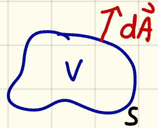
Gauss's Law Recap

$$\oint_S \vec{g}\cdot d\vec{A} = -4\pi G\, M_{enc}$$



$$M_{enc} = \int_V d^3r\, \rho(\vec{r})$$

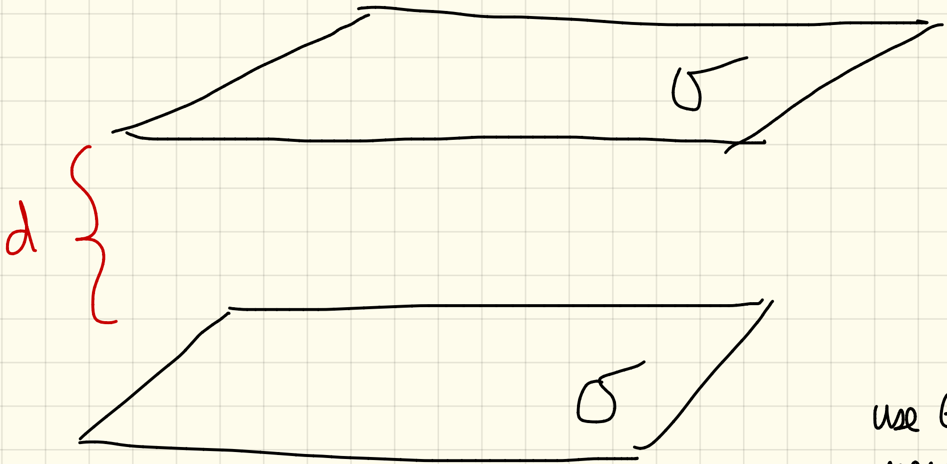
Useful to calc. $\vec{g}$ when symmetry + smart choice of S lets us "<u>pull $\vec{g}$ out of the integral</u>"

1.) $\rho(r,\phi,\theta) = \rho(r)$ Spherical symmetry S = gaussian sphere

2.) $\rho(r,\phi,z) = \rho(r)$ (cylindrical r!!) Axial or Cylindrical Symmetry S = gaussian cylinder

3.) Planar symmetry S = gaussian box

<u>Note</u>: Sometimes it looks like the problem is too asymmetric for Gauss to be helpful. <u>However</u>, sometimes you can use superposition to write the problem as a sum of symmetric pieces.

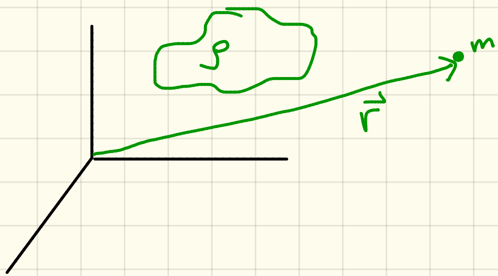


2 uniform ∞-planes

$\Downarrow$

use Gauss for each plane separately. Then $\vec{g} = \vec{g}_1 + \vec{g}_2$.

Gravitational Potential Field



$$\vec{F}_{\text{on } m} = m\,\vec{g}(\vec{r})$$

Question: Is $\vec{F} = -\vec{\nabla} U(\vec{r})$

i.e., **conservative**

$$\vec{g}(\vec{r}) = -G \int d^3r' \, \rho(\vec{r}\,') \, \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3}$$

* **trick:** $\dfrac{\vec{r}}{|\vec{r}|^3} = \dfrac{\hat{r}}{r^2} = -\vec{\nabla}\left(\dfrac{1}{r}\right)$

(recall, $\vec{\nabla} f(r) = \hat{r} \dfrac{d}{dr} f$)

* **Claim:** $$\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = -\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|}$$

(i.e., use previous result + let $\vec{\mathscr{r}} = \vec{r}-\vec{r}'$)

$\Downarrow$

$$\Rightarrow \vec{g}(\vec{r}) = G\int d^3r'\, \rho(\vec{r}')\, \vec{\nabla}\left(\frac{1}{|\vec{r}-\vec{r}'|}\right) = \vec{\nabla}\left(G\int d^3r'\, \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\right)$$

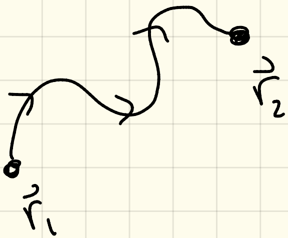
Df. Grav. Potential (aka Scalar potential)

$$\boxed{\phi(\vec{r}) \equiv -G\int d^3r'\, \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \quad \underline{\text{where}} \quad \vec{g}(\vec{r}) = -\vec{\nabla}\phi(\vec{r})}$$

* Trivially, the gravitational potential energy follows:

$$\boxed{\begin{aligned} \vec{F}_m(\vec{r}) &= m\vec{g}(\vec{r}) = -m\vec{\nabla}\phi(\vec{r}) \equiv -\vec{\nabla}U(\vec{r}) \\ U(\vec{r}) &= m\phi(\vec{r}) = -mG\int d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \end{aligned}}$$

* Another way to write $\phi(\vec{r})$ w/a nice Physical interpretation



Work you do to move m from $\vec{r}_1$ to $\vec{r}_2$ in a $\vec{g}$-field?

$\vec{g} = -\vec{\nabla}\phi$

$$W_{12} = -\int_{\vec{r}_1}^{\vec{r}_2} m\vec{g}\cdot d\vec{r} = +m\int_{\vec{r}_1}^{\vec{r}_2}(\vec{\nabla}\phi)\cdot d\vec{r} = m\int d\phi = \boxed{m\phi(2) - m\phi(1)}$$

$$\Rightarrow \frac{W_{12}}{m} \equiv W'_{12} = \phi(2) - \phi(1)$$

= work you do against gravity to move unit mass from $\vec{r}_1$ to $\vec{r}_2$.

(Path Independent)

Related....

$$\boxed{\phi(\vec{r}) = -\int_{\infty}^{\vec{r}} \vec{g} \cdot d\vec{r}} \quad (*)$$

equivalent to

$$\boxed{\phi(\vec{r}) = -G\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d^3r'} \quad (*)$$

when we normalize $\phi(\infty) = 0$.

(Recall, $U(\vec{r})$ thence $\phi(\vec{r})$ ambiguous up to arbitrary constant since

$$\Rightarrow \vec{F} = -\vec{\nabla}U(\vec{r}) = -\vec{\nabla}U(\vec{r}) - \cancelto{0}{\vec{\nabla}U_0}\ )$$

Comment: Why bother w/ $\phi(\bar{r})$? Because it's often easier to calculate $\phi(\bar{r})$ & then take $\vec{g} = -\vec{\nabla}\phi$ as opposed to a direct calculation of $\vec{g}$.

Rule of thumb for finding $\vec{g}$ (in order of increasing tedium)

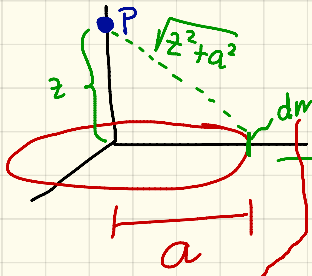
1) If highly symmetric ρ, use Gauss's Law

$$\oint_S \vec{g} \cdot d\vec{A} = -4\pi G M_{enc}$$

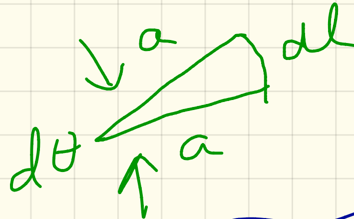
2) If not sufficiently symmetric for Gauss, calc. ϕ then $\vec{g} = -\vec{\nabla}\phi$

3) Brute force $\vec{g} = -G \int d^3r' \, \rho \, \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$

Example: Circular wire w/ const. λ. Find $\phi(0,0,z)$



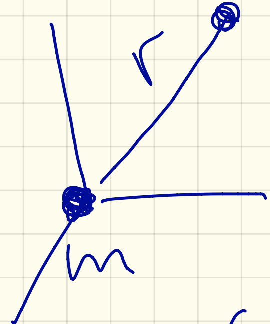
$dm = \lambda dl = \lambda a\, d\theta$



$$\int d\phi(0,0,z) = \frac{-G\,dm}{\sqrt{z^2+a^2}} = \int \frac{-G\lambda a\, d\theta}{\sqrt{z^2+a^2}}$$

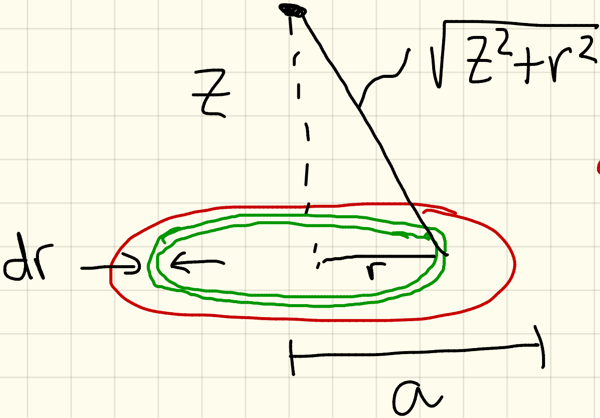
$$\therefore\ \phi(0,0,z) = \int d\phi(0,0,z) = \frac{-G\,2\pi a\lambda}{\sqrt{z^2+a^2}} = \boxed{\frac{-G M_{loop}}{\sqrt{z^2+a^2}}}$$

$$M_{loop} = 2\pi a\lambda$$



$$\phi(r) = \frac{-Gm}{r}$$

EX:



disk w/ uniform σ

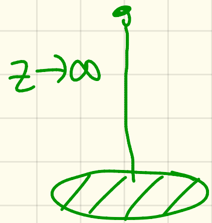
$$= \boxed{\frac{-G M_{loop}}{\sqrt{z^2+a^2}}}$$

$$d\phi_{\substack{ring \\ at\ r}} = \frac{-G\, dM_{ring}}{\sqrt{Z^2+r^2}} \qquad dM_{ring} = \sigma\, 2\pi r\, dr$$

$$\int d\phi = \int_0^a \frac{-G\sigma 2\pi r\, dr}{\sqrt{Z^2+r^2}} = -G 2\pi\sigma\left[\sqrt{a^2+z^2} - |Z|\right]$$

$$= \phi_{disk}$$

Check the result in some limit where answer is "obvious"



Expect: $\displaystyle \Phi_{disk}(0,0,z) \xrightarrow[z \to \infty]{} \frac{-G M_{disk}}{z} = \frac{-G \sigma \pi a^2}{z}$

check it: $\displaystyle \Phi_{disk} = -G 2\pi\sigma \left[z\sqrt{1+\frac{a^2}{z^2}} - z \right]$

$$\sqrt{1+\frac{a^2}{z^2}} \simeq 1 + \frac{1}{2}\frac{a^2}{z^2} + \cdots$$

$$\Phi \simeq -G \cancel{2}\pi\sigma \left[\cancel{z} + \frac{1}{\cancel{2}}\frac{a^2}{z} - \cancel{z} \right] = \frac{-G \pi a^2 \sigma}{z}$$

Very long cylinder

$$\rho = kr \qquad (0 \le r \le a)$$

1.) $\vec{g}(r,\phi,z) = g(r)\,\hat{r}$

r<a:

$$\oint_S \vec{g}\cdot d\vec{A} = \int_{\text{sides}} g(r)\, 2\pi r\, dz$$

$$= g(r)\, 2\pi r h$$

$$M_{\text{enclosed}} = \int_{\substack{\text{gaussian}\\ \text{cyl}}} \rho\, d^3r$$

$$= \rho\, V_{\substack{\text{gaussian}\\ \text{cylinder}}}$$

$$= \rho \cdot \pi r^2 h$$

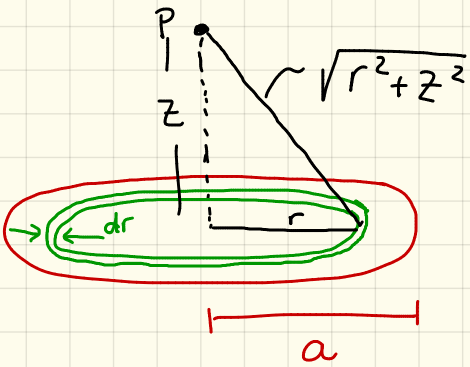
$$d^3r = r\,dr\,dz\,d\phi \Rightarrow V = \cancel{2}\pi \frac{r^2}{\cancel{2}} h$$

r>a:

$$M_{enc} = \rho \int_0^a r\,dr \int_0^h dz \int_0^{2\pi} d\phi$$

$$= \rho \cdot \pi a^2 h$$

Ex: Same as before, but for a uniform disk



* Build up the disk as infinitesimally thick rings of variable radii r

$$d\phi_{\substack{\text{ring}\\ \text{at } r}} = \frac{-G\, dM_{\text{ring}}}{\sqrt{r^2+z^2}} = \frac{-G\sigma\, dA_{\text{ring}}}{\sqrt{r^2+z^2}} = \frac{-G\sigma\, 2\pi r\, dr}{\sqrt{r^2+z^2}}$$

$$\therefore \phi_{\text{disk}} = -G\sigma 2\pi \int_0^a \frac{r\,dr}{\sqrt{r^2+z^2}} = -G 2\pi\sigma\left[\sqrt{a^2+z^2} - |z|\right]$$

* It's a good idea to check the result in some physical limit where answer is "obvious".

e.g.: for $z \gg a$, the disk "looks" like a point mass $M_{disk} = \sigma \pi a^2$ at the origin

expect: $\phi(0,0,z) \xrightarrow[z \to \infty]{} \dfrac{-GM_{disk}}{z}$

check it: $-G 2\pi\sigma \left[\sqrt{a^2+z^2} - |z|\right]$

$$= -G2\pi\sigma\left[z\sqrt{1+\frac{a^2}{z^2}} - z\right] \qquad \text{(take } z>0 \text{ WLOG)}$$

$$\simeq -G2\pi\sigma\left[\cancel{z} + \frac{1}{2}\frac{a^2}{z} + \cdots - \cancel{z}\right] \approx \frac{-GM_{disk}}{z} \text{ as it must.}$$