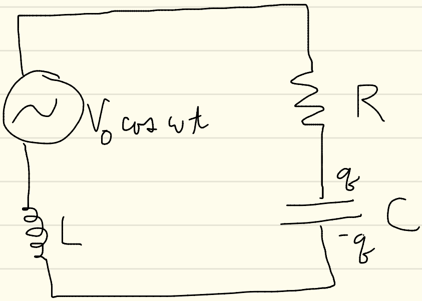
Example: Electrical analog of driven, damped HO "RLC circuit"



Voltages (Kirchoff's Law): $-L\frac{dI}{dt} + V_0 \cos \omega t = IR + \frac{q}{C}$ $\qquad I = \dot{q}$

$$-L\ddot{q} + V_0 \cos \omega t = \dot{q}R + \frac{q}{C}$$

$$\Downarrow$$

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{q}{LC} = \frac{V_0}{L}\cos \omega t$$

Compare vs. $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f\cos\omega t$

$x \leftrightarrow q$

$\beta \leftrightarrow \frac{1}{2}\frac{R}{L}$

$\omega_0^2 \leftrightarrow \frac{1}{LC}$

$f \leftrightarrow \frac{V_0}{L}$

Steady State

$$q(t) = \frac{V_0/L}{\sqrt{\left(\frac{1}{LC}-\omega^2\right)^2 + \omega^2\frac{R^2}{L^2}}}\cos(\omega t - \delta)$$

$$\delta = \tan^{-1}\left(\frac{\omega R/L}{\frac{1}{LC}-\omega^2}\right)$$

Amplitude Res: $\omega_R = \sqrt{\frac{1}{LC} - \frac{1}{2}\frac{R^2}{L^2}}$ maximizes $q(t)$

KE Res: $\omega = \frac{1}{LC}$ maximizes $\left\langle \frac{1}{2}LI^2 \right\rangle$

<u>Example</u>: Find ω for which avg. KE $\langle T \rangle$ is maximized ("KE resonance")

* Only use <u>Steady State</u> (i.e., $x_p(t) = D(\omega)\cos(\omega t - \delta)$)

$$\dot{x} = -\omega D \sin(\omega t - \delta)$$

$$\Rightarrow T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\omega^2 D^2 \sin^2(\omega t - \delta)$$

$$\langle T \rangle \equiv \frac{1}{\tau}\int_0^\tau \frac{1}{2}m\omega^2 D^2(\omega)\sin^2(\omega t - \delta)\,dt \qquad \left(\tau = \frac{2\pi}{\omega}\right)$$

$$\frac{1}{\tau}\int_0^\tau \sin^2(\omega t - \delta)\,dt = \frac{1}{\tau}\int_0^\tau \frac{1-\cos 2(\omega t - \delta)}{2}\,dt$$

$$= \frac{1}{2\tau}\left(\tau - \underbrace{\cancel{\int_0^\tau \cos 2(\omega t - \delta)\,dt}}_{\to 0}\right)$$

$$\Rightarrow \langle \sin^2(\omega t - \delta)\rangle_\tau = \frac{1}{2}$$

$$\boxed{\therefore \langle T \rangle = \frac{1}{4}m\omega^2 D^2(\omega)}$$

$$\langle T \rangle = \frac{m}{4}\frac{\omega^2 f^2}{(\omega_0^2-\omega^2)^2 + 4\beta^2\omega^2}$$

* could find ω that maximizes $\langle T \rangle$ by setting $\frac{d}{d\omega}\langle T \rangle = 0$.

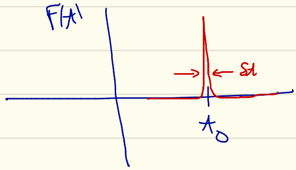
* However, just by re-writing $\langle T \rangle$ slightly as

$$\langle T \rangle = \frac{m}{4}\frac{f^2}{\left(\frac{\omega_0^2}{\omega}-\omega\right)^2 + 4\beta^2} \Rightarrow \text{want to minimize denominator}$$

$$\Rightarrow \frac{\omega_0^2}{\omega} - \omega = 0 \Rightarrow \boxed{\omega = \omega_0} \text{ Maximizes } \langle T \rangle$$

Impulsive Driving Forces

* Suppose we kick a damped HO at rest at some time t_0. This is an impulsive force lasting some $\delta t \ll \frac{2\pi}{\omega_0}$.



* Assume δt so small that m moves negligible distance while F is acting

$$\text{Impulse } mv_0 = \int_{t_0}^{t_0+\delta t} F\,dt \simeq F(t_0)\delta t$$

* After the kick, $\quad X(t) = Ae^{-\beta t}\cos(\omega_1 t + \delta) \quad$ (i.e., soln for damped HO)

IC's: ① $X(t_0) = 0 = Ae^{-\beta t_0}\cos(\omega_1 t_0 + \delta)$

$$\Rightarrow \omega_1 t_0 + \delta = \frac{\pi}{2} \Rightarrow \delta = \frac{\pi}{2} - \omega_1 t_0$$

$$\therefore X(t) = Ae^{-\beta t}\cos\left(\frac{\pi}{2} + \omega_1(t - t_0)\right)$$

$$X(t) = -Ae^{-\beta t}\sin(\omega_1(t-t_0)) \quad //$$

② $\dot{X}(t_0) = \frac{1}{m}\int F\,dt = \frac{\delta t F}{m} \equiv v_0 = -A\omega_1 e^{-\beta t_0}$

$$\therefore A = -\frac{v_0}{\omega_1}e^{\beta t_0}$$

$$\boxed{\Rightarrow X(t) = \begin{cases} 0 & t < x_0 \\ \frac{v_0}{\omega_1}e^{-\beta(t-t_0)}\sin[\omega_1(t-t_0)] & t \geq x_0 \end{cases}}$$

$$\Rightarrow x(t) = \begin{cases} 0 & t < x_0 \\ \frac{v_0}{\omega_1} e^{-\beta(t-t_0)} \sin[\omega_1(t-t_0)] & t \geq x_0 \end{cases}$$

Claim: This is independent of the precise form of the impulsive force. I leave it to you to show that the following examples for $F(t)$ all give the same answer as above in the $\delta t \to 0$ limit.

① $$F(t) = \begin{cases} 0 & t < t_0 \\ \frac{mv_0}{\delta t} & t_0 \leq t \leq t_0 + \delta t \\ 0 & t > t_0 + \delta t \end{cases}$$

② $$F(t) = \frac{mv_0}{\pi} \frac{\delta t}{(t-t_0)^2 + (\delta t)^2} \qquad -\infty < t < \infty$$

③ $$F(t) = \frac{mv_0}{\delta t \sqrt{\pi}} \exp\left(-\frac{(t-t_0)^2}{\delta t^2}\right) \qquad -\infty < t < \infty$$

✱ Superposition Principle: Driven HO w/ arbitrary driving force $F(t)$ (i.e., not a simple sin/cos or single impulse)

How to solve $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)$ for arbitrary $F(t) = mf(t)$?

Theorem: Let the set $x_n(t)$ $n = 1, 2, 3, \ldots$ solve the IHG ODEs

$$\ddot{x}_n + 2\beta\dot{x}_n + \omega_0^2 x_n = f_n(t)$$

where $f(t) = \sum_n f_n(t)$. Then $x(t) = \sum_n x_n(t)$ obeys

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t).$$

proof: Let $\hat{L} = \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$

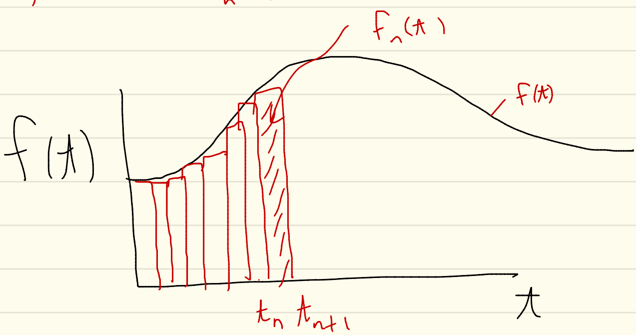
$$\hat{L}x = \hat{L}(x_1 + x_2 + \cdots) = \hat{L}x_1 + \hat{L}x_2 + \cdots$$

$$= f_1 + f_2 + \cdots$$

$$= \sum_n f_n = f \quad \checkmark.$$

Illustration of the theorem: Green's Function Method

Idea: try to write $F(t) = \sum_n F_n(t)$ where we know how to solve $\hat{L}x_n = f_n(t)$.



* approximate the smooth $f(t)$ as a sum over discrete impulses

$$f_n(t) = \begin{cases} 0 & t < t_n \\ f(t_n) & t_n \le t \le t_{n+1} \\ 0 & t > t_{n+1} \end{cases} \qquad \text{where } t_n = n\,\delta t \qquad n = -\infty, \ldots -1, 0, 1, 2, \ldots +\infty$$

$$\Rightarrow f(t) \simeq \sum_{n=-\infty}^{n_0} f_n(t) \qquad \text{where: } t_{n_0} \le t \le t_{n_0+1}$$

* but we already know the soln for a single impulse force $\quad x_n(t) = \frac{v_n}{\omega_1} e^{-\beta(t-t_n)} \sin[\omega_1(t-t_n)] \quad t_n \le t \le t_{n+1}$

$$= 0 \quad t < t_n \text{ or } t > t_{n+1}$$

$\therefore$ By the theorem, the particular sol'n of

$$\hat{L}x = f(t) \qquad \text{for } \underline{\text{general}}\ f(t) \text{ is}$$

$$x(t) = \sum_{n=-\infty}^{n_0} \frac{f_n \delta t}{\omega_1} e^{-\beta(t-t_n)} \sin[\omega_1(t-t_n)] \qquad \left(\text{used } v_n = \frac{\delta P_n}{m} = \frac{F_n \delta t}{m} = f(t_n)\delta t\right)$$

* take $\delta t \to 0$ & let $t_n \equiv t'$

$$\boxed{x(t) = \int_{-\infty}^{t} \frac{f(t')}{\omega_1} e^{-\beta(t-t')} \sin[\omega_1(t-t')]\, dt'}$$

* Define "Green's Function"

$$G(t,t') \equiv \begin{cases} 0 & \text{if } t' > t \\ \dfrac{e^{-\beta(t-t')}}{\omega_1} \sin[\omega_1(t-t')] & \text{if } t' \leq t \end{cases}$$

$$\boxed{x(t) \equiv \int_{-\infty}^{\infty} G(t,t') f(t')\, dt'}$$