* Phase Space Diagrams

* Motion of 1D HO uniquely specified if, at any time, x_0 and $\dot{x}_0$ are known.
  (Why? Because the ODE that gives $x(t)$ (+ $\dot{x}(t)$) at later times is 2nd-order)

* Useful to describe the state of the system as a point in 2D "phase space" $x(t), \dot{x}(t)$.

eg: SHO w/no damping

$$x(t) = A\cos(\omega_0 t - \delta)$$

$$\dot{x}(t) = -A\omega_0 \sin(\omega_0 t - \delta)$$

$$\boxed{\Rightarrow \left(\frac{x}{A}\right)^2 + \left(\frac{\dot{x}}{A\omega_0}\right)^2 = 1}$$

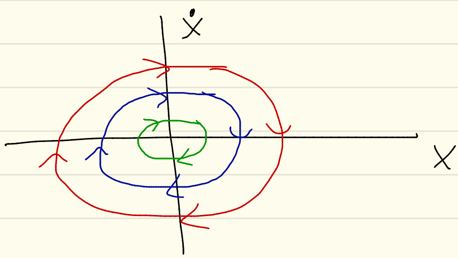
eqn. of an ellipse

* recall $E = \frac{1}{2}kA^2$ for SHO w/out damping

$$\Rightarrow A^2 = \frac{2E}{k} \quad \text{and} \quad A^2\omega_0^2 = \frac{2E}{m}$$

ellipse becomes:

$$\boxed{\frac{x^2}{2E/k} + \frac{\dot{x}^2}{2E/m} = 1}$$



* Ellipse size increases w/energy

$$x^2 + \frac{\dot{x}^2}{\omega_0^2} = \frac{2E}{k}$$

*<u>Note that</u>:

1) Curves in phasespace don't ever cross (Why? Bc. if 2 curves crossed, they have the same $x, \dot{x}$ as IC's. However, we know 2^{nd}-order ODE soln meeting a given set of IC's is <u>unique</u>.)

2) Motion CW ($x>0$ $\dot{x}<0$
$x<0$ $\dot{x}>0$)

* Damped HO w/ periodic driving forces

$$F = -kx - b\dot{x} + \underbrace{F_0 \cos\omega t}$$

(generally, $\omega \neq \omega_0$)

driving force to keep the system oscillating

$\therefore\ m\ddot{x} = -kx - b\dot{x} + F_0 \cos\omega t$

$$\Rightarrow \boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f\cos\omega t}$$

(ω_0 & β as before, $f \equiv \frac{F_0}{m}$)

Inhomogeneous 2^{nd}-order ODE. Read App. C !!

General way to solve such equations:

$$x(t) = x_{hg}(t) + x_p(t)$$

Sol'n to the Homogeneous eqn. (i.e., w/out driving term)

$$x_{hg} = e^{-\beta t}\left(A_1 e^{\sqrt{\beta^2-\omega_0^2}\,t} + A_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right)$$

$x_p(t)$ = "particular sol'n" that obeys

$$\ddot{x}_p + 2\beta\dot{x}_p + \omega_0^2 x_p = f\cos\omega t$$

let $\hat{L} = \frac{d^2}{dt^2} + 2\beta \frac{d}{dt} + \omega_0^2$ ("linear operator")

$\Rightarrow \hat{L}(x_{hg} + x_p) = \underbrace{\cancel{\hat{L} x_{hg}}}_{0} + \hat{L} x_p = f \cos\omega t$

$\Rightarrow x = x_{hg} + x_p$ is indeed a soln & x_{hg} has 2 constants to match $x(0), \dot{x}(0)$

Guess: $x_p(t) = D\cos(\omega t - \delta)$

$\dot{x}_p = -\omega D \sin(\omega t - \delta)$

$\ddot{x}_p = -\omega^2 D\cos(\omega t - \delta) = -\omega^2 x_p$

$\hat{L} x_p = (\omega_0^2 - \omega^2) D\cos(\omega t - \delta) - 2\beta\omega D \sin(\omega t - \delta) = f\cos\omega t$

* $\cos(\omega t - \delta) = \cos\omega t \cos\delta + \sin\omega t \sin\delta$

* $\sin(\omega t - \delta) = \sin\omega t \cos\delta - \cos\omega t \sin\delta$

$\Rightarrow (\omega_0^2 - \omega^2) D[\cos\omega t \cos\delta + \sin\omega t \sin\delta] - 2\beta\omega D[\sin\omega t \cos\delta - \cos\omega t \sin\delta] = f\cos\omega t$

$\Downarrow$ re-group $\cos\omega t$ & $\sin\omega t$ terms

$\cos\omega t \underline{\{(\omega_0^2 - \omega^2) D\cos\delta + 2\beta\omega D\sin\delta - f\}} + \sin\omega t \underline{\{(\omega_0^2 - \omega^2) D\sin\delta - 2\beta\omega D\cos\delta\}} = 0$

*Key Point: $\sin\omega t$ & $\cos\omega t$ are linearly independent functions

$A\cos\omega t + B\sin\omega t = 0$

only possible if $A = B = 0$

$$\cos\omega t\,\{\underline{(\omega_0^2-\omega^2)D\cos\delta + 2\beta\omega D\sin\delta - f}\} + \sin\omega t\,\{\underline{(\omega_0^2-\omega^2)D\sin\delta - 2\beta\omega D\cos\delta}\} = 0$$

$$\Downarrow$$

$$(\omega_0^2-\omega^2)D\sin\delta - 2\beta\omega D\cos\delta = 0 \quad (1)$$

2 eqs 2 unknowns (D, δ)

$$(\omega_0^2-\omega^2)D\cos\delta + 2\beta\omega D\sin\delta - f = 0 \quad (2)$$

eq.(1) $\Rightarrow$ $\boxed{\tan\delta = \dfrac{2\beta\omega}{\omega_0^2-\omega^2}}$ (*)

(2) $\Rightarrow$ $f^2 = \left[(\omega_0^2-\omega^2)^2\cos^2\delta + 4\beta^2\omega^2\sin^2\delta + 4\beta\omega(\omega_0^2-\omega^2)\sin\delta\cos\delta\right]D^2$

*but eq.(*) $\Rightarrow$ $\sin\delta = \dfrac{2\omega\beta}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}$

$$\cos\delta = \frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}$$

<u>plug into f^2 & simplify:</u>

$$f^2 = D^2\left[(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2\right]$$

$\Rightarrow$ $\boxed{D = \dfrac{f}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}}$ (**)