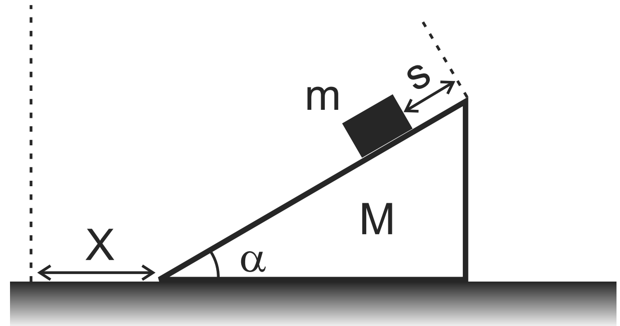
Solutions

PHY321 Homework Set 10

1. [5 pts] A small block of mass m slides without friction down a wedge-shaped block of mass M and of opening angle α. The triangular block itself slides along a horizontal floor, without friction. Your ultimate goal will be to find the horizontal acceleration $\ddot{X}$ of the triangular block, following the second and third Newton's laws. The cartesian coordinates for m are x (horizontal) and y (vertical).



   (a) Consider the forces acting on the mass m and the wedge block. Express the components of the acceleration $\ddot{x}$ and $\ddot{y}$ for m and $\ddot{X}$ for M in terms of the force components.

   (b) The system of equations you wrote cannot be closed without introducing an equation of constraint between x, y and X. Express the equation of constraint in terms of the angle α. Differentiate the equation to get an equation of constraint between the acceleration components.

   (c) Solve the system of equations for the acceleration components, to arrive at $\ddot{X}$.

2. [10 pts] Now solve the preceding problem following the Lagrangian method and using X and distance s traveled by m down the wedge as generalized coordinates.

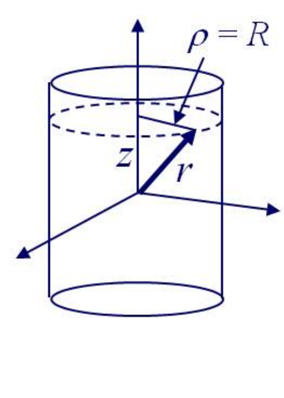
   (a) Express the total potential energy U of the block system in terms of the cartesian coordinates. Reexpress that energy in terms of the generalized coordinates. An arbitrary reference constant in the potential energy is of no relevance.

   (b) Express the total kinetic energy T of m and M in terms of the cartesian coordinates. Reexpress that energy in terms of the generalized coordinates.

   (c) Derive the Lagrange equations for the system. Manipulate them to obtain $\ddot{X}$. Does your result agree with that from the preceding problem?

   (d) Given that there is no external horizontal force component acting on the block system, the center of mass of that system should not accelerate in the horizontal direction. Demonstrate that the lack of acceleration for the center of mass follows from your Lagrange equations.

3. [10 pts] A particle of mass m is constrained to move on the cylindrical surface described in cylindrical coordinates (ρ, ϕ, z) by the constraint equation $\rho = R$. The *only* force acting on the particle is the central force directed towards the origin, $\vec{F} = -k\,\vec{r}$. Your task will be to obtain and solve the Lagrangian equations for the particle, with and without an explicit reference to the constraint equation.



(a) At first introduce the constraint inherently and express the Lagrangian for the particle only in terms of the generalized coordinates z and ϕ, and associated velocities, excluding the possibility of the particle leaving the cylindrical surface.

(b) Obtain and solve the Lagrange equations associated with the coordinates z and ϕ. Describe in words the motion that the particle executes.

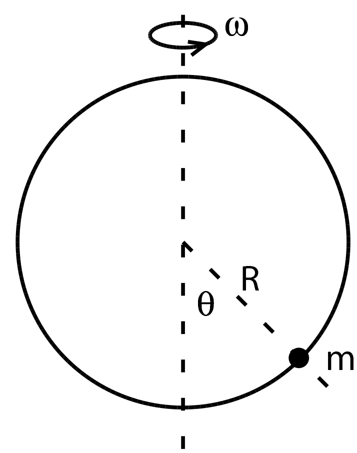
(c) Now restart the problem from scratch and construct the Lagrangian in terms of the coordinates ρ, ϕ and z, allowing at this stage for the possibility of the particle moving outside of the cylindrical surface.

(d) Obtain the Lagrange equations associated with the coordinates ρ, ϕ and z, incorporating explicitly the constraint equation with the accompanying, yet undetermined, multiplier λ.

(e) Solve the Lagrange equations. Arrive at the dependence of the multiplier λ on time, following from the requirement of the particle remaining on the cylindrical surface.

(f) What conclusion can you draw from the Lagrange equations on the normal force that the constraint surface exerts on the particle?

4. [5 pts] (adapted from a graduate qualifying exam) A bead of mass m can slide without friction on a vertical hoop of radius R. The hoop is rotating at constant angular speed ω about a vertical axis passing through the hoop's center.



(a) Determine the Lagrangian for the bead, using the indicated angle θ as the generalized coordinate, and including the effects of gravity.

(b) Obtain an equation of motion for the bead in terms of the angle θ.

(c) Determine any equilibrium angles $\theta_{\rm eq}(\omega)$ where the bead would remain if there initially at $\dot{\theta} = 0$.

5. [10 pts] Consider the central-force problem for a potential $U(r) = -k/r^2$ and a reduced mass μ. Here, k is a positive constant.

(a) Sketch the effective potential for large and small values of angular momentum $|\ell|$. Is there a difference in the sign of the potential? Use your graph and energy conservation and show that for large ℓ the orbits extend from some minimal value of radius $r_{\rm min}$ up to infinity. Obtain $r_{\rm min}$ as a function of energy E. Show that for small ℓ the orbits reach $r = 0$, but can either extend up to a finite $r_{\rm max}$ or up to infinity, depending on E. What value of $|\ell| = \ell_s$ separates the low and high value regions of ℓ, with different types of orbits? Obtain $r_{\rm max}$ as a function of E.

(b) Turn to the orbit equation and find general analytic solutions $r(\theta)$ for high and low values of $|\ell|$. (Note that, $\ell = \pm\ell_s$ requires a separate solution, but you are not asked to investigate it here.)

(c) As an example, adjust the arbitrary constants in the solution $r(\theta)$ for *high* $|\ell|$, so that the orbit is symmetric about the axis $\theta = 0$ and that the orbit extends from $r_{\rm min}$ on.

Note that at $r_{\min}$, $\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{r} = 0$. Sketch the orbit in the plane of motion. What are the asymptotic directions at $r \to \infty$, in terms of angle, for the orbit?

(d) As another example, adjust the arbitrary constants in the solution $r(\theta)$ for low $|\ell|$, so that the orbit is symmetric about the axis $\theta = 0$ and that it extends up to $r_{\max}$. Again, at $r_{\max}$, $\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{r} = 0$. Sketch the trajectory in the plane of motion for $\theta > 0$.

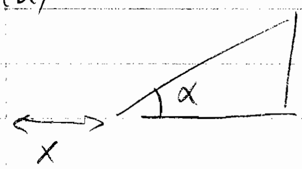
6. [5 pts] Consider objects of mass m moving under the influence of a central gravitational force characterized by potential energy $U = -k/r$.

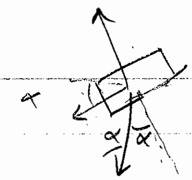
   (a) Demonstrate that when two orbits, circular and parabolic, have the same net angular momentum, the parabolic orbit has a perihelion that is half the radius of the circular orbit.

   (b) The speed of a particle at any point of a parabolic orbit is larger by a factor of $\sqrt{2}$ than the speed of particle on a circular orbit passing through the same point.

7. [5 pts] At perihelion, a particle of mass m in an elliptical orbit in a gravitational $1/r$ potential receives an impulse $m\,\Delta v$ in the radial direction. Find the semi-major axis a_{new} and eccentricity ϵ_{new} of the new elliptical orbit in terms of the old ellipse's parameters a_{old} and ϵ_{old}. Are these parameters increasing or decreasing in effect of the impulse. Does the direction of the impulse, inward or outward, matter for those specific parameters?

1.) (a)





$$m\ddot{x} = -N\sin\alpha$$

$$m\ddot{y} = N\cos\alpha - mg$$

$$M\ddot{X} = N\sin\alpha$$

(b) $y = (x - X)\tan\alpha \quad \Rightarrow \quad \ddot{y} = (\ddot{x} - \ddot{X})\tan\alpha$

(c) $$\begin{cases} m\ddot{x} = -N\sin\alpha \\ m(\ddot{x} - \ddot{X})\tan\alpha = N\cos\alpha - mg \\ M\ddot{X} = N\sin\alpha \end{cases}$$

First 2 eqs: $\quad -N\dfrac{\sin^2\alpha}{\cos\alpha} - m\ddot{X}\tan\alpha = N\cos\alpha - mg$

$$mg - m\ddot{X}\tan\alpha = N\frac{\sin^2\alpha + \cos^2\alpha}{\cos\alpha} = \frac{N}{\cos\alpha}$$

$$N = mg\cos\alpha - m\ddot{X}\sin\alpha$$

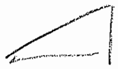
$$M\ddot{X} = mg\sin\alpha\cos\alpha - m\ddot{X}\sin^2\alpha$$

$$(M + m\sin^2\alpha)\ddot{X} = mg\sin\alpha\cos\alpha$$

$\triangle$ cont.

$$\ddot{X} = g \, \frac{m \sin\alpha \cos\alpha}{M + m \sin^2\alpha}$$

2.)



Lagrangian

(a) $U = mgy = -mgs \sin\alpha + \text{const}$

(b) $T = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$

$$\dot{y}^2 = \dot{s}^2 \sin^2\alpha$$

$$\dot{x}^2 = (\dot{X} - \dot{s}\cos\alpha)^2$$

$$T = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} m \left[\dot{s}^2 \sin^2\alpha + (\dot{X} - \dot{s}\cos\alpha)^2\right]$$

$$= \frac{1}{2} M \dot{X}^2 + \frac{1}{2} m \left[\dot{s}^2 \sin^2\alpha + \dot{X}^2 - 2\dot{X}\dot{s}\cos\alpha + \dot{s}^2\cos^2\alpha\right]$$

$$= \frac{1}{2}(m+M)\dot{X}^2 - m\dot{X}\dot{s}\cos\alpha + \frac{1}{2} m \dot{s}^2$$

(c) $L = T - U = \frac{1}{2}(m+M)\dot{X}^2 + \frac{1}{2} m\dot{s}^2 - m\dot{X}\dot{s}\cos\alpha$
$+ mgs \cdot \sin\alpha$

$$\frac{\partial L}{\partial X} = 0 \qquad \frac{\partial L}{\partial \dot{X}} = (m+M)\dot{X} - m\dot{s}\cos\alpha$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{X}} = 0 \quad \Rightarrow \quad (m+M)\ddot{X} - m\ddot{s}\cos\alpha = 0$$

 Lagrangian continued

$$\frac{\partial L}{\partial s} = mg \sin\alpha$$

$$\frac{\partial L}{\partial \dot{s}} = m\dot{s} - m\dot{X}\cos\alpha$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) = m\ddot{s} - m\ddot{X}\cos\alpha = mg\sin\alpha$$

$$\ddot{s} - \ddot{X}\cos\alpha = g\sin\alpha$$

$$\ddot{s} = \ddot{X}\cos\alpha + g\sin\alpha$$

$$(m+M)\ddot{X} = m\cos\alpha\,(\ddot{X}\cos\alpha + g\sin\alpha)$$

$$= m\ddot{X}\cos^2\alpha + mg\sin\alpha\cos\alpha$$

$$[m(1-\cos^2\alpha) + M]\ddot{X} = mg\sin\alpha\cos\alpha$$

$$[m\sin^2\alpha + M]\ddot{X} = mg\sin\alpha\cos\alpha$$

$$\ddot{X} = g\,\frac{m\sin\alpha\cos\alpha}{M + m\sin^2\alpha}$$

same result

(d)

$$\dot{X}_{CM} = \frac{m\dot{x} + M\dot{X}}{m+M} = \frac{M\dot{X} + m(\dot{X} - \dot{s}\cos\alpha)}{m+M}$$

$$(m+M)\ddot{X}_{CM} = M\ddot{X} + m(\ddot{X} - \ddot{s}\cos\alpha)$$

$$= (m+M)\ddot{X} - m\ddot{s}\cos\alpha = 0$$

according to an earlier result $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$

3)

(a) $U = \frac{1}{2}k\left(z^2 + R^2\right) \equiv \frac{1}{2}kz^2 + \text{const.}$

$T = \frac{1}{2}m\left(\dot{z}^2 + R^2\dot{\varphi}^2\right)$

$L = T - U = \frac{1}{2}m\left(\dot{z}^2 + R^2\dot{\varphi}^2\right) - \frac{1}{2}kz^2$

(b) $\frac{\partial L}{\partial \varphi} = 0 \qquad \frac{\partial L}{\partial \dot{\varphi}} = mR^2\dot{\varphi} \quad \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = 0$

$\Rightarrow mR^2\dot{\varphi} = \text{const} = \ell \qquad \varphi = \varphi_0 + \frac{\ell}{mR^2}t$

rotation in φ at constant angular velocity

$\frac{\partial L}{\partial z} = -kz \qquad \frac{\partial L}{\partial \dot{z}} = m\dot{z} \quad \Rightarrow \frac{d}{dt}m\dot{z} = m\ddot{z} = -kz$

$\Rightarrow z = A\cos(\omega t + \delta) \qquad \omega = \sqrt{\frac{k}{m}}$

harmonic oscillation in z at angular frequency ω

(c) $U = \frac{1}{2}k\left(z^2 + \rho^2\right)$

$T = \frac{1}{2}m\left(\dot{\rho}^2 + \dot{z}^2 + \rho^2\dot{\varphi}^2\right)$

$L = T - U = \frac{1}{2}m\left(\dot{\rho}^2 + \dot{z}^2 + \rho^2\dot{\varphi}^2\right) - \frac{1}{2}k\left(z^2 + \rho^2\right)$

(e) & (d) $\quad f = \rho - R = 0$

$$\frac{\partial L}{\partial \rho} = m\rho\dot{\varphi}^2 - k\rho \qquad \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho} \qquad \frac{\partial f}{\partial \rho} = 1$$

$$\frac{d}{dt} m\dot{\rho} = m\rho\dot{\varphi}^2 - k\rho + \lambda$$

ρ-L eq $\qquad m\ddot{\rho} = m\rho\dot{\varphi}^2 - k\rho + \lambda$

For $f=0 \quad \ddot{\rho} = 0 \quad \Rightarrow \quad \lambda = (k - m\dot{\varphi}^2)R$

$$\frac{\partial L}{\partial \varphi} = 0 \qquad \frac{\partial L}{\partial \dot{\varphi}} = m\rho^2\dot{\varphi} \qquad \frac{\partial f}{\partial \varphi} = 0$$

$$\frac{d}{dt}(m\rho^2\dot{\varphi}) = 0 \quad \Rightarrow \quad mR^2\ddot{\varphi} = 0$$ φ-L eq

$$\underbrace{mR^2\dot{\varphi} = \text{const}}_{= \ell} \qquad \dot{\varphi} = \frac{\ell}{mR^2}$$

$$\lambda = \left(k - \frac{\ell^2}{mR^4}\right)R = kR - \frac{\ell^2}{mR^3}$$

$$\frac{\partial L}{\partial z} = -kz \qquad \frac{\partial L}{\partial \dot{z}} = m\dot{z} \qquad \frac{\partial f}{\partial z} = 0$$

z-L eq

$$\frac{d}{dt} m\dot{z} = -kz \quad \Rightarrow \quad m\ddot{z} = -kz$$

$$z = A\cos(\omega t + \delta) \quad \text{with} \quad \omega = \sqrt{\frac{k}{m}}$$

$\lambda = kR - \frac{\ell^2}{mR^3}$ is time-independent

(f) Given that

$m\ddot{\rho} = m\rho\dot{\varphi}^2 - k\rho + \lambda$

effective centrifugal force

spring contribution

this must be the normal force produced by the constraint surface

$\underbrace{mR\dot{\varphi}^2}_{\text{centripetal acceleration in circular motion}} = k\rho - \lambda$ for $\ddot{\rho} = 0$

spring

surface

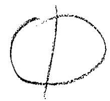
If the spring is weak $\lambda < 0$

force of the surface points inwards

If the spring is strong $\lambda > 0$,

force of the surface points outwards balancing the spring.

4.)



(a) $T = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} m R^2 \sin^2\theta\, \omega^2$

$U = -mgR\cos\theta \quad + \text{const}$

$L = T - U = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} m R^2 \sin^2\theta\, \omega^2 + mgR\cos\theta$

(b) $\frac{\partial L}{\partial \theta} = mR^2 \sin\theta\cos\theta\, \omega^2 - mgR\sin\theta$

$\frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta}$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mR^2\ddot{\theta} = mR^2\sin\theta\cos\theta\,\omega^2 - mgR\sin\theta$

$$\ddot{\theta} = \omega^2\sin\theta\cos\theta - \frac{g}{R}\sin\theta$$

(c) In equilibrium $\ddot{\theta} = 0$

$$0 = \sin\theta\left(\omega^2\cos\theta - \frac{g}{R}\right)$$

equilibrium points $\theta = 0$

or $\theta = \cos^{-1}\frac{g}{\omega^2 R}$ $\leftarrow$ that equilibrium angle only there if

$\frac{g}{\omega^2 R} < 1$ or $\omega^2 > \frac{g}{R}$

5)

$$u = -\frac{k}{r^2}$$

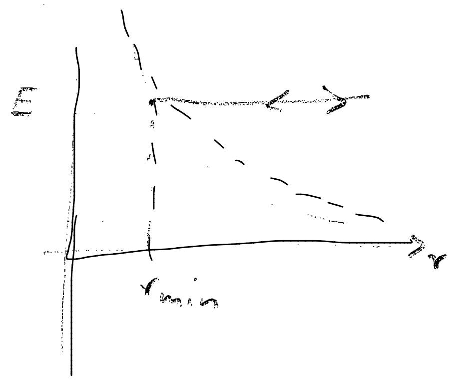
(a)

$$V = \frac{\ell^2}{2\mu r^2} - \frac{k}{r^2} = \frac{1}{r^2}\left(\frac{\ell^2}{2\mu} - k\right)$$

$$V = \frac{1}{2\mu r^2}\left(\ell^2 - \ell_s^2\right)$$

$$\ell_s = \sqrt{2\mu k}$$

$\ell > \ell_s$



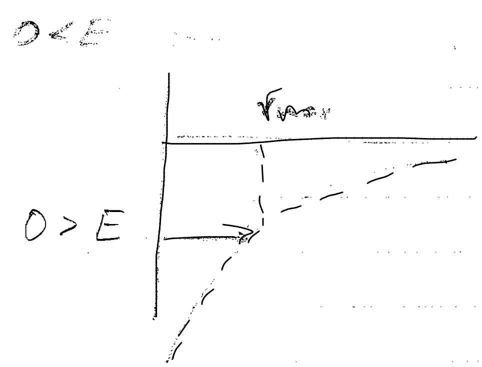
$$E = \frac{1}{r_{min}^2}\left(\frac{\ell^2}{2\mu} - k\right)$$

$$r_{min} = \sqrt{\frac{1}{E}\left(\frac{\ell^2}{2\mu} - k\right)}$$

$\ell < \ell_s$

$E < 0$

$$r_{max} = \sqrt{\frac{1}{E}\left(k - \frac{\ell^2}{2\mu}\right)}$$



$E > 0$ no bounds in r

(b) $$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{\ell^2} F = \frac{2\mu k}{\ell^2 r}$$

$$F = -\frac{dU}{dr} = -\frac{2k}{r^3}$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{1}{r} \times \left(\frac{2\mu k}{\ell^2} - 1\right) = \frac{1}{r}\left(\frac{\ell_s^2}{\ell^2} - 1\right)$$

$\ell > \ell_s$ $\quad \Omega^2 = 1 - \frac{\ell_s^2}{\ell^2}$ $\quad \Omega = \sqrt{1 - \frac{\ell_s^2}{\ell^2}} < 1$

$$\left(\frac{1}{r}\right)'' = -\Omega^2\left(\frac{1}{r}\right)$$

$$\frac{1}{r} = A\cos(\Omega\cdot\theta) + B\sin(\Omega\cdot\theta)$$

$\ell < \ell_s$ $\quad \beta^2 = \frac{\ell_s^2}{\ell^2} - 1$

$$\left(\frac{1}{r}\right)'' = \beta^2\left(\frac{1}{r}\right)$$

$$\frac{1}{r} = A e^{\beta\theta} + B e^{-\beta\theta}$$

(c) $L > L_s$ symmetric $\theta \leftrightarrow -\theta$

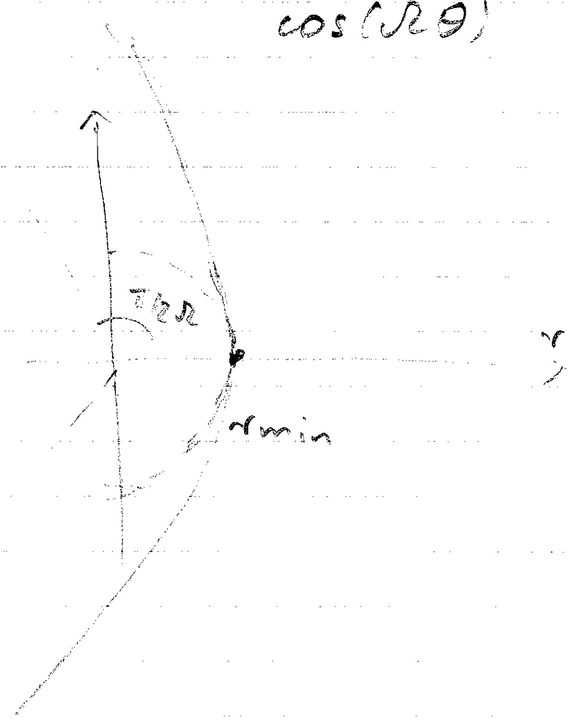
$B = 0$

$\frac{1}{r} = A\cos(\Omega\theta)$ $\frac{1}{r_{min}} = A$

$\frac{1}{r} = \frac{1}{r_{min}} \cos(\Omega\theta)$

$\Rightarrow \quad r = \frac{r_{min}}{\cos(\Omega\theta)}$

$r \to \infty$ when $\theta \to \frac{\pi}{2\Omega}$



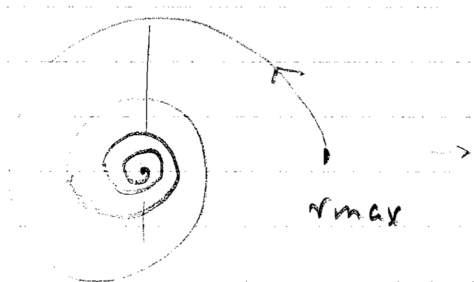
(d) symmetric $\Rightarrow$ $A = B$ $\qquad\qquad$ $\ell < \ell_c$

$$\frac{1}{r} = A\left(e^{\beta\theta} + e^{-\beta\theta}\right) = a \cosh \beta\theta$$

$$\frac{1}{r_{max}} = a$$

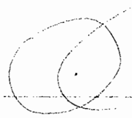
$$\frac{1}{r} = \frac{1}{r_{max}} \cosh \beta\theta$$

$$r = \frac{r_{max}}{\cosh \beta\theta}$$



Spiral

6.)



(a)

parabolic : $E = 0$ $\qquad \varepsilon = 1$

$$\frac{\alpha}{r_{min}} = 2 \qquad r_{min} = \frac{\alpha}{2} = \frac{1}{2} r_{circ}$$

circular : $\varepsilon = 0$ $\qquad r_{circ} = \alpha$

(b) parabolic $E = 0 \Rightarrow \frac{mv^2}{2} = \frac{k}{r}$

$$v_{par} = \sqrt{\frac{2k}{mr}}$$

circular $\frac{mv^2}{2} = \frac{1}{2}\frac{k}{r} \Rightarrow v_{circ} = \sqrt{\frac{k}{mr}}$

$$v_{par} = \sqrt{2}\, v_{circ}$$

7.)

$$\ell_{new} = \ell_{old}$$

$$E_{new} = E_{old} + \frac{m(\Delta v)^2}{2} \qquad E_{old} = -\frac{k}{2a_{old}}$$

$$a_{new} = \frac{k}{2\,|E_{new}|} = -\frac{k}{2E_{new}}$$

$$= -\frac{k}{2\left(E_{old} + \frac{m(\Delta v)^2}{2}\right)} = -\frac{k}{2\left(-\frac{k}{2a_{old}} + \frac{m(\Delta v)^2}{2}\right)}$$

$$= \frac{1}{\frac{1}{a_{old}} - \frac{m(\Delta v)^2}{k}} > a_{old}$$

$$\varepsilon_{new} = \sqrt{1 - \frac{\alpha}{a_{new}}} \qquad \varepsilon_{old}^2 = 1 - \frac{\alpha}{a_{old}}$$

$$\alpha = a_{old}\left(1 - \varepsilon_{old}^2\right)$$

$$\varepsilon_{new} = \sqrt{1 - \left(\frac{1}{a_{old}} - \frac{m(\Delta v)^2}{k}\right) a_{old}\left(1 - \varepsilon_{old}^2\right)}$$

$$= \sqrt{\cancel{1} - \cancel{1} + \varepsilon_{old}^2 + \frac{m(\Delta v)^2}{k} a_{old}\left(1 - \varepsilon_{old}^2\right)}$$

$$\varepsilon_{new} = \sqrt{\varepsilon^2_{old} + \frac{m(\Delta v)^2}{k} a_{old}(1-\varepsilon^2_{old})} > \varepsilon_{old}$$

Direction of the impulse does not matter.