Recap from last time

* Shortcomings of Newtonian $\vec{F} = m\vec{a}$ formulation of mechanics

  - Constrained motion (eg, particle on a sphere, bead on a wire, etc.) impossible if you don't know forces of constraint

  - Hard to connect to QM or classical field theories (E&M, general relativity, ...)

  - Vector eqns "hard" (Scalar qtys. "Nicer")

* Lagrangian formulation of Mechanics - Euler-Lagrange eqns. of motion

$$L \equiv T - U \quad (\text{"Lagrangian"})$$

$$T = \sum_{\alpha=1}^{N} \frac{m_\alpha}{2} \dot{\vec{r}}_\alpha^{\,2} \; ; \quad U = U(\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N)$$

Noticed that we can write $F_{\alpha,i} = \frac{dP_{\alpha,i}}{dt}$ in terms of L

$$\boxed{\frac{\partial L}{\partial r_{\alpha,i}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}_{\alpha,i}}\right) = 0 \qquad \begin{array}{l} \alpha = 1 \ldots N \\ i = x, y, z \end{array}} \qquad (*)$$

* We "derived" (*) for cartesian coordinates. Remarkably, (*) holds for non-cartesian coords.

e.g., $\quad r_{\alpha,i} \quad$ where $i = r, \theta, \phi \quad$ (spherical coords.)

etc...

Calculus of Variations (ch 6): Find $y(x)$ that minimizes or maximizes the integral

$$J[y] = \int_{x_1}^{x_2} f(y(x), y'(x); x)\, dx$$

* letting $y(x;\alpha) = y(x) + \alpha\, \eta(x)$

α = constant

$\eta(x)$ = arbitrary function that obeys $\eta(x_1) = \eta(x_2) = 0$

$\Downarrow$

extremum $y(x)$ found by

$$\frac{\partial J}{\partial \alpha} = 0 \Rightarrow \boxed{\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial y'}\right) = 0}$$

Euler-Lagrange eqn (solve for $y(x)$)

* Trivial extension to several dependent variables $y_1(x), y_2(x), \dots y_n(x)$

$$J[y_i] = \int_{x_1}^{x_2} f(y_i(x), y_i'(x); x)\, dx \;, \quad y_i(x;\alpha) = y_i(x) + \alpha\, \eta_i(x)$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial y_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial y_i'}\right) = 0 \qquad i = 1 \dots n}$$

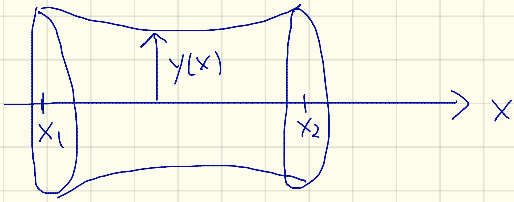
to find extremum $y_i(x)$

These eqns. should ring a bell. They are the same eqns as in Lagrangian mechanics!

Hamilton's Principle: $J[r_{\alpha,i}] = \int_{t_1}^{t_2} L(r_{\alpha,i}, \dot{r}_{\alpha,i})\, dt$ "The action" functional

extremum $r_{\alpha,i}(t)$ = physical trajectory of the system between t_1 + t_2.

*Example: Minimal surface area for an axially-symmetric soap film between 2 rings at x_1 & x_2



$$\text{area} = \int dA = \int 2\pi y \sqrt{dx^2+dy^2} = \int_{x_1}^{x_2} 2\pi y \, dx \sqrt{1+(y')^2}$$

$$J[y] = 2\pi \int_{x_1}^{x_2} y\sqrt{1+y'^2}\, dx$$

$$\Rightarrow f(y,y';x) = 2\pi y \sqrt{1+y'^2}$$

EL eqns:
$$\frac{\partial f}{\partial y} = 2\pi\sqrt{1+y'^2}$$

$$\frac{\partial f}{\partial y'} = \frac{1}{2}\frac{2\pi y \cdot 2y'}{\sqrt{1+y'^2}} = \frac{2\pi y y'}{\sqrt{1+y'^2}}$$

$\Rightarrow$ EL eqns become:
$$\sqrt{1+y'^2} - \frac{d}{dx}\left(\frac{yy'}{\sqrt{1+y'^2}}\right) = 0$$

$$\Rightarrow \frac{1}{\sqrt{1+y'^2}} - \frac{yy''}{(1+y'^2)^{3/2}} = 0$$

looks ugly, but can simplify w/a trick by multiplying thru by y'

$$\Rightarrow 0 = \frac{y'}{\sqrt{1+y'^2}} - \frac{y y' y''}{(1+y'^2)^{3/2}} = \frac{d}{dx}\left(\frac{y}{\sqrt{1+y'^2}}\right)$$

$$\Rightarrow \frac{y}{\sqrt{1+y'^2}} = k \quad (\text{constant})$$

Non-linear ODE but separable.

$$\Downarrow$$

$$\frac{dy}{dx} = \sqrt{\frac{y^2}{k^2} - 1}$$

$$\int \frac{dy}{\sqrt{\frac{y^2}{k^2}-1}} = \int dx$$

$\cosh^2 t - \sinh^2 t = 1$

$$\left.\begin{aligned} &* \text{let } y = k\cosh t \\ &dy = k \sinh t \, dt \end{aligned}\right\} \Rightarrow \int \frac{dy}{\sqrt{\frac{y^2}{k^2}-1}} = k\int \frac{\sinh t}{\sqrt{\cosh^2 t - 1}}\, dt$$

$C =$ constant

$$\Rightarrow \quad x = kt + C$$

$$x = k \operatorname{Arccosh}\left[\frac{y}{k}\right] + C$$

$$\Rightarrow \operatorname{Arccosh}\left(\frac{y}{k}\right) = \frac{x-C}{k}$$

$$\boxed{y = k\cosh\left(\frac{x-C}{k}\right)}$$

lastly, k & C fixed from $y(x_1) = y_1$ & $y(x_2) = y_2$

Useful 2nd-form of EL-eqns

* In the soap film example, how'd we know to multiply thru by y' to simplify the calculation?

Notice $f(y,y';x) = f(y',y)$ in the soap example (i.e., no <u>explicit</u> x-dependence)

$$\Rightarrow \frac{df(y,y')}{dx} = \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{dy'}{dx} + \cancel{\frac{\partial f}{\partial x}}^{\,0}$$

$$\boxed{\frac{df}{dx} = \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y''} \quad ①$$

also,
$$\boxed{\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = y''\frac{\partial f}{\partial y'} + y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)} \quad ②$$

Use ① to eliminate $y''\frac{\partial f}{\partial y'}$ in ②

$$\Rightarrow \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right) = \frac{df}{dx} - \frac{\partial f}{\partial y}y' + y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

$$= \frac{df}{dx} + y'\Big[\underbrace{\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y}}_{0 \text{ by EL eqns}}\Big]$$

$$\Rightarrow \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'} - f\right) = 0$$

$$\Rightarrow \left(y'\frac{\partial f}{\partial y'} - f\right) = \text{constant if } \frac{\partial f}{\partial x} = 0 \quad //$$

* You can verify this applied to the soap example gives us what we found after m. by y'.

~~Hamilton's Principle: Put another way, the equations of motion follow from~~

$$\cancel{\delta J = \delta \int_{t_1}^{t_2} L \, dt = 0}$$

~~(I use the books notation where~~ $\cancel{\delta J = 0}$ ~~is shorthand for~~ $\cancel{r_i(t;\alpha) = r_i(t) + \alpha \eta_i(t)}$ ~~& setting~~ $\cancel{\frac{dJ}{d\alpha} d\alpha = 0}$~~)~~

Euler's Eqns with Constraints

example: $J[y_1, y_2] = \int_{x_1}^{x_2} f(y_1, y_1', y_2, y_2'; x) \, dx$

* Minimize J subject to the constraint $g(y_1(x), y_2(x); x) = 0$

$$\Downarrow$$

as before:

$$y_1(x;\alpha) = y_1(x) + \alpha \eta_1(x) \qquad \eta_i(x_1) = \eta_i(x_2) = 0$$
$$y_2(x;\alpha) = y_2(x) + \alpha \eta_2(x)$$

* HOWEVER, $\eta_1(x)$ & $\eta_2(x)$ are no longer arbitrary & independent, as the constraint eqn. must hold for all α.

$$g(y_1(x;\alpha), y_2(x;\alpha), x) = 0$$

$$\Rightarrow \frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left[\left(\frac{\partial f}{\partial y_1} - \frac{d}{dx} \frac{\partial f}{\partial y_1'} \right) \eta_1(x) + \left(\frac{\partial f}{\partial y_2} - \frac{d}{dx} \frac{\partial f}{\partial y_2'} \right) \eta_2(x) \right] dx$$

Setting $\frac{\partial J}{\partial \alpha} = 0$, we can't set $\left(\frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial y_i'} \right) = 0$ since η_1 & η_2 are not independent!

What to do then?

$$g(y_i(x;\alpha), x) = 0$$

$$\Rightarrow dg = 0 = \left(\frac{\partial g}{\partial y_1}\frac{\partial y_1}{\partial \alpha} + \frac{\partial g}{\partial y_2}\frac{\partial y_2}{\partial \alpha}\right)d\alpha$$

$$0 = \left(\frac{\partial g}{\partial y_1}\eta_1 + \frac{\partial g}{\partial y_2}\eta_2\right)d\alpha$$

$$\Rightarrow \frac{\partial g}{\partial y_1}\eta_1(x) = -\frac{\partial g}{\partial y_2}\eta_2(x)$$

$$\Rightarrow \boxed{\eta_2(x) = -\eta_1(x)\frac{\frac{\partial g}{\partial y_1}}{\frac{\partial g}{\partial y_2}}} \longrightarrow \text{plug in to } \frac{\partial J}{\partial \alpha} \text{ expression}$$

$$\Rightarrow \frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left[\left(\frac{\partial f}{\partial y_1} - \frac{d}{dx}\frac{\partial f}{\partial y_1'}\right) - \left(\frac{\partial f}{\partial y_2} - \frac{d}{dt}\frac{\partial f}{\partial y_2'}\right)\left(\frac{\partial g/\partial y_1}{\partial g/\partial y_2}\right)\right]\eta_1(x)\,dx$$

$$\therefore \frac{\partial J}{\partial \alpha} = 0 \Rightarrow [\,\cdots\cdots\,] = 0$$

$$\boxed{\Rightarrow \left(\frac{\partial f}{\partial y_1} - \frac{d}{dx}\frac{\partial f}{\partial y_1'}\right)\left(\frac{\partial g}{\partial y_1}\right)^{-1} = \left(\frac{\partial f}{\partial y_2} - \frac{d}{dx}\frac{\partial f}{\partial y_2'}\right)\left(\frac{\partial g}{\partial y_2}\right)^{-1}}$$

* Only way this eqn can be met is if

$LHS = -\lambda(x)$ (− sign is customary but not necessary)

$RHS = -\lambda(x)$ where $\lambda(x)$ is some unknown function

"Lagrange undetermined multiplier"

$\Rightarrow$ We have 3 eqns. to fix the 3 unknowns (y_1, y_2, λ)

$$\boxed{\begin{aligned} &\frac{\partial f}{\partial y_1} - \frac{d}{dx}\frac{\partial f}{\partial y_1'} + \lambda(x)\frac{\partial g}{\partial y_1} = 0 \\ &\frac{\partial f}{\partial y_2} - \frac{d}{dx}\frac{\partial f}{\partial y_2'} + \lambda(x)\frac{\partial g}{\partial y_2} = 0 \\ &g(y_1, y_2; x) = 0 \end{aligned}}$$

* Generalizing to m-dependent variables $y_i(x)$ $i = 1 \ldots m$ and n constraints

$g_j(y_i; m) = 0 \quad j = 1 \ldots n$

$$\boxed{\begin{aligned} &\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} + \sum_{j=1}^{m}\lambda_j(x)\frac{\partial g_j}{\partial y_i} = 0 \qquad i = 1 \ldots m \\ &g_j(y_i; x) = 0 \qquad j = 1 \ldots n \end{aligned}}$$

Sidenote I: In mechanics, it's often easier to use the differential form of the $g_j(y_i; x) = 0$ eqns

$$\sum_{i=1}^{m} \frac{\partial g_j}{\partial y_i} dy_i = 0 \qquad j = 1 \ldots n$$

Sidenote 2: Sometimes, the constraint equation is an integral relation.

e.g.: Minimize $J[y] = \int_a^b f(y(x), y'(x); x)\, dx$

subject to the constraint

$$\int_a^b g(y, y'; x)\, dx = \text{constant}$$

Then one can show (similar steps as above) that

we minimize $$I[y] = \int_a^b (f + \lambda g)\, dx$$

$$\Rightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \lambda\left(\frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'}\right) = 0$$