# Chapter 7: Lagrangian + Hamiltonian Mechanics

* Now we turn to two related advanced re-formulations of Newtonian Mechanics due to Lagrange, Hamilton, + others in the late 1700's + early 1800's.

* We do this for several reasons. On the pragmatic side, these new formulations give a powerful general "recipe" to solve problems that are difficult to solve in $\vec{F} = m\vec{a}$ formulation. For example, problems with "forces of constraint" (e.g., a particle constrained to move on a spherical surface, or a bead sliding along some curved wire, etc...) are hard to solve in Newton's $\vec{F} = m\vec{a}$ formulation since **we often don't know the precise form of the often complicated constraining forces.**

* On a deeper level, the new Lagrangian + Hamiltonian formulations of classical mechanics are beneficial since

1.) They generalize easily to classical <u>field theories</u> (e.g., E+M, general relativity, etc.)

2) Quantum mechanics carries over/generalizes concepts from the Lagrangian + Hamiltonian formalisms

3.) Likewise, statistical mechanics (either classical/Quantum) borrows heavily from ideas of Hamiltonian Mechanics

## * Poor man's "derivation" of Lagrangian Mechanics

- Later on, we'll wrap this up in the elegant formulation called "Hamilton's Principle"

- For now, let's consider N-body conservative system

$$\frac{d}{dt}\vec{p}_1 = \vec{F}_1$$
$$\vdots$$
$$\frac{d}{dt}\vec{p}_N = \vec{F}_N$$

Vector eqns, 3N components

Can we reduce the complexity?

Conservative forces $\Rightarrow$ $$\vec{F}_\alpha = -\vec{\nabla}_\alpha U$$

1 scalar function $U$ takes care of all vector forces

** Is there some scalar function that encodes the N vector EOM?

Notice: $T = \sum_\alpha \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha^{\,2}$ $\qquad$ $\alpha$ labels particle

$$\frac{\partial T}{\partial \dot{r}_{\alpha,i}} = \frac{\partial}{\partial \dot{r}_{\alpha,i}} \sum_{\beta=1}^{N} \sum_{j=1}^{3} \frac{1}{2} m_\beta \dot{r}_{\beta,j} \dot{r}_{\beta,j}$$

$$= \sum_{\beta=1}^{N} \sum_{j=1}^{3} \frac{1}{2} m_\beta \cdot 2 \delta_{\alpha\beta} \delta_{ij} \dot{r}_{\beta,j} = m_\alpha \dot{r}_{\alpha,i} = p_{\alpha,i}$$

$\therefore$ Newton's 2nd Law $\qquad \frac{d}{dt} p_{\alpha,i} = F_{\alpha,i}$

$$\left.\begin{aligned} \Rightarrow \text{LHS} &= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}_{\alpha,i}}\right) \\ \text{RHS} &= -\frac{\partial}{\partial r_{\alpha,i}} U \end{aligned}\right\} \text{ for } \begin{aligned} &\alpha = 1 \dots N \\ &i = 1,2,3 \end{aligned}$$

* Suggests the scalar function $L = T - U$ ("Lagrangian")

correctly encodes Newton's $2^{nd}$ Law via

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_{\alpha,i}}\right) = \frac{\partial L}{\partial r_{\alpha,i}} \qquad \begin{matrix} \alpha = 1 \ldots N \\ i = 1,2,3 \end{matrix}}$$

"Euler-Lagrange EOM"

* Were we just lucky that the Euler-Lagrange eqns. are so simple? I.e., you might worry that if we choose non-cartesian coordinates, the eqns. might look alot uglier. Let's check it

Ex: Newtonian formulation for central force motion of a particle

$$\vec{F} = m\vec{a} \qquad \text{where} \quad \vec{F} = \hat{r}F(r) = -\hat{r}\frac{d}{dr}U(r)$$

$$\vec{r} = r\hat{r}$$

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\,\hat{\theta}$$

$$\ddot{\vec{r}} = \vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$$

$$\boxed{\Rightarrow m(\ddot{r} - r\dot{\theta}^2) = -\frac{d}{dr}U} \quad (1) \qquad (\text{radial comp. of } \vec{F} = m\vec{a})$$

$$\Rightarrow m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \Rightarrow m(r^2\ddot{\theta} + 2\dot{r}r\dot{\theta}) = 0 \qquad (\text{angular comp.})$$

$$\Rightarrow 0 = \frac{d}{dt}(mr^2\dot{\theta})$$

$$\boxed{\Rightarrow \ell \equiv mr^2\dot{\theta} = \text{constant}} \quad (2)$$

Question: Will Euler-Lagrange reproduce (1) + (2) if we blindly treat $r, \theta$ like cartesian?

Check it:

EL eqn. in r $\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial}{\partial r} L$

$$L = T - U = \tfrac{1}{2}(m\dot{r}^2 + mr^2\dot{\theta}^2) - U(r)$$

$\therefore \frac{\partial L}{\partial \dot{r}} = m\dot{r}$, $\frac{\partial L}{\partial r} = -\frac{\partial U}{\partial r}$ $\Big\} \Rightarrow$ EL gives $m\ddot{r} = -\frac{\partial U}{\partial r}$ as we'd hope!

EL eqn in $\theta$: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \cancel{\frac{\partial L}{\partial \theta}} \nearrow 0$

$\Rightarrow \frac{d}{dt}(mr^2\dot{\theta}) = 0$ just like Newton's EOM give us.

Moral: Euler-Lagrange eqns. of motion look the same for any set of independent coordinates $(q_1, q_2, q_3)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

Later on, we'll see the real power of the Lagrangian framework, when we'll see $(q_i)$ don't even have to be coordinates in the usual sense. They can be "generalized coordinates" (More later...), which can be complicated derived functions of the spatial coords.

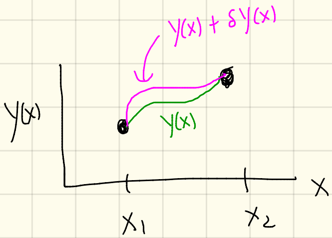
## * Calculus of Variations

Need some new mathematical methods to understand Hamilton's Principle. The new math is called "variational calculus" or "calculus of variations".

The basic problem (in its simplest form) is to find the $y(x)$ that minimize or maximize some integral expression

$$J[y] = \int_{x_1}^{x_2} f(y(x), y'(x); x)\, dx$$

(Jargon: $J[y]$ is a "functional". Functions map a # to a #. Functionals map a function (i.e., $x(t)$) to a number)




If $y(x)$ is an extremum (i.e., Min. or Max.), then any variation $y(x) + \delta y(x)$ where $\delta y(x_1) = \delta y(x_2) = 0$ but otherwise totally arbitrary will no longer be extremum of $J[y]$

## * How to find extremum $y(x)$?

* Let $y(x;\alpha) = y(x) + \alpha\, \eta(x)$

extremum

arbitrary function obeying

$$\eta(x_1) = \eta(x_2) = 0$$

Then in analogy w/ finding Min/Max of an ordinary function, we demand

$$\frac{\partial J}{\partial \alpha} = 0$$

# Derivation of Euler-Lagrange Eqn

$$\frac{\partial J}{\partial \alpha} = 0 = \frac{\partial}{\partial \alpha}\int_{x_1}^{x_2} f(y(x;\alpha),\ y'(x;\alpha)\ ;x)\,dx$$

$$0 = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right]dx \qquad \text{(by chain rule)}$$

$*$ but $\frac{\partial y}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[y(x) + \alpha\eta(x)\right] = \eta(x)$

$$\frac{\partial y'}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[y' + \alpha\eta'\right] = \eta'(x)$$

$$\therefore\ 0 = \int_{x_1}^{x_2}\left\{\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\frac{d}{dx}\eta(x)\right\}dx$$

$*$ write $\frac{\partial f}{\partial y'}\frac{d}{dx}\eta(x) = \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\eta(x)\right] - \eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$

$$\Rightarrow 0 = \int_{x_1}^{x_2}\eta(x)\left\{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right\} + \underbrace{\frac{\partial f}{\partial y'}\cancel{\eta(x)}\Big|_{x_1}^{x_2}}_{0}$$

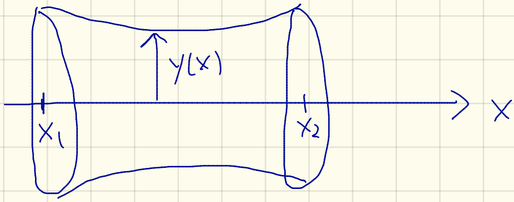
since $\eta(x_1) = \eta(x_2) = 0$

$$\Rightarrow 0 = \int_{x_1}^{x_2}\eta(x)\left\{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right\}dx \qquad \eta(x) \text{ totally arbitrary}$$

$$\therefore\ \boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0} \qquad \text{"Euler-Lagrange" eqn}$$

$\Rightarrow$ Necessary condition for $y(x)$ to min/maximize $J[y]$.

*Example: Minimal surface area for an axially-symmetric soap film between 2 rings at $x_1$ & $x_2$




$$\text{area} = \int dA = \int 2\pi y \sqrt{dx^2 + dy^2} = \int_{x_1}^{x_2} 2\pi y \, dx \sqrt{1 + (y')^2}$$

$$J[y] = 2\pi \int_{x_1}^{x_2} y \sqrt{1 + y'^2} \, dx$$

$$\Rightarrow f(y, y'; x) = 2\pi y \sqrt{1 + y'^2}$$

EL eqns:

$$\frac{\partial f}{\partial y} = 2\pi \sqrt{1 + y'^2}$$

$$\frac{\partial f}{\partial y'} = \frac{1}{2} \frac{2\pi y \cdot 2y'}{\sqrt{1 + y'^2}} = \frac{2\pi y y'}{\sqrt{1 + y'^2}}$$

$\Rightarrow$ EL eqns become: $$\sqrt{1 + y'^2} - \frac{d}{dx}\left(\frac{yy'}{\sqrt{1 + y'^2}}\right) = 0$$

$$\Rightarrow \frac{1}{\sqrt{1 + y'^2}} - \frac{yy''}{(1 + y'^2)^{3/2}} = 0$$

looks ugly, but can simplify w/a trick by multiplying thru by $y'$

$$\Rightarrow 0 = \frac{y'}{\sqrt{1+y'^2}} - \frac{yy'y''}{(1+y'^2)^{3/2}} = \frac{d}{dx}\left(\frac{y}{\sqrt{1+y'^2}}\right)$$

$$\Rightarrow \frac{y}{\sqrt{1+y'^2}} = k \quad (\text{constant})$$

Non-linear ODE but separable.

$$\Downarrow$$

$$\frac{dy}{dx} = \sqrt{\frac{y^2}{k^2} - 1}$$

$$\int \frac{dy}{\sqrt{\frac{y^2}{k^2}-1}} = \int dx$$

$$\left.\begin{aligned} *\text{let } y &= k\cosh t \\ dy &= k\sinh t \, dt \end{aligned}\right\} \Rightarrow \int \frac{dy}{\sqrt{\frac{y^2}{k^2}-1}} = k\int \frac{\sinh t}{\sqrt{\cosh^2 t - 1}}\, dt$$

$\cosh^2 t - \sinh^2 t = 1$

$C =$ constant

$$\Rightarrow x = kt + C$$

$$x = k \operatorname{Arccosh}\left[\frac{y}{k}\right] + C$$

$$\Rightarrow \operatorname{Arccosh}\left(\frac{y}{k}\right) = \frac{x-C}{k}$$

$$\boxed{y = k\cosh\left(\frac{x-C}{k}\right)}$$

lastly, $k$ & $C$ fixed from $y(x_1) = y_1$ & $y(x_2) = y_2$

# Useful $2^{nd}$-form of EL-eqns

* In the soap film example, how'd we know to multiply thru by $y'$ to simplify the calculation?

Notice $f(y,y';x) = f(y',y)$ in the soap example (i.e., no <u>explicit</u> x-dependence)

$$\Rightarrow \frac{df(y,y')}{dx} = \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} + \cancel{\frac{\partial f}{\partial x}}^{\,0}$$

$$\boxed{\frac{df}{dx} = \frac{\partial f}{\partial y}y' + \frac{\partial f}{\partial y'}y''} \quad ①$$

also, $$\boxed{\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = y''\frac{\partial f}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)} \quad ②$$

Use ① to eliminate $y''\frac{\partial f}{\partial y'}$ in ②

$$\Rightarrow \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{df}{dx} - \frac{\partial f}{\partial y}y' + y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

$$= \frac{df}{dx} + y'\left[\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y}\right]$$

0 by EL eqns

$$\Rightarrow \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'} - f\right) = 0$$

$$\Rightarrow \left(y'\frac{\partial f}{\partial y'} - f\right) = \text{constant if } \frac{\partial f}{\partial x} = 0$$

* You can verify this applied to the soap example gives us what we found after m. by $y'$.

# * Generalization to several dependent Variables

$$J[y_1, y_2, \dots y_N] = \int_{x_1}^{x_2} dx\, f(y_1, y_1', y_2, y_2', \dots y_N, y_N'; x)$$

* Find extremum, i.e., the $y_i(x)$ $i=1,\dots N$ that extremize $J$ like we did in the 1 variable case

let $y_i(x;\alpha) = y_i(x) + \alpha \eta_i(x)$ $\quad i = 1, \dots N$

$\eta_i(x_1) = \eta_i(x_2) = 0$ but otherwise arbitrary & independent

Demand: $\dfrac{\partial J}{\partial \alpha} = 0$

$$= \int_{x_1}^{x_2} dx \sum_{i=1}^{N} \left( \frac{\partial f}{\partial y_i} \frac{\partial y_i}{\partial \alpha} + \frac{\partial f}{\partial y_i'} \frac{\partial y_i'}{\partial \alpha} \right)$$

$$= \int_{x_1}^{x_2} dx \sum_{i=1}^{N} \left( \frac{\partial f}{\partial y_i} \eta_i + \frac{\partial f}{\partial y_i'} \eta_i' \right)$$

$$\frac{\partial f}{\partial y_i'} \eta_i' = \frac{d}{dx}\left( \frac{\partial f}{\partial y_i'} \eta_i \right) - \eta_i \frac{d}{dx}\left( \frac{\partial f}{\partial y_i'} \right)$$

$$\Rightarrow 0 = \int_{x_1}^{x_2} dx \sum_{i=1}^{N} \eta_i(x) \left\{ \frac{\partial f}{\partial y_i} - \frac{d}{dt}\left( \frac{\partial f}{\partial y_i'} \right) \right\} + \sum_{i=1}^{N} \overset{0}{\cancel{\eta_i}} \frac{\partial f}{\partial y_i'} \Bigg|_{x_1}^{x_2}$$

The $\eta_i(x)$ are independent & totally arbitrary.

$$\boxed{\Rightarrow \frac{\partial f}{\partial y_i} = \frac{d}{dt}\left( \frac{\partial f}{\partial y_i'} \right) \quad i = 1 \dots N}$$