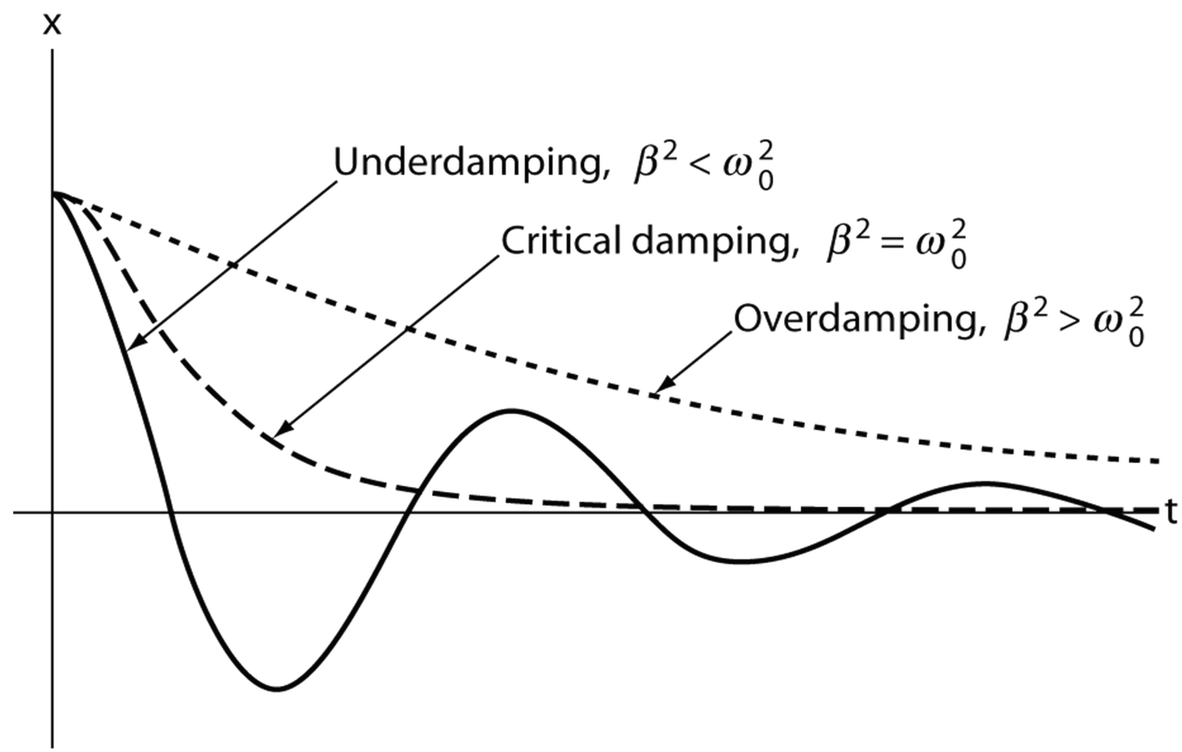
Plug into ODE: $\quad \ddot{X} + 2\beta \dot{X} + \beta^2 X = 0$

$$\cancel{B}(-2\beta + \beta^2 t)\cancel{e^{-\beta t}} + \cancel{B}(2\beta - 2\beta^2 t)\cancel{e^{-\beta t}} + \beta^2 t\, \cancel{B e^{-\beta t}} \overset{?}{=} 0$$

$$-2\beta + \beta^2 t + 2\beta - 2\beta^2 t + \beta^2 t = 0 \quad \checkmark.$$

$\Rightarrow$ General sol'n for $\omega_0^2 = \beta^2$ case: $\quad \boxed{X(t) = e^{-\beta t}(A + Bt)}$

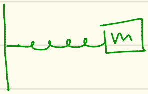
SUMMARY OF DAMPING RESPONSES



Note: Critical yields fastest relaxation of the system to the vicinity of neutral position.

In technical design, proximity to critical damping is often desired. Examples: class door, car suspension, structures such as bridges and tall buildings.

*Example: Electric Circuit analogs of SHO motion



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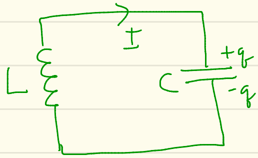


"LC circuit"

are identical (mathematically)

Voltages: $-L\frac{dI}{dt} = \frac{q}{C}$ (Voltage drop across inductor = drop across capacitor)

$I = \frac{dq}{dt}$

$\Downarrow$

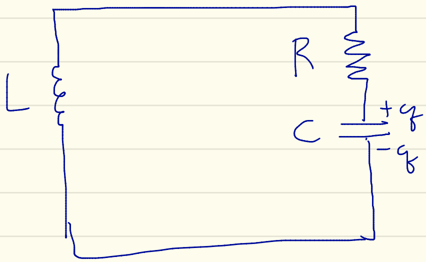
$-L\ddot{q} = \frac{1}{C}q \Rightarrow \boxed{\ddot{q} = -\frac{1}{LC}q}$

* compare to the mass-spring system $\boxed{\ddot{x} = -\frac{k}{m}x \equiv -\omega_0^2 x}$

mechanical	electrical
$x(t)$	$q(t)$
k	$\frac{1}{C}$
m	L
ω_0^2	$\frac{1}{LC}$

$\Rightarrow$ Can often build circuit analogs of mechanical systems that might be too hard/costly to prototype

* Another example of circuit analogy to SHO motion



"LCR circuit"

$$-L\frac{dI}{dt} = IR + \frac{q}{C}$$

$$\Downarrow$$

$$-L\ddot{q} = R\dot{q} + \frac{q}{C}$$

$$\boxed{\ddot{q} + \frac{R}{L}\dot{q} + \frac{q}{LC} = 0}$$

* compare vs. damped HO ODE

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$$

or

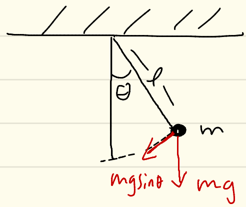
$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

$R \leftrightarrow b$

$L \leftrightarrow m$

$\frac{1}{C} \leftrightarrow k$

Example: Find natural ω_0 for a simple pendulum (rod ℓ is massless)



* assume small θ.

Polar coordinates: $\vec{v} = \dot{\vec{r}} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$ (see ch. 1)

$$\vec{a} = \dot{\vec{v}} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

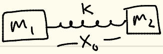
here, $r = \ell$ for all t, $\Rightarrow \dot{r} = \ddot{r} = 0$

$$m\vec{a} = -m\ell\dot{\theta}^2\,\hat{e}_r + m\ell\ddot{\theta}\,\hat{e}_\theta$$

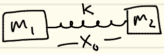
θ-components: $m\ell\ddot{\theta} = -mg\sin\theta \approx -mg\theta$

$$\Rightarrow \ddot{\theta} = -\frac{g}{\ell}\theta$$

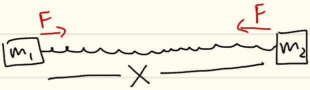
$$\boxed{\begin{gathered}\Rightarrow \theta(t) = \theta_0 \cos\omega_0 t \\ \omega_0 = \sqrt{\frac{g}{\ell}}\end{gathered}}$$

Example: 

Find Natural ω_0 for this system (no friction, massless spring, etc.)



Unstretched



Stretched

let x_1 = instantaneous position of m_1

x_2 = " " m_2

$$\left.\begin{array}{l} x_1 \\ x_2 \end{array}\right\} \Rightarrow X(t) = x_2(t) - x_1(t)$$

$$F_1 = -k(X - x_0) = -k(x_2 - x_1 - x_0) = -F_2$$

$$m_1 \ddot{x}_1 = -k(x_2 - x_1 - x_0) \qquad (a)$$

$$m_2 \ddot{x}_2 = -k(x_1 - x_2 + x_0) \qquad (b)$$

$$(b) \Rightarrow \quad kx_1 = kx_2 - kx_0 - m_2\ddot{x}_2 \Rightarrow \boxed{x_1 = x_2 - x_0 - \frac{m_2}{k}\ddot{x}_2}$$

plug into (a)

$$m_1 \ddot{x}_1 = -k(x_2 - x_1 - x_0) \qquad (a)$$

$$\boxed{x_1 = x_2 - x_0 - \frac{m_2}{k}\ddot{x}_2}$$

plug into (a)

$$m_1 \frac{d^2}{dt^2}\left(x_2 - x_0 - \frac{m_2}{k}\ddot{x}_2\right) = -k\left(\cancel{x_2} - \cancel{x_2} + \cancel{x_0} + \frac{m_2}{k}\ddot{x}_2 - \cancel{x_0}\right) = -m_2\ddot{x}_2$$

$$\Rightarrow \quad m_1\ddot{x}_2 - \frac{m_2}{k}\frac{d^2}{dt^2}\ddot{x}_2 + m_2\ddot{x}_2 = 0 \qquad \otimes$$

Now, $\ddot{x}_2 = -\omega_0^2 x_2$ (we want to solve for ω_0^2)

$$\therefore \frac{d^2}{dt^2}(\ddot{x}_2) = (-\omega_0^2)\ddot{x}_2 = \omega_0^4 x_2$$

$\therefore$ Eq. $\otimes$ becomes

$$-m_1 \cancel{\omega_0^2} \cancel{x_2} - \frac{m_2}{k}\omega_0^{\cancel{4}\,2} \cancel{x_2} - m_2 \cancel{\omega_0^2} \cancel{x_2} = 0$$

$$-m_1 - \frac{m_2\omega_0^2}{k} - m_2 = 0$$

* Phase Space Diagrams

* Motion of 1D HO uniquely specified if, at any time, x_0 and $\dot{x}_0$ are known.
(Why? Because the ODE that gives $x(t)$ (+ $\dot{x}(t)$) at later times is 2^{nd}-order)

* Useful to describe the state of the system as a point in 2D "phase space" $x(t), \dot{x}(t)$.

eg: SHO w/no damping

$$x(t) = A\cos(\omega_0 t - \delta)$$

$$\dot{x}(t) = -A\omega_0 \sin(\omega_0 t - \delta)$$

$$\boxed{\Rightarrow \left(\frac{x}{A}\right)^2 + \left(\frac{\dot{x}}{A\omega_0}\right)^2 = 1}$$

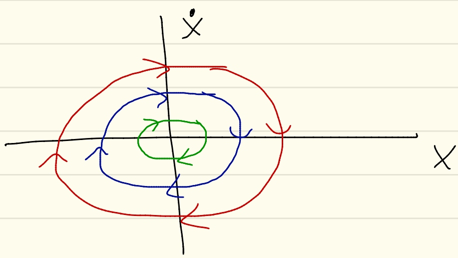
eqn. of an ellipse

* recall $E = \frac{1}{2}kA^2$ for SHO w/out damping

$$\Rightarrow A^2 = \frac{2E}{k} \quad \text{and} \quad A^2\omega_0^2 = \frac{2E}{m}$$

ellipse becomes:

$$\boxed{\frac{x^2}{2E/k} + \frac{\dot{x}^2}{2E/m} = 1}$$



* Ellipse size increases w/energy $\qquad x^2 + \frac{\dot{x}^2}{\omega_0^2} = \frac{2E}{k}$