S12 PHY321: Practice Final

Contextual information

Damped harmonic oscillator equation: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$

General solution: $x(t) = \mathrm{e}^{-\beta t}\left[A_1\,\exp\left(\sqrt{\beta^2-\omega_0^2}t\right) + A_2\,\exp\left(-\sqrt{\beta^2-\omega_0^2}t\right)\right]$

Driven harmonic oscillator equation: $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos\omega t$

Amplitude of stationary oscillations: $D = \frac{A}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\omega^2\beta^2}}$

Phase lag: $\delta = \tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2-\omega^2}\right)$

$G = 6.673\times 10^{-11}\,\frac{\mathrm{m}^3}{\mathrm{kg\,s}^2}$

Calculus in spherical coordinates:

$\vec{\mathrm{grad}}\,\Psi = \vec{\mathrm{e}}_r\,\frac{\partial\Psi}{\partial r} + \vec{\mathrm{e}}_\theta\,\frac{1}{r}\,\frac{\partial\Psi}{\partial\theta} + \vec{\mathrm{e}}_\phi\,\frac{1}{r\sin\theta}\,\frac{\partial\Psi}{\partial\phi}$

$\vec{\mathrm{div}}\,\vec{A} = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,A_r\right) + \frac{1}{r\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,A_\theta\right) + \frac{1}{r\sin\theta}\,\frac{\partial A_\phi}{\partial\phi}$

$\nabla^2\,\Psi = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2\Psi}{\partial\phi^2}$

Orbits in a gravitational field:

$$\frac{\alpha}{r} = 1 + \epsilon\cos\theta \qquad a = \frac{k}{2|E|} \qquad b = \frac{\ell}{\sqrt{2\mu|E|}} \qquad \alpha = \frac{\ell^2}{\mu k}$$

1. Consider the vector field $\vec{C} = (x + 2, y + 1, xy - 3)$.

   (a) [2 pts] Find div $\vec{C}$.

$$\frac{\partial C_x}{\partial x} + \frac{\partial C_y}{\partial y} + \frac{\partial C_z}{\partial z} = 1 + 1 + 0 = 2$$

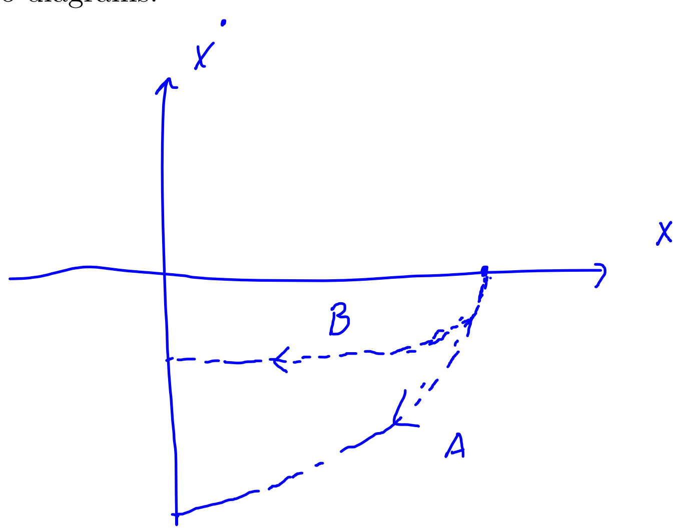
   (b) [2 pts] What is the magnitude of $\vec{C}$ at the location $(x, y, z) = (1, 3, -4)$?

$$\vec{C} = (3, 4, 0) \qquad C = \sqrt{3^2 + 4^2} = 5$$

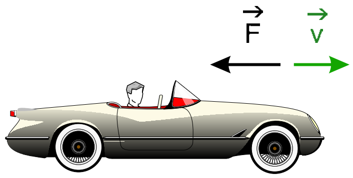
   (c) [2 pts] Determine the angle that the field $\vec{C}$ makes with the x-axis at the location $(x, y, z) = (1, 3, -4)$.

$$\cos\theta = \frac{C_x}{C} = \frac{3}{5} \qquad \theta = \cos^{-1}(0.6) = 53^\circ$$

2. [3 pts] Two objects A and B are simultaneously released from the same elevation H above the ground. Object A is heavy and compact and falls to the ground without being much affected by air resistance. Object B, on the other hand, is light and extended and it quickly approaches its terminal velocity when falling. Draw the phase diagrams for the motion of the objects until they reach the ground, marking which diagram is for which object and emphasizing similarities and differences between the two diagrams.



3. Suppose a coasting car of mass m is subject to a resistance from air and ground that is proportional to the *square* of velocity, $F = -a\,v^2$, where a is a constant.



(a) [4 pts] Find the dependence of the car's velocity v on time t, when the above force is acting. At time $t = 0$, the car's velocity is v_0.

$$m\frac{dv}{dt} = -av^2 \qquad \frac{m}{a}\frac{dv}{v^2} = -dt$$

$$-\frac{m}{a}\frac{1}{v}\Big|_{v_0}^{v} = -t\Big|_0^t \Rightarrow \frac{m}{a}\left(\frac{1}{v} - \frac{1}{v_0}\right) = t$$

$$\frac{1}{v} - \frac{1}{v_0} = \frac{at}{m} \qquad \frac{1}{v} = \frac{at}{m} + \frac{1}{v_0} \qquad v = \frac{1}{\frac{1}{v_0} + \frac{at}{m}}$$

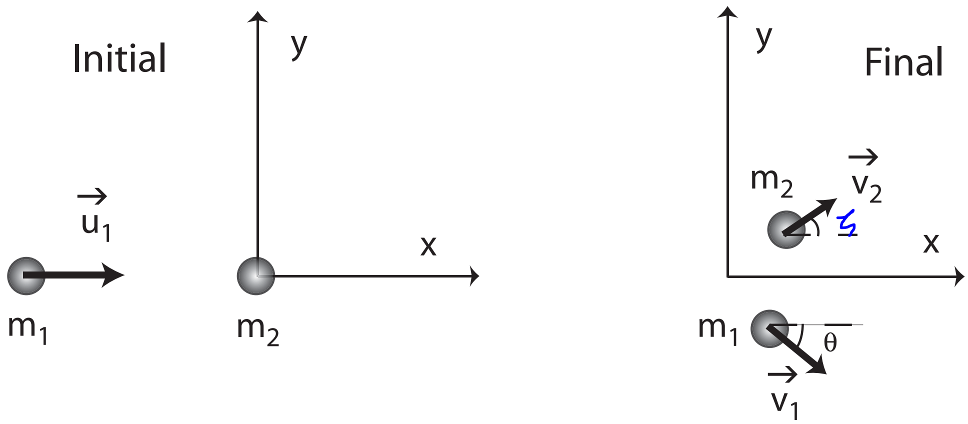
$$v = \frac{mv_0}{m + av_0 t}$$

(b) [3 pts] Determine the time after which the car's velocity drops to $\frac{1}{2}\,v_0$.

$$\frac{v_0}{2} = \frac{mv_0}{m + av_0 t} \qquad \frac{1}{2} = \frac{1}{1 + \frac{av_0 t}{m}}$$

$$\frac{av_0 t}{m} = 1 \Rightarrow t = \frac{m}{av_0}$$

4. [8 pts] A ball of mass $m_1 = 3.0\,\mathrm{kg}$ and velocity $u_1 = 10.0\,\mathrm{m/s}$ strikes a ball of mass $m_2 = 2.0\,\mathrm{m/s}$ at rest. After the collision the first ball (mass m_1) emerges at a velocity $v_1 = 6.0\,\mathrm{m/s}$ at an angle $\theta = 30.0°$ relative to the original direction. Find magnitude of the velocity v_2 of the second ball (mass m_2) after the collision. The collision is INELASTIC.



x: $3 \times 10 = 3 \times 6 \times \cos 30^\circ + 2\, v_2 \cos\zeta$

$$2 v_2 \cos\zeta = 30 - 18\frac{\sqrt{3}}{2} = 30 - 9\sqrt{3} = 3(10 - 3\sqrt{3}).$$

$$v_2 \cos\zeta = 7,21$$

y: $2 \times v_2 \sin\zeta = 3 \times 6 \sin 30^\circ = 9$

$$v_2 \sin\zeta = 4.5$$

$$v_2 = \sqrt{(7.21)^2 + (4.5)^2} = 8,50$$

5. A small object of mass $m = 6.5\,\mathrm{g}$ is suspended from a light spring with force constant $k = 4.4\,\mathrm{N/m}$.

   (a) [3 pts] If the object is pulled a short distance down and released, how many oscillations per second is it going to execute?

$$\omega_0 = \sqrt{\frac{k}{m}} \qquad \nu_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{4.4}{.0065}}$$

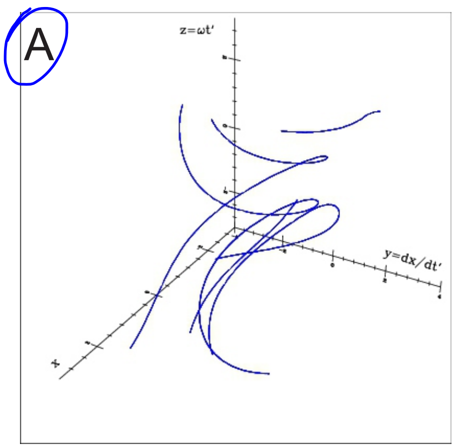
$$= 4.14\,s^{-1}$$

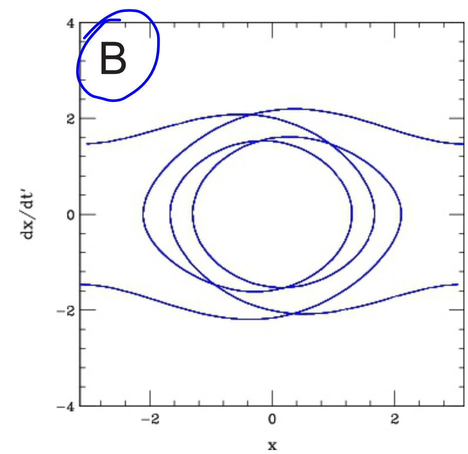
   (b) [2 pts] The system is next immersed in a viscous fluid that resists the motion of the object with a resistance force $\vec{F}_r = -b\,\vec{v}$, where $\vec{v}$ is the velocity of the object and $b = 0.34\,\mathrm{kg/s}$. Establish whether a vertically displaced and released object will be now executing a motion that is underdamped, overdamped or critically damped.

$$? \; \omega_0^2 - \beta^2 = \frac{k}{m} - \left(\frac{b}{2m}\right)^2 = \frac{4.4}{.0065} - \left(\frac{0.34}{2 \times .0065}\right)^2$$

$$= 677 - 684 < 0 \qquad \text{OVERDAMPED!}$$

6. [3 pts] We discussed chaos in nonlinear systems in the context of a driven pendulum. Which of the following plots (A-E) can be principally used to asses the maximal velocity for a driven pendulum?





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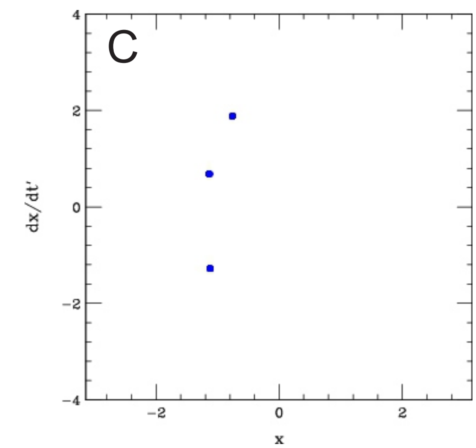

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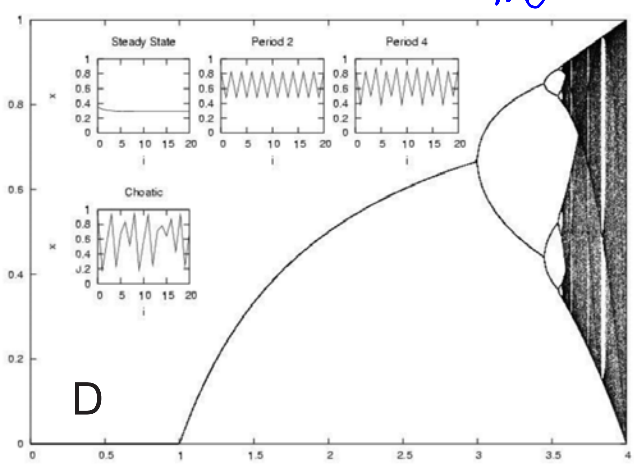
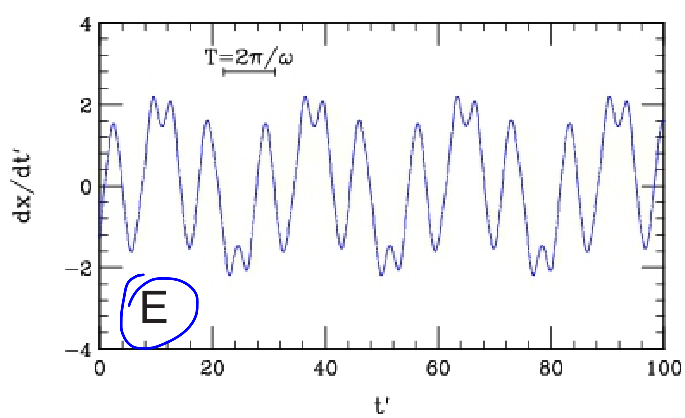




7. A spherical planet of radius R has a density that is largest at its center, and decreases with distance r from the center as $\rho(r) = B\,(R^2 - r^2)$, where B is a constant with the appropriate units.

(a) [3 pts] Determine the mass of the planet.

$$M = \int \rho\, dV = \int \rho\, 4\pi r^2 dr = 4\pi \int_0^R r^2 B(R^2 - r^2)\, dr$$

$$= 4\pi B \left[R^2 \tfrac{1}{3} r^3 - \tfrac{1}{5} r^5\right]_0^R = 4\pi B R^5 \left[\tfrac{1}{3} - \tfrac{1}{5}\right] = \frac{8\pi}{15} B R^5$$

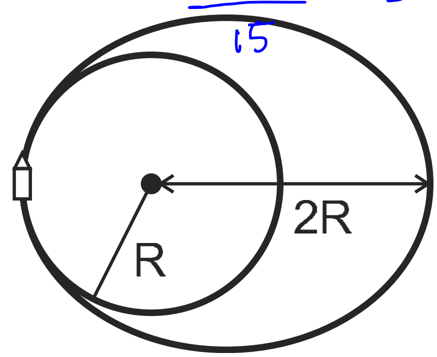
(b) [5 pts] Determine the gravitational field at distance $r < R$ from the center of the planet.

$$g = -\frac{G}{r^2} M_{inside} = -\frac{G}{r^2} 4\pi \int_0^r r'^2 B(R^2 - r'^2)\, dr'$$

$$= -\frac{G4\pi}{r^2} B\left[R^2 \tfrac{1}{3} r'^3 - \tfrac{1}{5} r'^5\right]_0^r = -4\pi G B r\left[\frac{R^2}{3} - \frac{r^2}{5}\right]$$

$$= -\frac{4\pi G B}{15} r\left[5R^2 - 3r^2\right]$$

8. [7 pts] A satellite moves on a circular orbit of radius R around Earth. At some instant the satellite's boosters are briefly activated changing then the magnitude, but not direction, of the satellite's velocity. But what factor should the satellite's velocity be changed by the boosters to put the satellite on an orbit with apogee $r_{\max} = 2R$? If you cannot find the value of the required factor, determine at least whether that factor needs to be larger or smaller than 1.



$$E_i = -\frac{k}{2R} = \frac{mv_i^2}{2} - \frac{k}{R}$$

$$\frac{mv_i^2}{2} = +\frac{k}{2R}$$

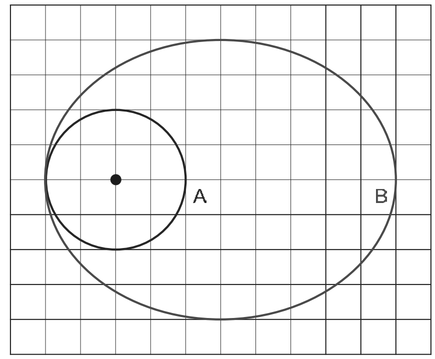
$$E_f = -\frac{k}{3R} = \frac{mv_f^2}{2} - \frac{k}{R}$$

$$\frac{mv_f^2}{2} = \frac{k}{R} - \frac{k}{3R} = \frac{2k}{3R}$$

$$v_f^2 = v_i^2 \frac{4}{3}$$

$$v_f = \frac{2}{\sqrt{3}} v_i$$

9. Consider the two possible orbits, one circular A and one elliptical B, drawn to scale, for the motion of a satellite of mass m around Earth. Exploit geometry of these orbits in the tasks that follow. As elsewhere, *no credit is given for unsupported answers.*



(a) [2 pts] Establish the eccentricities ϵ_A and ϵ_B for the orbits.

$\epsilon_A = 0$

$$r_{min} = a(1-\varepsilon) \qquad 2 = 5(1-\varepsilon) \Rightarrow 1-\varepsilon = \frac{2}{5}$$

$$\varepsilon_B = \frac{3}{5}$$

(b) [2 pts] Determine the ratio of energies E_B/E_A for the orbits.

$$\frac{E_B}{E_A} = \frac{a_A}{a_B} = \frac{2}{5}$$

(c) [1 pt] Determine the ratio of periods T_B/T_A.

$$\frac{T_B}{T_A} = \left(\frac{a_B}{a_A}\right)^{3/2} = \left(\frac{5}{2}\right)^{3/2}$$

(d) [4 pts] Determine the ratio of velocities v_B/v_A at the perigee ($r = r_{\min}$) of B. If you have difficulty determining the value of this ratio, establish at least whether this ratio is larger or smaller than 1.

$$T = E - U = -\frac{k}{2a} + \frac{k}{R} = \frac{k}{2}\left(\frac{2}{R} - \frac{1}{a}\right)$$

$$\frac{v_B}{v_A} = \left(\frac{T_B}{T_A}\right)^{1/2} = \left(\frac{\frac{2}{R} - \frac{1}{2.5R}}{\frac{2}{R} - \frac{1}{R}}\right)^{1/2} = \left(\frac{2 - \frac{1}{2.5}}{2-1}\right)^{1/2} = \left(2 - \frac{4}{10}\right)^{1/2}$$

$$= (1.6)^{1/2} = 1.26$$