Clicker Question

$$\vec{A} = (x+y)\,\hat{\imath} - y^2\,\hat{\jmath} + (y+z)\,\hat{k}$$

Find $\vec{\nabla}\cdot\vec{A}$ at $(1,1,1)$

A) -2

B) 0

C) 1

D) 2

E) 3

Clicker Question (ANS)

$$\vec{A} = (x+y)\hat{i} - y^2\hat{j} + (y+z)\hat{k}$$

Find $\vec{\nabla}\cdot\vec{A}$ at $(1,1,1)$

A) -2

$$\frac{\partial A_x}{\partial x} = 1$$

B) 0

$$\frac{\partial A_y}{\partial y} = -2y = -2$$

C) 1

$$\frac{\partial A_z}{\partial z} = 1$$

D) 2

$$\therefore\ 1 - 2 + 1 = 0.$$

E) 3

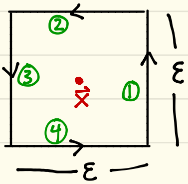
The Curl + Stokes Theorem

* $\vec{\nabla}\cdot\vec{F}(\vec{x})$ measures how $\vec{F}$ "spreads out" from $\vec{x}$

* $\vec{\nabla}\times\vec{F}(\vec{x})$ (aka "curl $\vec{F}$") measures how $\vec{F}$ circulates or curls around $\vec{x}$

$$\boxed{(\vec{\nabla}\times\vec{F})_i = \sum_{jk} \varepsilon_{ijk}\nabla_j F_k \quad \text{or} \quad \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \nabla_1 & \nabla_2 & \nabla_3 \\ F_1 & F_2 & F_3 \end{vmatrix} \qquad \left(\nabla_j \equiv \frac{\partial}{\partial x_j}\right)}$$

* Let $\vec{V}(\vec{x})$ = velocity field in a fluid



Loop in xy plane

$$\text{"circulation"} = \oint_C \vec{V}\cdot d\vec{x} = \sum_{i=1}^{4} \int_{C_i} \vec{V}\cdot d\vec{x}$$

<u>$C_①$ integral</u>: $\quad d\vec{x} = \hat{j}\,dy \quad \Rightarrow \displaystyle\int_{y-\frac{\varepsilon}{2}}^{y+\frac{\varepsilon}{2}} V_y(\vec{x}+\tfrac{\varepsilon}{2}\hat{\imath})\,dy \approx \varepsilon V_y(\vec{x}+\tfrac{\varepsilon}{2}\hat{\imath})$

<u>$C_③$ integral</u>: $\quad d\vec{x} = \hat{j}\,dy \quad \Rightarrow \displaystyle\int_{y+\frac{\varepsilon}{2}}^{y-\frac{\varepsilon}{2}} V_y(\vec{x}+\tfrac{\varepsilon}{2}\hat{\imath})\,dy \approx -\varepsilon V_y(\vec{x}-\tfrac{\varepsilon}{2}\hat{\imath})$

$$\Rightarrow \int_{C_1} + \int_{C_3} = \varepsilon\left[V_y(\vec{x}+\tfrac{\varepsilon}{2}\hat{i}) - V_y(\vec{x}-\tfrac{\varepsilon}{2}\hat{i})\right] \underset{\varepsilon\to 0}{=} \varepsilon^2 \frac{\partial V_y}{\partial x}$$

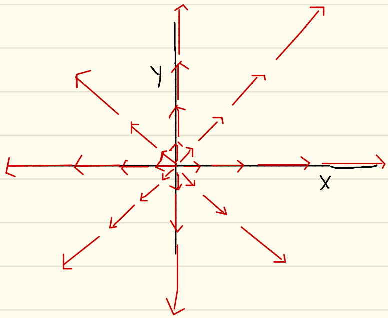
* A similar calculation for $C_2 + C_4$ gives

$$\int_{C_2} + \int_{C_4} = -\varepsilon\left[V_x(\vec{x}+\tfrac{\varepsilon}{2}\hat{j}) - V_x(\vec{x}-\tfrac{\varepsilon}{2}\hat{j})\right] \underset{\varepsilon\to 0}{=} -\varepsilon^2 \frac{\partial V_x}{\partial y}$$

$$\therefore \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) = \lim_{\delta A\to 0} \frac{1}{\delta A} \oint_C \vec{V}\cdot d\vec{x}$$

$$\boxed{\therefore (\vec{\nabla}\times\vec{V})_k = \lim_{\delta A\to 0} \frac{1}{\delta A} \oint_C \vec{V}\cdot d\vec{x} \quad \text{where } C \text{ in } x_i\text{-}x_j \text{ plane}}$$

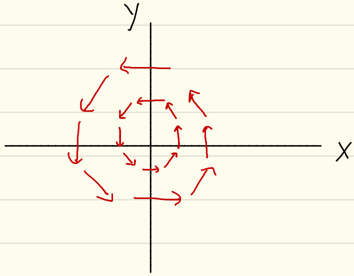
Ex 1: $\vec{F}(\vec{x}) = \vec{x} = x\hat{i} + y\hat{j}$



$$\vec{\nabla}\times\vec{F} = 0$$

(as expected)

Ex 2: $\vec{F}(\vec{x}) = -y\hat{i} + x\hat{j}$

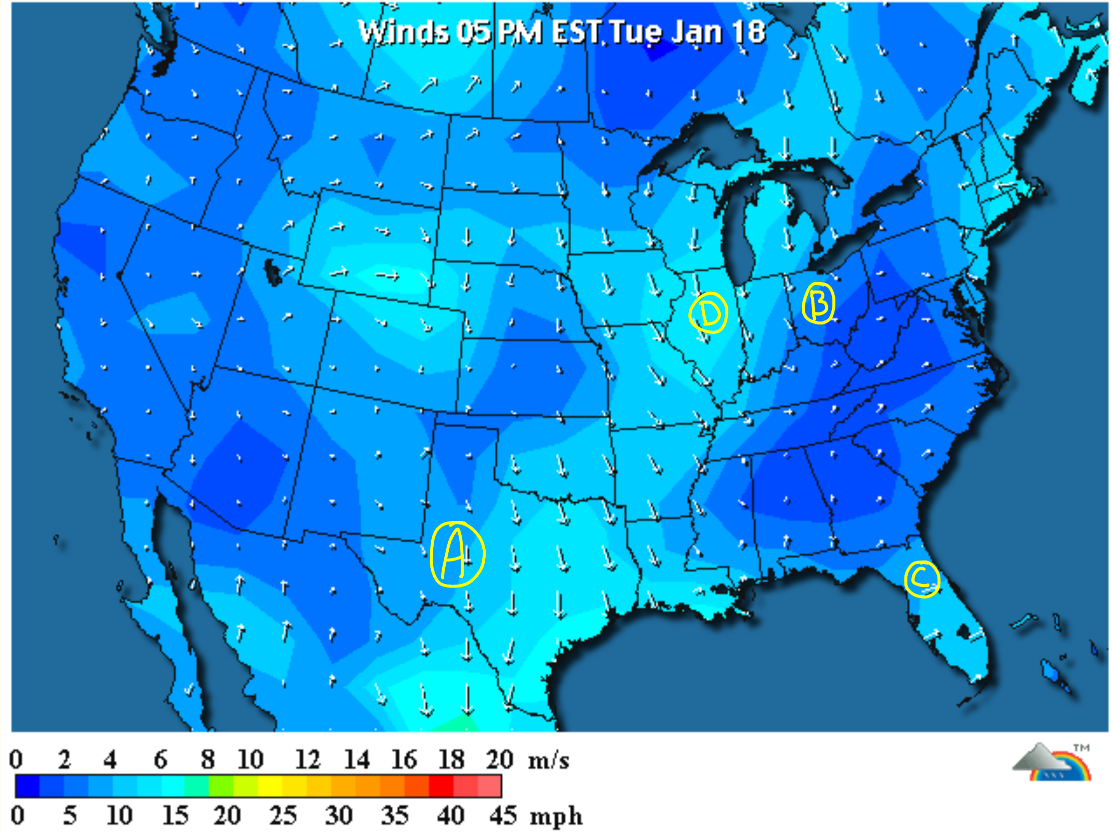


$$\vec{\nabla}\times\vec{F} = 2\hat{k}$$

(c.f. right hand rule)

Which state has both 1.) $\vec{\nabla}\cdot\vec{F} = 0$

2.) $(\vec{\nabla}\times\vec{F})_z < 0$

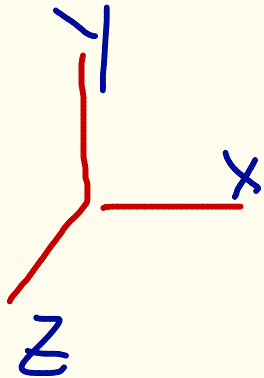


A ✓

B

C

D



Multiple derivative Example

Example: What is $\vec{\nabla} \times \vec{\nabla} f$?

$$(\vec{\nabla} \times \vec{\nabla} f)_i = \sum_{jk} \varepsilon_{ijk} \nabla_j \nabla_k f$$

$*$ but $\nabla_j \nabla_k f = \nabla_k \nabla_j f$

$$= \sum_{jk} \varepsilon_{ijk} \nabla_k \nabla_j f$$

$$= -\sum_{jk} \varepsilon_{ikj} \nabla_k \nabla_j f \qquad (\text{since } \varepsilon_{ijk} = -\varepsilon_{ikj})$$

$$= -(\vec{\nabla} \times \vec{\nabla} f)_i$$

$$\therefore (\vec{\nabla} \times \vec{\nabla} f)_i = -(\vec{\nabla} \times \vec{\nabla} f)_i \Rightarrow \vec{\nabla} \times \vec{\nabla} f = 0 \ \checkmark$$

a side remark:

$$\sum_{jk} \varepsilon_{ijk} S_{jk} = 0 \quad \text{if } S_{jk} = S_{kj}$$

$$\sum_{ijk} \varepsilon_{ijk} S_{ijk} = 0 \quad \text{if } S_{ijk} = S_{kij} = S_{jki} = S_{jik} = S_{kji} = S_{ikj}$$

(ie, symmetric under permutation of indices)

Example:

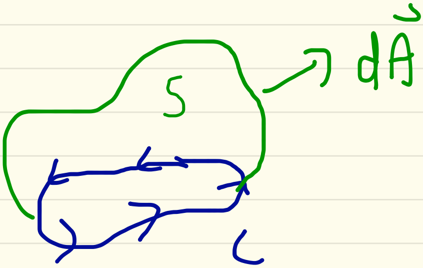
$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = \sum_i \nabla_i (\vec{\nabla} \times \vec{F})_i$$

$$= \sum_{ijk} \varepsilon_{ijk} \nabla_i \nabla_j F_k$$

$= 0$ by above remark since $\nabla_i \nabla_j F_k$ totally symmetric under permutations of its indices.

FTC for $\vec{\nabla}\times\vec{F}$: Stokes's Theorem

$$\boxed{\int_S (\vec{\nabla}\times\vec{F})\cdot d\vec{A} = \oint_C \vec{F}\cdot d\vec{x}}$$

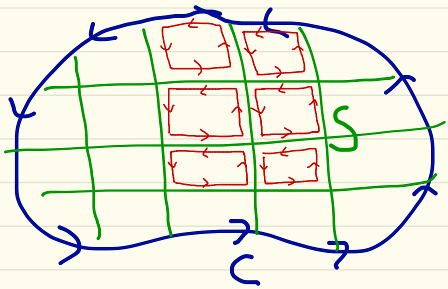


- S <u>any</u> open surface w/ C as its opening
- $d\vec{A}$ + direction of C correlated by RH rule.

Proof of Stokes's Theorem

divide $S = \sum_i \delta S_i$

$\Downarrow$



$$\int_S (\vec{\nabla}\times\vec{F})\cdot d\vec{A} = \sum_i \int_{\delta S_i} (\vec{\nabla}\times\vec{F})\cdot d\vec{A}(i) \approx \sum_i (\vec{\nabla}\times\vec{F})\cdot \delta\vec{A}(i) = \sum_i \oint_{\delta C(i)} \vec{F}\cdot d\vec{x}$$

(shared boundaries cancel, see fig.)
Only outer boundary remains

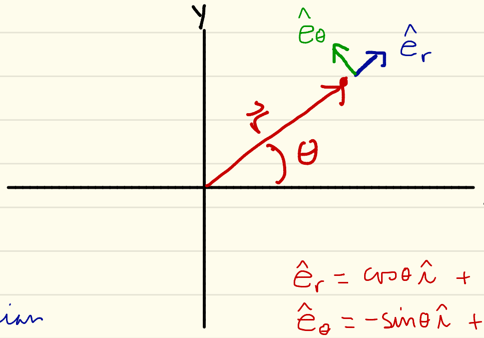
$$\Rightarrow \int_S (\vec{\nabla}\times\vec{F})\cdot d\vec{A} = \oint_C \vec{F}\cdot d\vec{x} \quad \underline{\text{QED}}.$$

Vector Calculus in Curvilinear Coordinates

(plane-polar, spherical, cylindrical coords.)

Example: $\vec{r}(t)$, $\vec{v}(t)$, $\vec{a}(t)$ in polar coords

$x = r\cos\theta$
$y = r\sin\theta$



$\hat{e}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$
$\hat{e}_\theta = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$

* Subtle point compared to cartesian coords: $\hat{e}_r = \hat{e}_r(\theta)$, $\hat{e}_\theta = \hat{e}_\theta(\theta)$

$$\therefore \vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\,\hat{e}_r) = \dot{r}\,\hat{e}_r + r\frac{d}{dt}\hat{e}_r\,)$$

$$\frac{d}{dt}\hat{e}_r = \frac{d\theta}{dt}\frac{d}{d\theta}\hat{e}_r = \dot{\theta}\,\hat{e}_\theta$$

$$\Longrightarrow \boxed{\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta}$$

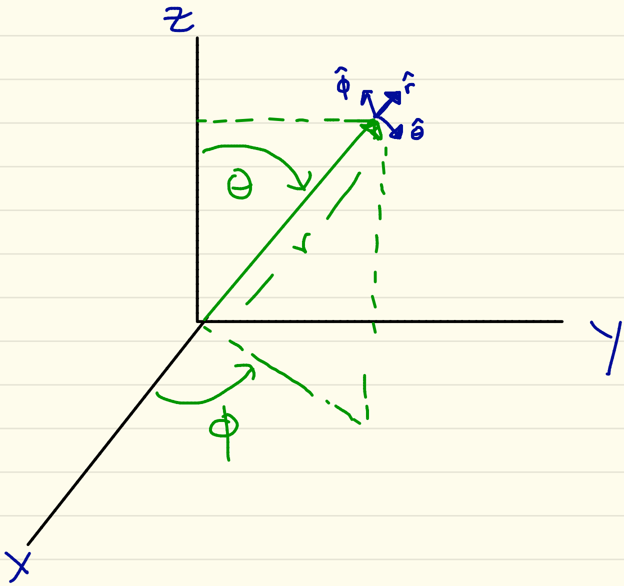
$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \ddot{r}\,\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta - r\dot{\theta}^2\,\hat{e}_r \qquad \left(\frac{d}{dt}\hat{e}_\theta = \dot{\theta}\frac{d}{d\theta}\hat{e}_\theta = -\dot{\theta}\,\hat{e}_r\right)$$

$$\boxed{\Rightarrow \vec{a} = (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{e}_\theta}$$

* also, $d\vec{r}$ (aka $d\vec{x}$ or $d\vec{s}$ in book) is $\boxed{d\vec{r} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta}$

Spherical Coords

$$\boxed{\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}}$$



$(0 \leq r \leq \infty,\ 0 \leq \phi \leq 2\pi,\ 0 \leq \theta \leq \pi)$

$\hat{e}_r \perp \hat{e}_\theta \perp \hat{e}_\phi$ at each point,

$\Downarrow$

$$\vec{A} = A_r \hat{e}_r + A_\theta \hat{e}_\theta + A_\phi \hat{e}_\phi$$

$$d\vec{r} = dr\,\hat{e}_r + r\,d\theta\,\hat{e}_\theta + r\sin\theta\,d\phi\,\hat{e}_\phi$$

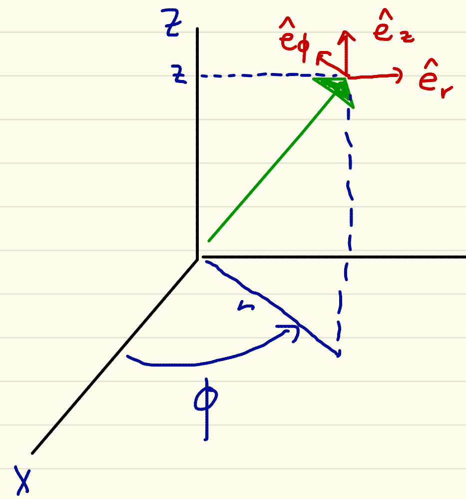
and

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta + r\sin\theta\,\dot{\phi}\,\hat{e}_\phi$$

$$d^3x = r^2 dr\,d\phi\,\sin\theta\,d\theta$$

Cylindrical Coords.

$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = z$$



$$\vec{r} = r\hat{e}_r + z\hat{e}_z$$

$$d\vec{r} = dr\,\hat{e}_r + r\,d\phi\,\hat{e}_\phi + dz\,\hat{e}_z$$
$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\phi}\,\hat{e}_\phi + \dot{z}\,\hat{e}_z$$

$$d^3\vec{r} = r\,dr\,d\phi\,dz$$

Translating $\vec{\nabla} f$, $\vec{\nabla}\cdot\vec{F}$, $\vec{\nabla}\times\vec{F}$ to non-cartesian coords.

* Any orthogonal coord. system w/coords u_1, u_2, u_3, we can write

$$\boxed{\begin{aligned} d\vec{r} &= h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3 \\ \text{"Scale factors"} \quad h_i &= \frac{dr_i}{du_i} \\ d^3 r &= h_1 h_2 h_3 \, du_1 du_2 du_3 \end{aligned}}$$

* **Claim** (see any good Calculus book or E&M book)

$$(\vec{\nabla} f)_i = \frac{1}{h_i}\frac{\partial f}{\partial u_i}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(F_1 h_2 h_3) + \frac{\partial}{\partial u_2}(F_2 h_3 h_1) + \frac{\partial}{\partial u_3}(F_3 h_1 h_2)\right]$$

$$(\vec{\nabla}\times\vec{F})_i = \frac{1}{h_j h_k}\left[\frac{\partial}{\partial u_j}(F_k h_k) - \frac{\partial}{\partial u_k}(F_j h_j)\right] \qquad ijk = 123, 312, 231$$

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3} \sum_{\substack{(ijk) \\ \text{cyclic}}} \frac{\partial}{\partial u_i}\left(\frac{h_j h_k}{h_i}\frac{\partial f}{\partial u_i}\right)$$

<u>ex</u>: spherical $u_1 = r$, $u_2 = \theta$, $u_3 = \phi$ $\Rightarrow h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$

cylindrical $u_1 = r$, $u_2 = \phi$, $u_3 = z$ $\Rightarrow h_1 = 1$, $h_2 = r$, $h_3 = 1$