

Practice Midterm Exam #1

Total points = 25. Show all of your work!

1. [6 points] If $\mathbf{A} = 5\mathbf{i}$ and $\mathbf{B} = 3\mathbf{i} + 4\mathbf{j}$ find

(a) [2] $\mathbf{A} \cdot \mathbf{B}$ $5\hat{i} \cdot (3\hat{i} + 4\hat{j}) = 15$

(b) [2] $\mathbf{A} \times \mathbf{B}$ $5\hat{i} \times (3\hat{i} + 4\hat{j}) = 20\hat{k}$

(c) [2] The angle θ_{AB} between \mathbf{A} and \mathbf{B} .

$$\cos \theta_{AB} = \frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}| |\mathbf{B}|} = \frac{15}{5 \cdot \sqrt{3^2 + 4^2}} = \frac{15}{25} = \frac{3}{5}$$

$$\theta_{AB} = \cos^{-1} 0.6 = 53^\circ$$

2. [7 points] Suppose that the frictional force on an object of mass m traveling through a fluid is proportional to the cube of the velocity: $F = -mkv^3$ where k is a constant (and m is included to make the math a bit easier).

(a) [4] Find the velocity as a function of time, assuming that the initial velocity is v_0 at time $t = 0$. Neglect gravity.

$$F = ma = m \frac{dv}{dt} \quad m \frac{dv}{dt} = -mkv^3 \quad \frac{dv}{dt} = -kv^3$$

$$\frac{dv}{v^3} = -k dt \quad \int_{v_0}^v \frac{dv}{v^3} = -k \int_0^t dt = -\frac{1}{2v^2} \Big|_{v_0}^v = -kt$$

(b) [3] After what time has the velocity slowed to half the initial velocity?

$$-\frac{1}{2v^2} + \frac{1}{2v_0^2} = -kt \quad \frac{1}{v^2} - \frac{1}{v_0^2} = 2kt$$

$$\frac{1}{v^2} = \frac{1}{v_0^2} + 2kt \quad \Rightarrow v = \frac{1}{\sqrt{\frac{1}{v_0^2} + 2kt}}$$

$$v = \frac{v_0}{2} \quad \left. \begin{array}{l} 2kt = \frac{3}{v_0^2} \\ t = \frac{3}{2kv_0^2} \end{array} \right\}$$

$$\frac{1}{\sqrt{\frac{1}{v_0^2} + 2kt}} = \frac{v_0}{2}$$

Note: There is another question on the next page!


$$\frac{1}{v_0^2} + 2kt = \frac{4}{v_0^2}$$

3. [12 points] An object of mass $m_0 = 30$ kg. is launched at time $t = 0$ with a horizontal velocity of 40.0 m/s. (There is initially no vertical component to the velocity.)

(a) [2] What is the kinetic energy of the object, K_i (in Joules)?

$$K = \frac{1}{2} m v_0^2 = \frac{1}{2} 30 \text{ kg} \left(40 \frac{\text{m}}{\text{s}} \right)^2 = 30 \times 40 \times 20 = 24000 \text{ J}$$

(b) [4] If the initial height of the object is $h = 1000$ m, what is the expected range, R (in meters), before it hits the ground? (Use the x origin as the point of launch, and use $g = 9.81$ m.s⁻².)



The diagram shows a dashed parabolic path of an object starting from a height h and landing at a horizontal distance R from the launch point. The path is symmetric about the vertical line through the launch point.

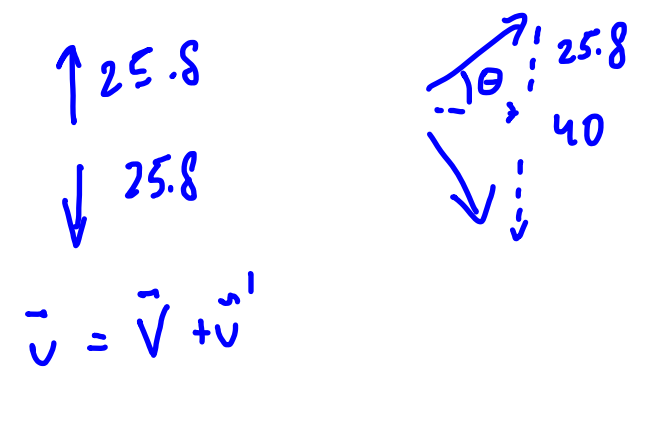
$$R = v_0 t \quad h = \frac{1}{2} g t^2 \quad t = \sqrt{\frac{2h}{g}}$$

$$R = v_0 \sqrt{\frac{2h}{g}} = 40 \frac{\text{m}}{\text{s}} \sqrt{\frac{2 \times 1000 \text{ m}}{9.81 \text{ m/s}^2}} = 571 \text{ m}$$

Unfortunately, immediately after the launch, the object explodes into two fragments (each of mass equal to one-half of the original object ($m_1 = m_2 = m_0/2 = 15$ kg) i.e. we are neglecting the mass of the explosive material). The explosion contributes an additional energy of $E_{\text{ex}} = 10.0$ kJ (10000 Joules). The two fragments are ejected at right angles to the original line of flight of the initial object i.e. vertically in the CM frame, fragment m_1 straight up and fragment m_2 straight down.

(c) [6] Immediately after the explosion, what is the velocity (magnitude and angle relative to the horizontal) of fragment m_1 relative to an observer on the ground?

$$\frac{1}{2} m_1 v'^2 = \frac{1}{2} E_{\text{ex}} \quad m_1 v'^2 = E_{\text{ex}} \quad v' = \sqrt{\frac{E_{\text{ex}}}{m_1}}$$

$$= \sqrt{\frac{10000 \text{ J}}{15 \text{ kg}}} = 25.8 \frac{\text{m}}{\text{s}}$$


The diagrams show the velocity vectors for fragment m_1 . On the left, a vertical vector pointing up is labeled 25.8 and a vertical vector pointing down is labeled 25.8. In the middle, a vector pointing up and to the right is labeled 25.8, and a vertical vector pointing down is labeled 40. The angle between the 25.8 vector and the horizontal is labeled θ . Below these, the vector equation $\vec{v} = \vec{v} + \vec{v}'$ is written.

$$v_y = v' = 25.8$$

$$v_x = v = 40$$

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{25.8^2 + 40^2} = 47.6$$

$$\tan \theta = \frac{v_y}{v_x} = \frac{25.8}{40}$$

$$\theta = \tan^{-1} (25.8/40) = 32.8^\circ$$