* Recap from last time

Center of Mass (CM) of n-body system $\boxed{\vec{R} = \dfrac{\sum_{\alpha=1}^{n} m_\alpha \vec{r}_\alpha}{\sum_\alpha m_\alpha}}$

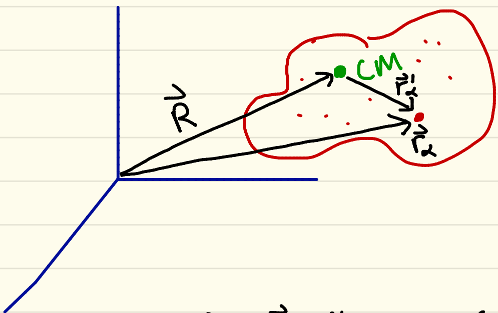
* CM has simple dynamics: $M\ddot{\vec{R}} = \dfrac{d\vec{P}}{dt} = \vec{F}^{ext}$ w/ $\vec{P} = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha$ (assuming Newton's 3rd Law for internal forces $\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha}$)

(internal forces play no role)

$\Rightarrow$ CM of n-body system moves like a point particle of $M = \sum_\alpha m_\alpha$ and position vector $\vec{R}$ under the action of $\vec{F} = \vec{F}^{ext} = \sum_\alpha \vec{F}^{ext}_\alpha$

$\Rightarrow \vec{P} = \sum_\alpha \vec{p}_\alpha = \text{const}$ if $\vec{F}^{ext} = 0$

* Coordinates with respect to the CM



* let $\vec{r}_\alpha = \vec{R} + \vec{r}\,'_\alpha$ (i.e., $\vec{r}\,'_\alpha$ = position vector of m_α w.r.t. the CM)

* Some Simplification for coords. in the CM

$$\cancel{M}\vec{R} = \sum_\alpha m_\alpha \vec{r}_\alpha = \sum_\alpha m_\alpha(\vec{R}+\vec{r}_\alpha{}') = \cancel{M\vec{R}} + \sum_\alpha m_\alpha \vec{r}_\alpha{}'$$

$$\boxed{\Rightarrow \sum_\alpha m_\alpha \vec{r}_\alpha{}' = 0}$$

"obvious" since $\vec{R}' = \dfrac{\sum_\alpha m_\alpha \vec{r}_\alpha{}'}{M}$ is just the CM wrt the CM!

* momentum wrt CM:

$$\dot{\vec{r}}_\alpha = \dot{\vec{R}} + \dot{\vec{r}}_\alpha{}'$$

$$\Downarrow$$

$$\vec{p}_\alpha = m_\alpha \dot{\vec{r}}_\alpha = m_\alpha \dot{\vec{R}} + m_\alpha \dot{\vec{r}}_\alpha{}'$$

$$\boxed{\Rightarrow \vec{p}_\alpha = m_\alpha \vec{V} + \vec{p}_\alpha{}'}$$

* also, note that

$$\boxed{\sum_\alpha \vec{p}_\alpha{}' = \frac{d}{dt}\Big(\cancelto{0}{\sum_\alpha m_\alpha \vec{r}_\alpha{}'}\Big) = 0}$$

* Total Angular Momentum of n-body System

$$\vec{L} \equiv \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha = \sum_\alpha (\vec{R}+\vec{r}_\alpha{}') \times (m_\alpha \vec{V} + \vec{p}_\alpha{}')$$

$$= \vec{R} \times \Big(\sum_\alpha m_\alpha\Big)\vec{V} + \vec{R} \times \cancelto{0}{\sum_\alpha \vec{p}_\alpha{}'} + \Big(\cancelto{0}{\sum_\alpha m_\alpha \vec{r}_\alpha{}'}\Big) \times \vec{V} + \sum_\alpha \vec{r}_\alpha{}' \times \vec{p}_\alpha{}'$$

$$\boxed{\Rightarrow \vec{L} = \underbrace{\vec{R}\times\vec{P}}_{\text{CM rotation about origin}} + \underbrace{\sum_\alpha \vec{r}_\alpha{}' \times \vec{p}_\alpha{}'}_{\text{rotation about the CM}}}$$

e.g.: $\vec{L}_{earth} = \vec{R}\times\vec{P} + \sum_\alpha \vec{r}_\alpha{}'\times\vec{p}_\alpha{}' \equiv \vec{L}_{cm} + \vec{L}'$

rot. around Sun

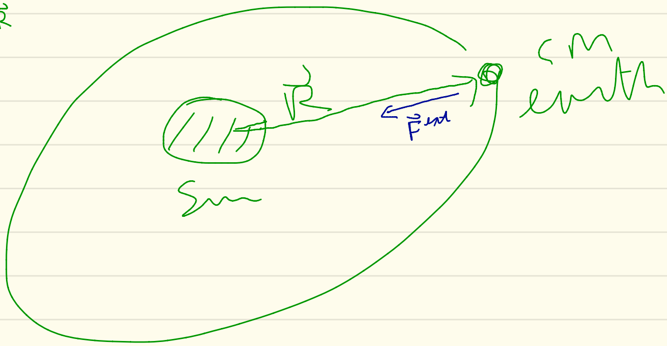
rot around its own axis

* Dynamics of $\vec{L}$ for the system

$$\frac{d\vec{L}}{dt} = \frac{d\vec{L}_{cm}}{dt} + \frac{d\vec{L}'}{dt}$$

$$\frac{d}{dt}(\vec{R}\times\vec{P}) = \cancel{\dot{\vec{R}}\times\vec{P}} + \vec{R}\times\dot{\vec{P}} = \vec{R}\times\vec{F}^{ext}$$

0 since $\vec{P} = M\dot{\vec{R}}$



$\Rightarrow \dot{\vec{L}}_{cm} = 0$ for earth

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\sum_\alpha \vec{r}_\alpha\times\vec{p}_\alpha = \sum_\alpha \vec{r}_\alpha\times\vec{F}_\alpha = \sum_\alpha \vec{N}_\alpha \equiv \vec{N} \quad \text{(Net torque)}$$

* let $\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}'$, $\vec{p}_\alpha = m_\alpha\vec{V} + \vec{p}_\alpha{}'$

$$\dot{\vec{L}} = \sum_\alpha(\vec{R}+\vec{r}_\alpha{}')\times\frac{d}{dt}(m_\alpha\vec{V} + \vec{p}_\alpha{}') = \vec{R}\times\Big(\underbrace{\sum_\alpha m_\alpha}_{M}\Big)\dot{\vec{V}} + \vec{R}\times\frac{d}{dt}\Big(\underbrace{\sum_\alpha \vec{p}_\alpha{}'}_{0}\Big) + \Big(\underbrace{\sum_\alpha m_\alpha\vec{r}_\alpha{}'}_{0}\Big)\times\dot{\vec{V}} + \sum_\alpha \vec{r}_\alpha{}'\times\dot{\vec{p}}_\alpha{}'$$

$$= \vec{R}\times\dot{\vec{P}} + \sum_\alpha \vec{r}_\alpha{}'\times\dot{\vec{p}}_\alpha{}'$$

$$\Rightarrow \quad \frac{d\vec{L}}{dt} = \vec{R}\times\dot{\vec{P}} + \sum_\alpha \vec{r}\,'_\alpha \times \dot{\vec{p}}\,'_\alpha$$

$$= \underbrace{\vec{R}\times\vec{F}^{ext}}_{\text{torque on CM about origin}} + \underbrace{\sum_\alpha \vec{r}\,'_\alpha\times\vec{F}'_\alpha}_{\text{torque wrt CM}} = \vec{N} \equiv \vec{N}_{cm} + \vec{N}'$$

$$\boxed{\begin{aligned} &\Rightarrow \frac{d\vec{L}_{cm}}{dt} = \vec{N}_{cm} = \vec{R}\times\vec{F}^{ext} \\ &\frac{d\vec{L}'}{dt} = \vec{N}' = \sum_\alpha \vec{r}\,'_\alpha\times\vec{F}'_\alpha = \sum_\alpha \vec{r}\,'_\alpha\times\vec{F}_\alpha \end{aligned}}$$

$$\sum_\alpha \vec{r}\,'_\alpha\times\vec{F}_\alpha = \sum_\alpha \vec{r}\,'_\alpha\times\dot{\vec{p}}_\alpha = \sum_\alpha \vec{r}\,'_\alpha\times(m_\alpha\dot{\vec{V}} + \dot{\vec{p}}\,'_\alpha)$$

$$= \Big(\sum_\alpha \cancel{m_\alpha \vec{r}\,'_\alpha}^{\,0}\Big)\times\dot{\vec{V}} + \sum_\alpha \vec{r}\,'_\alpha\times\dot{\vec{p}}\,'_\alpha$$

*** Simplify $\dot{\vec{L}}'$:**

$$\sum_\alpha \vec{r}\,'_\alpha\times\vec{F}_\alpha = \sum_\alpha \vec{r}\,'_\alpha\times\vec{F}^{ext}_\alpha + \sum_{\substack{\alpha\beta \\ (\alpha\neq\beta)}} \vec{r}\,'_\alpha\times\vec{f}_{\alpha\beta}$$

torque about CM coming from F ext

torque about CM from int

$$\Rightarrow \vec{N}' = \vec{N}'_{ext} + \vec{N}'_{int}$$

*** Massage $\vec{N}'_{int}$:**

$$\sum_{\alpha\beta} \vec{r}_\alpha\times\vec{f}_{\alpha\beta} = \frac{1}{2}\sum_{\alpha\beta}\left(\vec{r}_\alpha\times\vec{f}_{\alpha\beta} + \vec{r}_\beta\times\vec{f}_{\beta\alpha}\right)$$

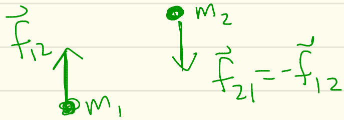
(i.e., swapped dummy indices α,β)

$$= \frac{1}{2}\sum_{\alpha\beta}(\vec{r}_\alpha - \vec{r}_\beta)\times\vec{f}_{\alpha\beta}$$

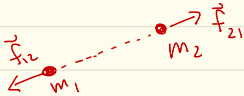
($\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha}$ Newton's 3rd)

Digression: $\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha}$ called the "weak form" of Newton's 3rd Law

e.g.:



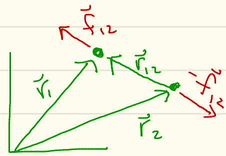
"Strong form" of 3rd Law $\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha}$ AND



"Central Forces"

* If we now assume $\vec{f}_{\alpha\beta}$ are central forces (gravity, electrostatic, elastic (springs))

$$\Rightarrow \vec{N}'_{int} = \frac{1}{2}\sum_{\alpha\beta}\frac{1}{2}(\vec{r}_\alpha - \vec{r}_\beta)\times\vec{f}_{\alpha\beta} = 0$$

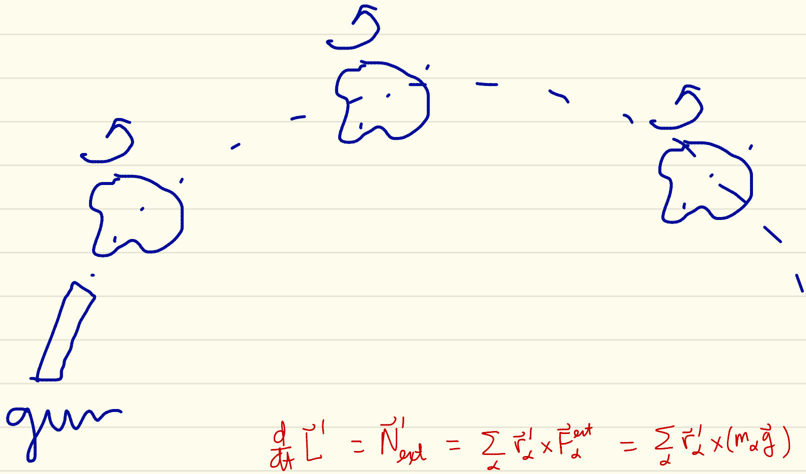


$$\frac{d\vec{L}}{dt} = \frac{d\vec{L}_{cm}}{dt} + \frac{d\vec{L}'}{dt} = \vec{N} = \vec{N}_{cm} + \vec{N}'$$

$$\frac{d\vec{L}_{cm}}{dt} = \vec{N}_{cm} = \vec{R}\times\vec{F}^{ext}$$

$$\frac{d\vec{L}'}{dt} = \vec{N}' = \vec{N}'_{ext} + \vec{N}'_{int} = \sum_\alpha \vec{r}'_\alpha\times\vec{F}^{ext}_\alpha + \underbrace{\cancel{\frac{1}{2}\sum_{\substack{\alpha\beta \\ \alpha\neq\beta}}(\vec{r}_\alpha - \vec{r}_\beta)\times\vec{f}_{\alpha\beta}}}_{0 \text{ central forces}}$$

Ex: Uniform gravity $\vec{F}_\alpha^{ext} = m_\alpha \vec{g}$
w/ central internal forces

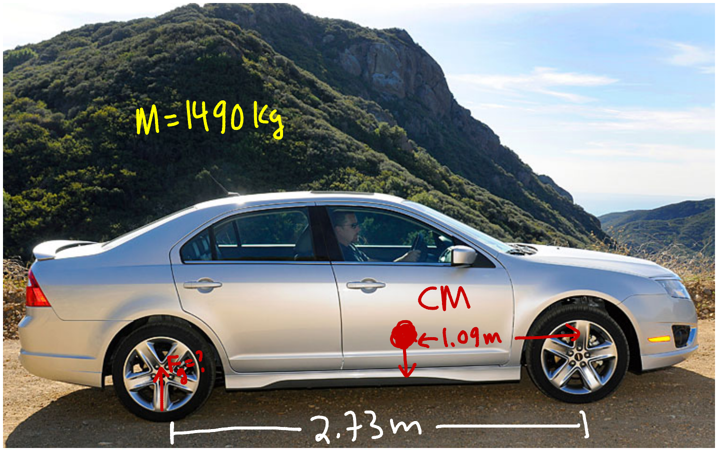


$$\frac{d}{dt}\vec{L}' = \vec{N}'_{ext} = \sum_\alpha \vec{r}'_\alpha \times \vec{F}_\alpha^{ext} = \sum_\alpha \vec{r}'_\alpha \times (m_\alpha \vec{g})$$

$$= \Big(\underbrace{\sum_\alpha m_\alpha \vec{r}'_\alpha}_{0}\Big) \times \vec{g}$$

$\Rightarrow \vec{L}' = \text{const.}$

FORD FUSION HAS A MASS OF 1490 kg AND A WHEELBASE OF 2.73 m. ITS CENTER OF MASS IS LOCATED 1.09 m BEHIND THE FRONT AXLE. FIND THE NET FORCE EXERTED BY THE GROUND ON THE REAR WHEELS, WHEN THE CAR STANDS ON HORIZONTAL GROUND.



A) 595 N

B) 895 N

C) 5840 N

D) 8780 N

E) 14620 N

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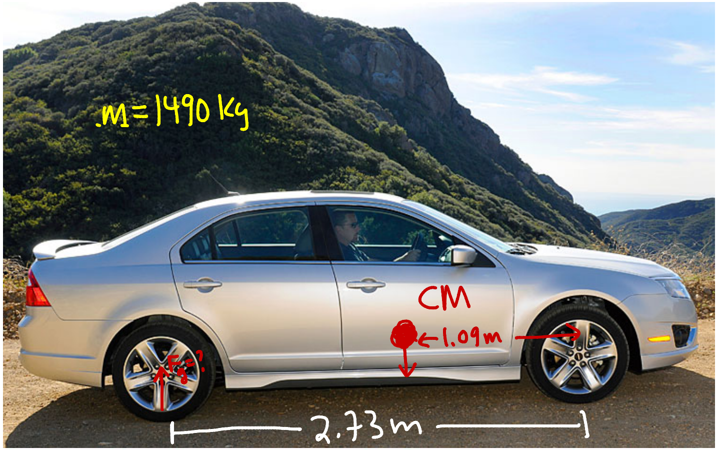

A) 595 N

B) 895 N

C) 5840 N

D) 8780 N

E) 14620 N

Sol'n: Does the car have $\frac{d\vec{L}}{dt} \neq 0$?

No.

$\therefore \vec{N} = 0$

$= Mg(1.09m) - F_{gr}(2.73m)$

$\approx (1500)(10)(1) - F_{gr}(3.0)$

$\Rightarrow F_{gr} \simeq \frac{15000}{3} \simeq 5000\,N$

(Sorry, I should've chosen nice round #'s!)