* Recap from last time

Problems w/ Constraints: $L(q_i, \dot{q}_i; t)$ $i = 1, \dots, s$ generalized coordinates

plus m eqns of constraint $f_K(q_1 \dots q_s) = 0$ $K = 1 \dots m$

(e.g., motion confined to some surface)

2 ways to deal w/ these situations

① Use $f_K(q_1 \dots q_s; t) = 0$ to express m of the q_i in terms of the remaining $(s-m)$ q_j.

$$\Rightarrow L(q_1, \dot{q}_1, \dots q_s, \dot{q}_s) \longrightarrow L(q_1, \dot{q}_1, \dots q_{s-m}, \dot{q}_{s-m})$$

$$\Rightarrow \quad \frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0 \quad j = 1, \dots, s-m$$

- EASY ☺
- Can't say anything about $\vec{F}_{constraint}$ ☹

② Lagrange Multipliers: Keep $L = L(q_i, \dot{q}_i)$ $i = 1 \dots s$, but solve

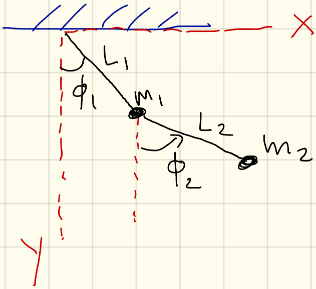
$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \sum_{K=1}^{m} \lambda_K(t) \frac{\partial f_K}{\partial q_i} = 0 \qquad i = 1 \dots s$$

$$f_K(q_1 \dots q_s; t) = 0 \qquad K = 1 \dots m$$

($s+m$ eqns. for $s+m$ unknowns $q_1 \dots q_s$ + $\lambda_1 \dots \lambda_m$)

- More eqns ($s+m$) ☹
- Allows you to calculate $F_{constraint}$ ☺

Example: Co-planar double pendulum Find L



2 masses $\Rightarrow$ Naively, 6 DOF

$\Downarrow$

Constrained motion
- planar z_1, z_2
- $L_1 + L_2$ fixed r_1, r_2

$\therefore$ Really only 2 independent generalized coords. Needed.

$\Rightarrow$ Try $\phi_1 + \phi_2$

* Construct $L(\phi_1, \phi_2, \dot{\phi}_1, \dot{\phi}_2)$: $\quad L = (T_1 + T_2) - (U_1 + U_2)$

$$T_1 = \frac{1}{2} m_1 L_1^2 \dot{\phi}_1^2 \qquad T_2 = \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2)$$

$$U_1 = -m_1 g L_1 \cos\phi_1 \qquad U_2 = -m_2 g\, y_2$$

but $x_2 = L_1 \sin\phi_1 + L_2 \sin\phi_2$

$y_2 = L_1 \cos\phi_1 + L_2 \cos\phi_2$

$$\Rightarrow T_2 = \frac{1}{2} m_2 \left[L_1^2 \dot{\phi}_1^2 + L_2^2 \dot{\phi}_2^2 + 2 L_1 L_2 \cos(\phi_1 - \phi_2)\, \dot{\phi}_1 \dot{\phi}_2 \right]$$

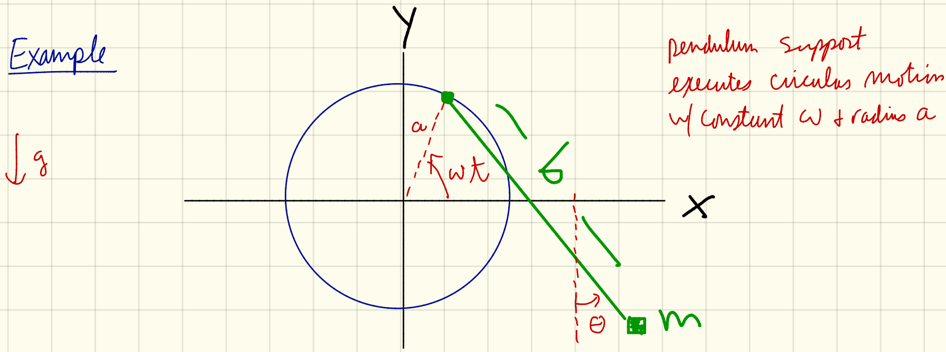
$$\Downarrow$$

$$L(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) = \frac{1}{2}(m_1+m_2) L_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 L_2^2 \dot{\phi}_2^2 + m_2 L_1 L_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2)$$

$$+ (m_1+m_2) g L_1 \cos\phi_1 + m_2 g L_2 \cos\phi_2$$

To find EOM, simply take $\quad \frac{\partial L}{\partial \phi_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_i}\right) = 0 \quad i = 1, 2$

Example



* Particle in 3D $\Rightarrow S = 3$

* But planar motion + fixed b $\Rightarrow$ only $S - 2 = 1$ independent coordinate θ

$$x = a\cos\omega t + b\sin\theta$$
$$y = a\sin\omega t - b\cos\theta$$

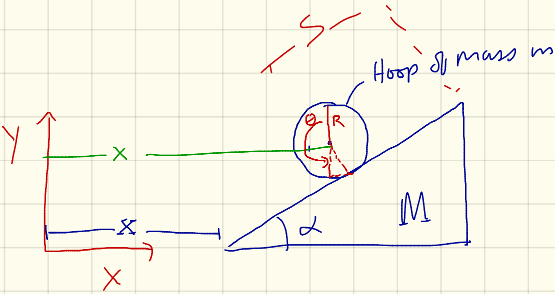
$$\dot{x} = -a\omega\sin\omega t + b\dot{\theta}\cos\theta$$
$$\dot{y} = a\omega\cos\omega t + b\dot{\theta}\sin\theta$$

$$\Rightarrow T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}\left[a^2\omega^2 + b^2\dot{\theta}^2 + 2b\dot{\theta}a\omega\sin(\theta - \omega t)\right]$$

$$U = mgy = mg(a\sin\omega t - b\cos\theta)$$

$$\boxed{\Rightarrow L = T - U = \frac{m}{2}\left[a^2\omega^2 + b^2\dot{\theta}^2 + 2b\dot{\theta}a\omega\sin(\theta - \omega t)\right] - mg(a\sin\omega t - b\cos\theta)}$$

Example (#7.6)



* Hoop rolls without slipping

$\Rightarrow S = R\theta$

* Wedge can slide w/out friction

* Find Lagrange eqns. of motion

- Based on example from last class, expect S, θ, X suitable set of generalized coords

$$L = T_{hoop} + T_{wedge} - U_{hoop} - \cancel{U_{wedge}}^{\,0}$$

$$T_{hoop} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{I}{2}\dot{\theta}^2$$

but
$$x = X + (l - S)\cos\alpha - R\sin\alpha$$
$$y = (l - S)\sin\alpha + R\cos\alpha$$

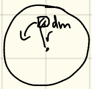
$$I = mR^2$$

Quick aside on moment of inertia for T_{rot} for the hoop

$$dT = \frac{dm}{2} v_{rot}^2 = \frac{dm}{2}\left(\frac{R\,d\theta}{dt}\right)^2 = \frac{R^2}{2}\dot{\theta}^2 \int dm$$

$$T_{rot} = \frac{mR^2}{2}\dot{\theta}^2 = \frac{I}{2}\dot{\theta}^2 \Rightarrow I = mR^2$$

Likewise, for the rolling disk example last time



$$dT = \frac{dm}{2} r^2 \dot{\theta}^2$$

$$dm = \sigma\, 2\pi r\, dr = \frac{M}{\pi R^2} \cdot 2\pi r\, dr$$

$$dm = \frac{2M}{R^2} r\, dr$$

$$\Rightarrow T_{rot} = \int dT_{rot} = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}M}{R^2} \dot{\theta}^2 \int_0^R r^3\, dr$$

$$= \frac{M\dot{\theta}^2 R^2}{4} \equiv \frac{I}{2}\dot{\theta}^2$$

$$\Rightarrow I_{disk} = \frac{1}{2} M R^2$$

$$L = T_{hoop} + T_{wedge} - U_{hoop} - \cancel{U_{wedge}}^{\to 0}$$

$$T_{hoop} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{mR^2}{2}\dot{\theta}^2$$

but $x = X + (l-s)\cos\alpha - R\sin\alpha$

$y = (l-s)\sin\alpha + R\cos\alpha$

$$\Downarrow$$

$$\dot{x} = \dot{X} - \dot{s}\cos\alpha$$

$$\dot{y} = -\dot{s}\sin\alpha$$

$$\boxed{T_{wedge} = \frac{M}{2}\dot{X}^2} \quad (1)$$

$$\therefore T_{hoop} = \frac{m}{2}\left((\dot{X} - \dot{s}\cos\alpha)^2 + \dot{s}^2\sin^2\alpha\right) + \frac{mR^2}{2}\dot{\theta}^2$$

$$* \text{ but } \underline{s = R\theta} \implies \dot{\theta} = \frac{\dot{s}}{R}$$

$$\implies T_{hoop} = \frac{m}{2}\left[\dot{X}^2 - 2\dot{X}\dot{s}\cos\alpha + \dot{s}^2\right] + \frac{m}{2}\dot{s}^2$$

$$\boxed{T_{hoop} = \frac{m}{2}\left[\dot{X}^2 - 2\dot{X}\dot{s}\cos\alpha + 2\dot{s}^2\right]} \quad (2)$$

$$\boxed{U_{hoop} = mgy = mg\left[(l-s)\sin\alpha + R\cos\alpha\right]} \quad (3)$$

$$\therefore L = (1) + (2) + (3) = m\dot{s}^2 + \frac{1}{2}(m+M)\dot{X}^2 - m\dot{X}\dot{s}\cos\alpha - mgs\sin\alpha + \text{const}$$

$$L = ① + ② + ③ = m\dot{s}^2 + \frac{1}{2}(m+M)\dot{X}^2 - m\dot{X}\dot{s}\cos\alpha - mgs\sin\alpha + \text{const}$$

* Now apply EL eqns.

$$\frac{\partial L}{\partial s} = -mg\sin\alpha$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = \frac{d}{dt}\left(2m\dot{s} - m\dot{X}\cos\alpha\right) = 2m\ddot{s} - m\ddot{X}\cos\alpha$$

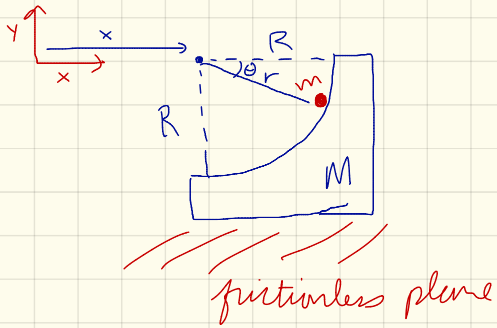
$$\boxed{\Rightarrow 2m\ddot{s} - m\ddot{X}\cos\alpha + mg\sin\alpha = 0}$$

$$\frac{\partial L}{\partial X} = 0$$

$$\frac{\partial L}{\partial \dot{X}} = (m+M)\dot{X} - m\dot{s}\cos\alpha$$

$$\boxed{\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) = 0 = (m+M)\ddot{X} - m\ddot{s}\cos\alpha}$$

Ex: 7.34



Find EOM for m + M and find reaction force of wedge on particle

⟹ Use Lagrange multipliers

$X_M = x$ $\qquad X_m = x + r\cos\theta$ $\qquad \otimes$

$Y_M = 0$ $\qquad Y_m = -r\sin\theta$

$$L = \frac{M}{2}\dot{X}_M^2 + \frac{m}{2}(\dot{X}_m^2 + \dot{Y}_m^2) - mgY_m$$ plug in $\otimes$

$$\dot{X}_M^2 = \dot{x}^2$$

$$\dot{X}_m^2 + \dot{Y}_m^2 = (\dot{x} + \dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2 + (-\dot{r}\sin\theta - r\dot{\theta}\cos\theta)^2$$

etc.

$\Big\}$ tedious algebra

$$L = \frac{(M+m)}{2}\dot{x}^2 + \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + 2\dot{x}\dot{r}\cos\theta - 2\dot{x}r\dot{\theta}\sin\theta\right) + mgr\sin\theta$$

Constraint: $f(r,\theta,x) = r - R$

$$\Rightarrow \frac{\partial f}{\partial r} = 1$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x} = 0$$

$\Rightarrow$ Need to solve then

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) + \lambda = 0$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0$$

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$$

plus $r = R$