Reminders: 1) HW3 due Monday 2/4
2) Midterm #1 on Friday 2/8 (Ch 2, 9 ... probably NOT section 9.9)
3) Read Ch. 9

Ang. Momentum Recap from last lecture

$$\vec{L} = \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha = \sum_\alpha \vec{L}_\alpha$$

$$= \underbrace{\vec{L}_{cm}}_{\text{CM Motion}} + \underbrace{\vec{L}'}_{\text{motion about the CM}}$$

$$\vec{L}_{cm} = \vec{R} \times \vec{P}$$

$$\vec{L}' = \sum_\alpha \vec{r}_\alpha' \times \vec{p}_\alpha'$$

$$\frac{d\vec{L}}{dt} = \vec{N} = \sum_\alpha \vec{r}_\alpha \times \dot{\vec{p}}_\alpha = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha$$

$$= \vec{N}_{cm} + \vec{N}'$$

$$\vec{N}_{cm} = \vec{R} \times \vec{F}^{ext}$$

$$\vec{N}' = \sum_\alpha \vec{r}_\alpha' \times \vec{F}_\alpha^{ext} + \cancel{\frac{1}{2}\sum_{\alpha\beta}(\vec{r}_\alpha' - \vec{r}_\beta') \times \vec{f}_{\alpha\beta}}$$

0 central forces

Uniform Gravity: $\vec{F}^{ext} = \sum_\alpha m_\alpha \vec{g} = M\vec{g}$

$$\therefore \vec{N}_{cm}^{grav} = \vec{R} \times M\vec{g}$$

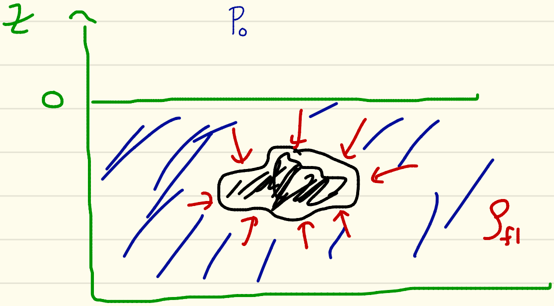
(acts as if applied at $\vec{R}$, hence the term "center of gravity")

$$\vec{N}'^{grav} = \left(\sum_\alpha m_\alpha \vec{r}_\alpha'\right) \times \vec{g} = 0$$

$$\therefore \frac{d\vec{L}'}{dt} = 0 \Rightarrow \vec{L}' = \text{const}$$

$\Rightarrow$ thrown object freely rotates about its CM.

*Another example of External Force on extended body: Buoyant Force



Archimedes Principle: Buoyant force = weight of displaced fluid applied @ CM of displaced fluid

pressure: $p(z) = P_0 - \rho_{fl} z g$

$$\vec{F}_{object} = -\int_S p\, d\vec{a}$$

pressure points in

outward normal of surface of the object

i^{th} component: $F_i = -\int_S p\, \hat{e}_i \cdot d\vec{a} = -\int_V \vec{\nabla}\cdot(p\hat{e}_i)\, d^3x$ (Gauss thm)

$$\vec{\nabla}\cdot(p\hat{e}_i) = (\vec{\nabla}p)\cdot\hat{e}_i + p\left(\cancelto{0}{\vec{\nabla}\cdot\hat{e}_i}\right)$$

$$= (-\rho_{fl}\, g)\,\hat{e}_3\cdot\hat{e}_i = -\rho_{fl}\, g\, \delta_{3,i}$$

$$\Rightarrow F_i = \rho_{fl}\, g\, \delta_{3,i} \int_V d^3x = \rho_{fl} V g\, \delta_{3,i} = M_{\substack{\text{displaced}\\\text{fluid}}}\, g\, \delta_{3,i}$$ (upward, as expected)

Work + Energy for n-body system

* let $\{\vec{r}_\alpha^{(1)}\}$ = particle positions in state 1

$\{\vec{r}_\alpha^{(2)}\}$ = " " state 2

$$\Rightarrow W_{12} = \sum_\alpha \int_1^2 \vec{F}_\alpha \cdot d\vec{r}_\alpha = \sum_\alpha \int_1^2 d\left(\tfrac{1}{2} m_\alpha v_\alpha^2\right)$$ (using the same steps as the 1-particle case in ch 2)

$$\boxed{\therefore W_{12} = T_2 - T_1 \quad \text{where} \quad T = \sum_\alpha \tfrac{1}{2} m_\alpha v_\alpha^2}$$

* Use the relation between "lab" and "CM" coordinates

$$\vec{r}_\alpha = \vec{R} + \vec{r}_\alpha{}' \Rightarrow \vec{v}_\alpha = \vec{V} + \vec{v}_\alpha{}' \qquad (\vec{V} = \dot{\vec{R}} \text{ etc})$$

$$\therefore T = \sum_\alpha \tfrac{1}{2} m_\alpha v_\alpha^2 = \sum_\alpha \tfrac{1}{2} m_\alpha (\vec{V} + \vec{v}_\alpha{}') \cdot (\vec{V} + \vec{v}_\alpha{}')$$

$$= \tfrac{1}{2}\left(\sum_\alpha m_\alpha\right) V^2 + \left(\sum_\alpha m_\alpha \vec{v}_\alpha{}'\right) \cdot \vec{V} + \sum_\alpha \tfrac{1}{2} m_\alpha v_\alpha'^2$$

$$= \tfrac{1}{2} M V^2 + \left(\underbrace{\cancel{\sum_\alpha \vec{p}_\alpha{}'}}_{0 \text{ (see Lect 8)}}\right) \cdot \vec{V} + \sum_\alpha \tfrac{1}{2} m_\alpha v_\alpha'^2$$

$$\boxed{\Rightarrow T = \sum_\alpha \tfrac{1}{2} m_\alpha v_\alpha^2 = \underbrace{\tfrac{1}{2} M V^2}_{\text{CM motion}} + \underbrace{\sum_\alpha \tfrac{1}{2} m_\alpha v_\alpha'^2}_{\text{relative motion W.R.T. the CM}}}$$

* i.e., we see the same decomposition as we did for $\vec{L}$

* Following similar steps as in ch 2 (cons. of energy)

$$\vec{F}_\alpha = \vec{F}_\alpha^{ext} + \sum_{\beta \neq \alpha} \vec{f}_{\alpha\beta}$$

$$\Rightarrow \quad W_{12} = \sum_\alpha \int_1^2 \vec{F}_\alpha^{ext} \cdot d\vec{r}_\alpha + \sum_{\substack{\alpha,\beta \\ (\alpha \neq \beta)}} \int_1^2 \vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha$$

Assume: $\vec{F}_\alpha^{ext} = -\vec{\nabla}_\alpha U_\alpha$ and $\vec{f}_{\alpha\beta} = -\vec{\nabla}_\alpha \bar{U}_\alpha$

$$\boxed{\sum_\alpha \int_1^2 \vec{F}_\alpha^{ext} \cdot d\vec{r}_\alpha = -\sum_\alpha \int_1^2 dU_\alpha = -\sum_\alpha U_\alpha \Big|_1^2}$$

$$\sum_{\substack{\alpha\beta \\ \alpha \neq \beta}} \int_1^2 \vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha = \frac{1}{2} \sum_{\alpha\beta}{}' \left[\int \vec{f}_{\alpha\beta} \cdot d\vec{r}_\alpha + \int \vec{f}_{\beta\alpha} \cdot d\vec{r}_\beta \right]$$

* $\vec{f}_{\beta\alpha} = -\vec{f}_{\alpha\beta}$ (2nd Law)

$$= \frac{1}{2} \sum_{\alpha\beta}{}' \int \vec{f}_{\alpha\beta} \cdot d\vec{r}_{\alpha\beta}$$

$[\, d\vec{r}_{\alpha\beta} \equiv d(\vec{r}_\alpha - \vec{r}_\beta) \,]$

$$\therefore \frac{1}{2} \sum_{\alpha\beta}{}' \int \vec{f}_{\alpha\beta} \cdot d\vec{r}_{\alpha\beta} = -\frac{1}{2} \sum_{\alpha\beta}{}' \int (\vec{\nabla}_\alpha \bar{U}_{\alpha\beta}) \cdot d\vec{r}_{\alpha\beta}$$

* Now, physically, $\bar{U}_{\alpha\beta} = \bar{U}_{\alpha\beta}(\vec{r}_\alpha, \vec{r}_\beta)$; (let $x_{\alpha,i}$ = i^{th} cart. component of $\vec{r}_\alpha$)

$$\therefore d\bar{U}_{\alpha\beta} = \sum_{i=1}^3 \frac{\partial \bar{U}_{\alpha\beta}}{\partial x_{\alpha,i}} dx_{\alpha,i} + \frac{\partial \bar{U}_{\alpha\beta}}{\partial x_{\beta,i}} dx_{\beta,i}$$

$$= (\vec{\nabla}_\alpha \bar{U}_{\alpha\beta}) \cdot d\vec{r}_\alpha + (\vec{\nabla}_\beta \bar{U}_{\alpha\beta}) \cdot d\vec{r}_\beta$$

* but $\vec{f}_{\alpha\beta} = -\vec{f}_{\beta\alpha} \Rightarrow -\vec{\nabla}_\alpha \bar{U}_{\alpha\beta} = \vec{\nabla}_\beta \bar{U}_{\beta\alpha}$

* Moreover, in general $\bar{U}_{\alpha\beta} = \bar{U}_{\beta\alpha}$

$$\therefore -\vec{\nabla}_\alpha \bar{U}_{\alpha\beta} = \vec{\nabla}_\beta \bar{U}_{\beta\alpha} = \vec{\nabla}_\beta \bar{U}_{\alpha\beta}$$

$$\Rightarrow d\bar{U}_{\alpha\beta} = (\vec{\nabla}_\alpha \bar{U}_{\alpha\beta}) \cdot (d\vec{r}_\alpha - d\vec{r}_\beta) \equiv (\vec{\nabla}_\alpha \bar{U}_{\alpha\beta}) \cdot d\vec{r}_{\alpha\beta}$$

Therefore, we have

$$\therefore \frac{1}{2}\sum_{\alpha\beta}{}' \int \vec{f}_{\alpha\beta}\cdot d\vec{r}_{\alpha\beta} = -\frac{1}{2}\sum_{\alpha\beta}{}' \int \left(\vec{\nabla}_\alpha \bar{U}_{\alpha\beta}\right)\cdot d\vec{r}_{\alpha\beta} = -\frac{1}{2}\sum_{\alpha\beta}{}' \int d\bar{U}_{\alpha\beta}$$

$$= -\frac{1}{2}\sum_{\alpha\beta}{}' \bar{U}_{\alpha\beta}\Big|_1^2$$

$$= -\sum_{\alpha<\beta} \bar{U}_{\alpha\beta}\Big|_1^2$$

So at long last we find!

$$\boxed{W_{12} = T_2 - T_1 = -\sum_\alpha U_\alpha\Big|_1^2 - \sum_{\alpha<\beta}\bar{U}_{\alpha\beta}\Big|_1^2}$$

df total (internal + external) Potential Energy

$$\boxed{U \equiv \sum_\alpha U_\alpha + \sum_{\alpha<\beta}\bar{U}_{\alpha\beta}}$$

$$\Rightarrow W_{12} = T_2 - T_1 = U_1 - U_2$$

$$\boxed{\begin{gathered}\therefore\ T_1 + U_1 = T_2 + U_2 \\ \text{or} \\ E_1 = E_2\end{gathered}}$$

energy conservation w/ conservative forces