Energy Conservation Recap

$$\boxed{W_{12} = \int_1^2 \vec{F}\cdot d\vec{r} = \int_1^2 d\left(\tfrac{1}{2}mv^2\right) = T_2 - T_1 \qquad \left(T = \tfrac{1}{2}mv^2\right)}$$

(always true, indep. of type of force $\vec{F}$)

* If $\vec{F} = -\vec{\nabla}U(\vec{r},t)$ (*):

$$\boxed{W_{12} = T_2 - T_1 = -\int_1^2 dU = U_1 - U_2}$$

(*) Necessary + Sufficient condition is $\vec{\nabla}\times\vec{F} = 0$

and

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dU}{dt} = \overbrace{\cancel{(\vec{F} + \vec{\nabla}U)}}^{0}\cdot\dot{\vec{r}} + \frac{\partial U}{\partial t}$$

$$\Rightarrow \boxed{\frac{dE}{dt} = \frac{\partial U}{\partial t}}$$

* Finally, if $\vec{F} = U(\vec{r})$:

$$\boxed{\frac{d}{dt}E = 0 \Rightarrow E = T + U = \text{constant}}$$

"Conservative force $\vec{F}$"

Non-conservative forces? e.g., $\vec{F} = \vec{F}_c + \vec{F}_{NC}$ where $\vec{F}_c = -\vec{\nabla}U^c$, $\vec{F}_{NC} \neq -\vec{\nabla}U$

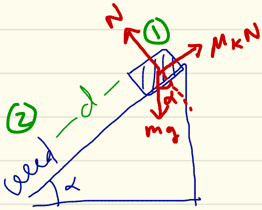
$$\Rightarrow W_{12} = T_2 - T_1 = \int_1^2 \vec{F}_c\cdot d\vec{r} + \int_1^2 \vec{F}_{NC}\cdot d\vec{r}$$

$$= U_1^c - U_2^c + W_{12}^{NC}$$

$$\Rightarrow \Delta T + \Delta U = W_{12}^{NC}$$

Example (HW2):

find v_2:



$$W_{12} = T_2 - \cancel{T_1}^{\,0} = \int_1^2 (\vec{F}_g + \vec{F}_{fric}) \cdot d\vec{r}$$

Conservative   Not conservative

$$\therefore \quad \frac{1}{2} m v_2^2 = U_1^g - \cancel{U_2^g}^{\,0} + W_{12}^{fric}$$

$$= mgd\sin\alpha - \mu_K N d$$

$$\frac{1}{2} \cancel{m} v_2^2 = \cancel{m} g d \sin\alpha - \mu_K \cancel{m} g \cos\alpha \, d$$

$$\Rightarrow v_2 = \sqrt{2gd\left[\sin\alpha - \mu_K \cos\alpha\right]}$$

Dynamics of a System of Particles (Reading: Ch. 9)

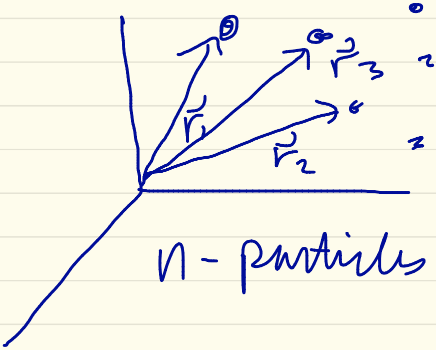
* Exact solutions of the eqns. of motion can almost always be done (sometimes analytically, other times numerically) for $n=1$ or 2 bodies

* Beyond this, problems can only (typically) be solved on a computer. The complexity/computational cost (in CM as well as QM) looks like

* large n => statistical methods simplify the problem
* small n => Analytic soln's or "cheap" numerical soln's
* medium n => HARD

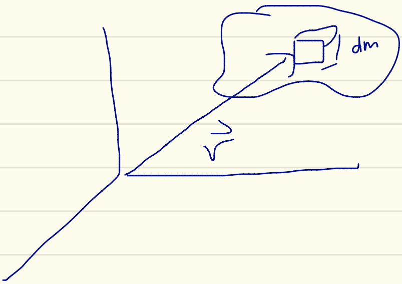
Nevertheless, we can often times make general statements about a system of n-bodies by appealing to conservation laws, etc., even if the direct solution of the eqns. of motion is impossible/impractical.

Center of Mass & total $\vec{P}$ of a System



$$\boxed{*\text{let } M = \sum_{\alpha=1}^{n} m_\alpha \quad (\text{total mass}) \qquad \text{and} \qquad \vec{R} \equiv \frac{\sum_\alpha m_\alpha \vec{r}_\alpha}{M} \qquad \text{"Center of Mass"}}$$

For a continuous distribution of mass



$$\boxed{\vec{R} = \frac{\int \vec{r}\, dm}{\int dm}}$$

density $\rho(\vec{r})$: $\qquad dm = \rho(\vec{r})\, d^3r$

$$\boxed{\Rightarrow \vec{R} = \frac{\int \vec{r}\rho(\vec{r})\, d^3r}{\int \rho(\vec{r})\, d^3r}}$$

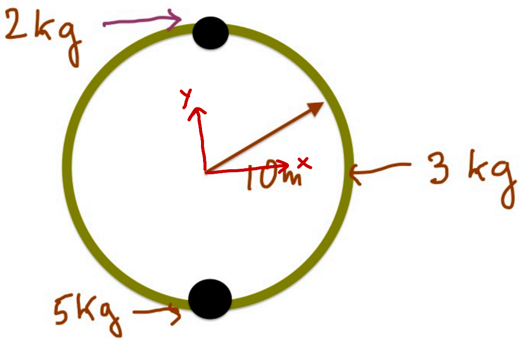
* Obvious generalization for 2d (i.e. surface) or 1d (linear) continuous mass distributions

2d: $\sigma(\vec{r})\,dA = dm \Rightarrow \vec{R} = \dfrac{\int \vec{r}\sigma(\vec{r})\,dA}{\int \sigma(\vec{r})\,dA}$

1d: $\lambda(\vec{r})\,d\ell = dm \Rightarrow \vec{R} = \dfrac{\int \vec{r}\lambda(\vec{r})\,d\ell}{\int \lambda(\vec{r})\,d\ell}$

Clicker Question

A THIN UNIFORM RING OF RADIUS 10m AND MASS 3 kg HAS TWO POINT MASSES OF 2kg AND 5 kg ATTACHED ON THE OPPOSITE SIDES OF ITS CIRCUMFERENCE. HOW FAR IS THE CM OF THE SYSTEM AWAY FROM THE CENTER OF THE RING?



A) 0.0m

B) 2.0m

C) 3.0m

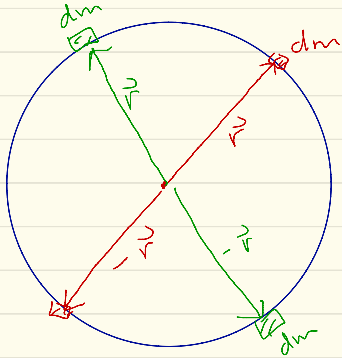
D) 4.3m

E) 7.0m

Solution:

$$\vec{R} = \frac{\sum_{\alpha=1}^{2} \vec{r}_\alpha m_\alpha}{\sum_\alpha m_\alpha} + \int_{ring} \vec{r}\, dm \Big/ M_{ring}$$

For the ring



$\vec{r}\,dm - \vec{r}\,dm = 0$

(cancel pairwise)

$\Rightarrow \int_{ring} \vec{r}\, dm = 0$ by circular symmetry

$$\therefore \vec{R} = \frac{\vec{r}_1 m_1 + \vec{r}_2 m_2}{m_1 + m_2} = \frac{(2\text{kg})(10\text{m})\hat{j} + (5\text{kg})(-10\text{m})\hat{j}}{7\text{ kg}} = -\frac{30}{7}\hat{j}$$

$\therefore R \simeq 4.3$ m

* Linear Momentum of the system (n-bodies)

* let $\vec{f}_{\alpha\beta}$ = force on particle α by particle β

$$\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha} \quad \text{(Newton's 3rd Law)}$$

* let $\vec{F}_\alpha$ = total force on particle α

$$= \vec{F}_\alpha^{ext} + \vec{f}_\alpha$$

external force originating from outside the n-body system

sum of forces from all the others (n-1) bodies of the system

$$\vec{f}_\alpha = \sum_\beta \vec{f}_{\alpha\beta}$$

* Newton's 2nd =>

$$\vec{F}_\alpha = \frac{d^2}{dt^2}(m_\alpha \vec{r}_\alpha) = \vec{F}_\alpha^{ext} + \sum_\beta \vec{f}_{\alpha\beta}$$

$$\therefore \sum_\alpha \vec{F}_\alpha = \sum_\alpha \vec{F}_\alpha^{ext} + \sum_{\substack{\alpha,\beta \\ (\alpha\neq\beta)}} \vec{f}_{\alpha\beta}$$

* but $\displaystyle\sum_{\substack{\alpha,\beta \\ (\alpha\neq\beta)}} \vec{f}_{\alpha\beta} = \sum_{\alpha<\beta} (\vec{f}_{\alpha\beta} + \vec{f}_{\beta\alpha}) = 0$ (Newton's 3rd => $\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha}$)

$$\Rightarrow \frac{d^2}{dt^2} \sum_\alpha m_\alpha \vec{r}_\alpha = \sum_\alpha \vec{F}_\alpha^{ext} \equiv \vec{F}$$

* but $\sum_\alpha m_\alpha \vec{r}_\alpha = M\vec{R}$ $\Rightarrow$ $\boxed{M\ddot{\vec{R}} = \vec{F}}$ **

So then, provided the internal forces obey Newton's 3rd Law ($\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha}$)

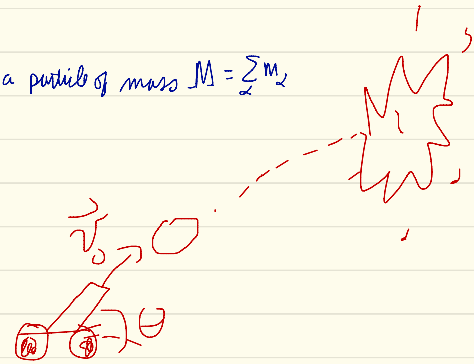
$$M\ddot{\vec{R}} = \sum_\alpha \vec{F}_\alpha^{ext} \equiv \vec{F}$$

df $\vec{P} = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha = \sum_\alpha \vec{p}_\alpha \Rightarrow \vec{F} = \dot{\vec{P}} = M\ddot{\vec{R}}$

$$\Downarrow$$

1.) COM moves as if it were a particle of mass $M = \sum_\alpha m_\alpha$

2.) Total $\vec{P}$ = const if $\vec{F} = 0$



* Example: Exploding artillery shell

$M\ddot{\vec{R}} = \sum_\alpha m_\alpha \vec{g} = M\vec{g}$ $\Rightarrow$ same as a point mass (see, e.g., ex 2.6)

$\Rightarrow$ COM $\vec{R}$ follows parabolic traj. just as in ex 2.6.