* Recap from last time

1) Generalized coordinates: Not limited to formulating the problem in terms of orthogonal coordinate systems (cartesian, spherical, cyl., etc)

$$r_{\alpha,i} = r_{\alpha,i}(q_1, \dots q_{3N}, t) \qquad \alpha = 1 \dots N$$
$$i = x, y, z \text{ or } r, \theta, \phi \text{ or } r, \phi, z, \text{ or} \dots$$

$\Downarrow$ invertible

$$q_j = q_j(r_{\alpha,i}, t) \qquad j = 1 \dots s \qquad (s = 3N)$$
$$\dot{q}_j = \dot{q}_j(r_{\alpha,i}, \dot{r}_{\alpha,i}, t)$$

* Still have the same Euler-Lagrange eqns. of motion

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0$$

generalized force: $\quad Q_j \equiv \dfrac{\partial L}{\partial q_j}$

generalized momentum: $\quad p_j \equiv \dfrac{\partial L}{\partial \dot{q}_j}$

ex: Consider 1 particle in 3d

$$Q_j = \frac{\partial L}{\partial q_j} = \sum_{i=x,y,z} \frac{\partial L}{\partial r_i}\frac{\partial r_i}{\partial q_j} = \sum_{i=x,y,z} F_i \frac{\partial r_i}{\partial q_j}$$

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \sum_i \frac{\partial L}{\partial \dot{r}_i}\frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \sum_{i=x,y,z} p_i \frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \sum_i p_i \frac{\partial r_i}{\partial q_j}$$

Used $\dot{r}_i = \sum_{j=1}^{s} \frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t} \Rightarrow \frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}$

* Problems w/ constraints

e.g. $J[q_1,q_2] = \int_{t_1}^{t_2} L(q_j, \dot{q}_j; t)\, dt$

w/ constraint $f(q_1,q_2;t) = 0$

Issue: Can't blindly apply Hamilton's principle

let $q_j(t;\alpha) = q_j(t) + \alpha\eta_j(t)$ $\qquad \eta_j(t_1) = \eta(t_2) = 0$

$$\frac{\partial J}{\partial \alpha} = 0 = \int_{t_1}^{t_2} dt \left[\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} \right)\eta_1(t) + \left(\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} \right)\eta_2(t) \right]$$

* but $f(q_1,q_2) = 0 \Rightarrow df = 0 = \left(\frac{\partial f}{\partial q_1}\eta_1(t) + \frac{\partial f}{\partial q_2}\eta_2(t) \right) d\alpha$

$\Rightarrow \eta_2(t) = -\frac{\partial f/\partial q_1}{\partial f/\partial q_2}\eta_1(t)$ NOT INDEP! Can't set coeffs. of η_1 & η_2 to zero!

* 2 ways to deal w/ constraints in Lagrangian Mechanics

1.) Use constraint eqns. to eliminate "redundant" variables

e.g., for $s=2$ case above, use $f(q_1,q_2;t) = 0 \Rightarrow$ find $q_2 = q_2(q_1,t)$

$\Rightarrow L(q_1,q_2,\dot{q}_1,\dot{q}_2) \longrightarrow L(q_1,\dot{q}_1)$ expressed in the 1 independent coordinate

$\Rightarrow$ Now apply Hamilton's Principle: $\frac{\partial L}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) = 0$

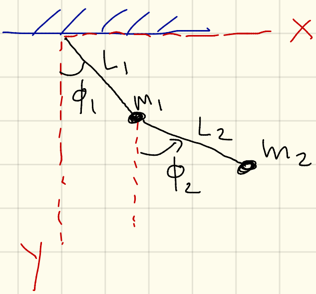
* Generalizing to $L(q_j, \dot{q}_j; t)$ $j = 1, \ldots, S$

w/ m eqns. of constraint $\quad f_k(q_j, t) = 0 \quad k = 1, \ldots, m$

$\Rightarrow$ Use m constraint eqns to re-write L in terms of $(S-m)$ independent generalized coords

$\Rightarrow$ then solve $\quad \dfrac{\partial L}{\partial q_j} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_j} = 0 \quad j = 1, \ldots, (S-m)$

Example: Co-planar double pendulum Find L



2 masses $\Rightarrow$ Naively, 6 DOF

$\Downarrow$

Constrained motion
- planar $\quad z_1, z_2$
- $L_1 + L_2$ fixed $\quad r_1, r_2$

$\therefore$ Really only 2 independent generalized coords. Needed.

$\Rightarrow$ Try $\phi_1 + \phi_2$

$T_1 = \frac{1}{2} m_1 L_1^2 \dot{\phi}_1^2 \qquad T_2 = \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2)$

$U_1 = -m_1 g L_1 \cos\phi_1 \qquad U_2 = -m_2 g y_2$

but $\quad x_2 = L_1 \sin\phi_1 + L_2 \sin\phi_2$

$\quad y_2 = L_1 \cos\phi_1 + L_2 \cos\phi_2$

$$\Rightarrow T_2 = \frac{1}{2} m_2 \left[L_1^2 \dot{\phi}_1^2 + L_2^2 \dot{\phi}_2^2 + 2 L_1 L_2 \cos(\phi_1 - \phi_2) \dot{\phi}_1 \dot{\phi}_2 \right]$$

$$\Downarrow$$

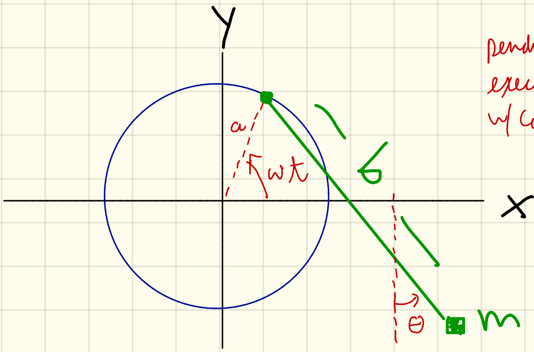
$$L(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) = \frac{1}{2}(m_1 + m_2) L_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 L_2^2 \dot{\phi}_2^2 + m_2 L_1 L_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$

$$+ (m_1 + m_2) g L_1 \cos\phi_1 + m_2 g L_2 \cos\phi_2$$

To find EOM, simply take $\quad \frac{\partial L}{\partial \phi_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_i}\right) = 0 \quad i = 1, 2$

Example

$\downarrow g$



pendulum support executes circular motion w/ constant ω + radius a

* Particle in 3D $\Rightarrow S = 3$

* But planar motion + fixed b $\Rightarrow$ only $S - 2 = 1$ independent coordinate θ

$$X = a\cos\omega t + b\sin\theta$$
$$Y = a\sin\omega t - b\cos\theta$$

$$\dot{X} = -a\omega\sin\omega t + b\dot{\theta}\cos\theta$$
$$\dot{Y} = a\omega\cos\omega t + b\dot{\theta}\sin\theta$$

$$\Rightarrow T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}\left[a^2\omega^2 + b^2\dot{\theta}^2 + 2b\dot{\theta}a\omega\sin(\theta - \omega t)\right]$$

$$U = mgy = mg(a\sin\omega t - b\cos\theta)$$

$$\boxed{\Rightarrow L = T - U = \frac{m}{2}\left[a^2\omega^2 + b^2\dot{\theta}^2 + 2b\dot{\theta}a\omega\sin(\theta - \omega t)\right] - mg(a\sin\omega t - b\cos\theta)}$$

* 2nd Way to treat Constraints: Lagrange Multipliers

* back to our generic constrained $s=2$ system

$$\frac{\partial J[q_1,q_2]}{\partial \alpha} = 0 = \int_{t_1}^{t_2}\left[\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1}\right)\eta_1(t) + \left(\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2}\right)\eta_2(t)\right]dt$$

where $\quad f(q_1, q_2; t) = 0 \Rightarrow \eta_2(t) = -\frac{\partial f/\partial q_1}{\partial f/\partial q_2}\eta_1(t)$

$\therefore$ eliminating η_2 for η_1 $\Rightarrow$

$$\frac{\partial J}{\partial \alpha} = 0 = \int_{t_1}^{t_2}\left[\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1}\right) - \left(\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2}\right)\frac{\partial f/\partial q_1}{\partial f/\partial q_2}\right]\eta_1(t)\,dt$$

$$\Rightarrow [\quad] = 0 \Rightarrow \left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1}\right)\left(\frac{\partial f}{\partial q_1}\right)^{-1} = \left(\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2}\right)\left(\frac{\partial f}{\partial q_2}\right)^{-1}$$

$$\underbrace{\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1}\right)}_{\text{derivs. wrt } q_1 + \dot{q}_1}\left(\frac{\partial f}{\partial q_1}\right)^{-1} = \underbrace{\left(\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2}\right)}_{\text{derivs wrt } q_2 + \dot{q}_2}\left(\frac{\partial f}{\partial q_2}\right)^{-1}$$

$L = L(q_1(t), \dot{q}_1(t), q_2(t), \dot{q}_2(t); t)$

$f = f(q_1(t), q_2(t); t)$

$\Rightarrow$ Only way this makes sense is if

$LHS = -\lambda(t)$

$RHS = LHS = -\lambda(t)$

"Lagrange undetermined Multiplier"

$\Downarrow$

$$\boxed{\begin{aligned} \frac{\partial L}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) + \lambda(t)\frac{\partial f}{\partial q_1} &= 0 \\ \frac{\partial L}{\partial q_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) + \lambda(t)\frac{\partial f}{\partial q_2} &= 0 \end{aligned}}$$

Plus the constraint eqn

$$\boxed{f(q_1, q_2; t) = 0}$$

$\Rightarrow$ 3 eqns for 3 unknowns (q_1, q_2, λ)

* Claims (See the graduate text of Goldstein for a proof) that

$$Q_j^{\text{Constraint}} = \lambda \frac{\partial f}{\partial q_j}$$ = j^{th} component of the generalized force of constraint

* Generalizing to higher # of DOF

$$L = L(q_j, \dot{q}_j; t) \qquad j = 1, \dots S$$

$$f_K(q_j; t) = 0 \qquad K = 1, \dots, m$$

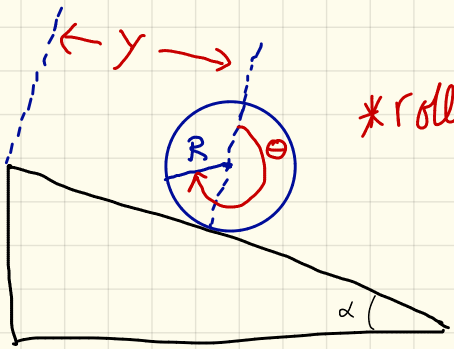
$$\Downarrow$$

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_{K=1}^{m} \lambda_K(t) \frac{\partial f_K}{\partial q_j} = 0 \qquad j = 1, \dots S$$

=> (S+m) eqns for (S+m) unknowns $(q_1, \dots q_S; \lambda_1, \dots \lambda_m)$

$$Q_j^{\text{const.}} = \sum_{K=1}^{m} \lambda_K \frac{\partial f_K}{\partial q_j}$$ generalized forces of constraint

Example: let's revisit the problem we solved with "Way 1" last time



* rolling disk without slipping

$\Downarrow$

Gen. Coords. y, θ

Constraint: $y = R\theta$

* Solve this w/ Lagrange Multiplier + find Gen. Constraining Force

$$T = T_{trans} + T_{rot} = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 \qquad \left(I = \frac{1}{2}MR^2\right)$$

$$U = Mg(l-y)\sin\alpha \qquad (l = \text{length of incline})$$

① $$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} + \lambda\frac{\partial f}{\partial y} = 0$$

and $f(y,\theta) = y - R\theta = 0$

② $$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial f}{\partial \theta} = 0$$

$\Rightarrow \frac{\partial f}{\partial y} = 1, \quad \frac{\partial f}{\partial \theta} = R$

$$\boxed{\begin{aligned} &① \Rightarrow \quad Mg\sin\alpha - M\ddot{y} + \lambda = 0 \\ &② \Rightarrow \quad -\tfrac{1}{2}MR^2\ddot{\theta} - \lambda R = 0 \end{aligned}}$$

2 eqns plus $y = R\theta$ for y, θ, λ

$y = R\theta \Rightarrow \ddot{\theta} = \frac{\ddot{y}}{R}$ plug into ②

② $\Rightarrow -\frac{1}{2}M\cancel{R}\ddot{y} = \lambda \cancel{R} \Rightarrow \boxed{\lambda = \frac{-M}{2}\ddot{y}}$

plug into eq. ①

$$Mg\sin\alpha - M\ddot{y} - \frac{M}{2}\ddot{y} = 0$$

$$\Downarrow$$

$$\boxed{\ddot{y} = \frac{2g}{3}\sin\alpha}$$

Same ODE we got in the Simpler way#1 of solving the problem

plug into λ + $\ddot{\theta}$ eqns

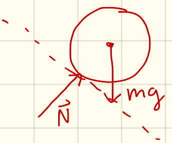
$$\boxed{\lambda = -\frac{Mg\sin\alpha}{3}} \quad \text{and} \quad \boxed{\ddot{\theta} = \frac{2g\sin\alpha}{3R}}$$

* Forces of Constraint

$$Q_y = \lambda\frac{\partial f}{\partial y} = \lambda = -\frac{Mg}{3}\sin\alpha \qquad \text{(Force)}$$

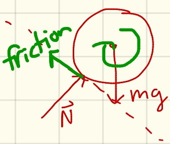
$$Q_\theta = \lambda\frac{\partial f}{\partial \theta} = -\lambda R = \frac{MgR}{3}\sin\alpha \qquad \text{(Torque)}$$

Physical Interp: Frictional force is what implements the "rolling w/out slipping" $y = R\theta$ constraint



No friction

disk will slide (No rotation since no torques)



No friction

torque from friction