$\boxed{\text{Solutions}}$

PHY321 Take-home Midterm 3

Due Monday April 21, 2014

1. A satellite is in a circular orbit of radius R around Earth (mass $M = 5.97 \times 10^{24}$ kg).

   (a) [3 pt] How is the velocity v of the satellite related to the radius R, mass M and gravitational constant G?

$$\frac{\cancel{m}v^2}{\cancel{R}} = \frac{\cancel{m}MG}{R^{\cancel{2}}} \Rightarrow \boxed{v = \sqrt{\frac{MG}{R}}}$$

   (b) [3 pts] What needs to be the radius R to make the orbit semisynchronous, i.e. with a period of 12h? (GPS satellites move on such orbits.) Obtain a value for R.

v = const for circular motion

$$\Rightarrow \quad T = \frac{2\pi R}{v} \Rightarrow T^2 = \frac{4\pi^2 R^2}{v^2} = \frac{4\pi^2 R^3}{MG}$$

$$\Rightarrow \boxed{R = \left[\frac{MGT^2}{4\pi^2}\right]^{1/3} = (MG)^{1/3}\left(\frac{T}{2\pi}\right)^{2/3}}$$

$$\Downarrow$$

$M_{earth} = 5.974 \times 10^{24}$ Kg

$G = 6.67 \times 10^{-11} \frac{N \cdot m^2}{Kg^2}$

$T = 12\cancel{h} \times \frac{3600s}{\cancel{h}} = 4.32 \times 10^4 s$

$$\boxed{R = 2.66 \times 10^7 m}$$

   (c) [3 pts] Now imagine that the satellite has a parabolic orbit. If the satellite has the same angular momentum as it did for the circular orbit, how big is r_{min} (i.e., the perihelion) compared to R in part a.

Parabola $\Rightarrow E = 0 \Rightarrow 0 = \underbrace{\tfrac{1}{2}m\dot{r}^2}_{0 \text{ at } r = r_{min}} + \frac{\ell^2}{2mr^2} - \frac{GmM}{r}$

$$\Rightarrow \frac{\ell^2}{2mr_{min}^2} = \frac{GmM}{r_{min}} \Rightarrow r_{min} = \frac{\ell^2}{2m^2MG} = \frac{\cancel{m^2}v_{circ}^2 R_{circ}^2}{2\cancel{m^2}MG}$$

(Since $\ell_{circ} = \ell_{parabola}$ by assumption)

$$\boxed{\Rightarrow r_{min} = \frac{v_{circ}^2 R_{circ}^2}{2MG} \underset{\text{part a)}}{=} \frac{\frac{\cancel{MG}}{\cancel{R_{circ}}} R_{circ}^{\cancel{2}}}{2\cancel{MG}} = \frac{1}{2} R_{circ}}$$

Alternative sol'n to 1c)

general sol'n for $U = -\frac{k}{r}$ potential is

$$\frac{\alpha}{r} = 1 + \varepsilon \cos\theta \qquad \alpha = \frac{\ell^2}{mk} \qquad \text{and } \theta = 0 \text{ at } r_{min}$$

$$\varepsilon = \sqrt{1 + \frac{2E\ell^2}{mk^2}}$$

Parabola: $\varepsilon = 1$

$$\therefore \frac{\alpha}{r_{min}} = 2 \Rightarrow \boxed{r_{min} = \frac{\alpha}{2}}$$

Circle: $\varepsilon = 0 \Rightarrow \frac{\alpha}{R} = 1 \Rightarrow \boxed{R_{circle} = \alpha}$

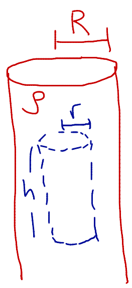
$$\Rightarrow \boxed{r_{min} = \frac{R_{circle}}{2}}$$

2. In the year 8050, a space voyage discovers a novel cylindrical planet in deep outer space of radius R and constant mass density ρ. The height of the cylindrical planet is much larger than R, so to a first approximation you can treat it as being infinitely long.

   (a) [4 pts] Find the gravitational field vector for $r \leq R$.

   (b) [4 pts] A very narrow hole is drilled in the radial direction from one side of the cylinder to the other. Show that if a mass m is dropped into the hole, it will execute simple harmonic motion. What is the angular frequency ω_0 of this oscillation?

a)



Gauss's law: $\underset{\text{cylinder radius } r}{\oint} \vec{g}\cdot d\vec{A} = g(r)2\pi r h = -4\pi G M_{enc}$

$M_{enc} = \underset{\text{gaussian cylinder volume}}{\int} \rho(r)\, d^3r \overset{\rho = \text{const}}{=} \rho \pi r^2 h$

$= \rho$

$\Rightarrow g(r)\cancel{2\pi} r\cancel{h} = -\overset{2\pi}{\cancel{4\pi^2}} G\rho r^{\cancel{2}}\cancel{h}$

$\vec{g} = g(r)\hat{r}$

$\Rightarrow \boxed{g(r) = -2\pi G\rho r \text{ for } r < R}$

b) $m\ddot{r} = F_{grav} = -2\pi G\rho r$

$\Rightarrow \ddot{r} = -\frac{2\pi G\rho}{m} r \Rightarrow$ SHO eqn w/ $\boxed{\omega_0^2 = \frac{2\pi G\rho}{m}}$

3. A particle of mass m and angular momentum ℓ moves under the influence of a central force for which the potential energy is given by

$$U(r) = \frac{c}{r^2} - \frac{k}{r},$$

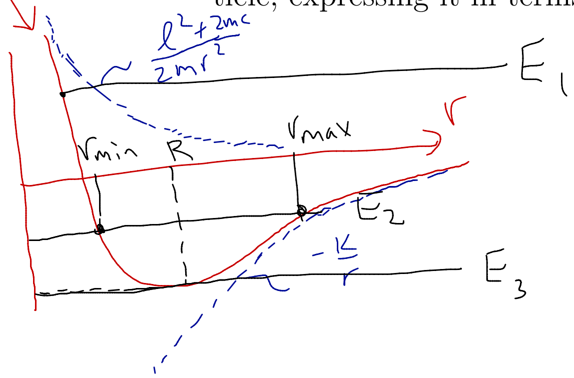
where c and k are positive constants.

(a) [3 pts] Sketch the effective potential energy $V_{eff} = \frac{l^2}{2mr^2} + U$, and indicate on the graph what the particle's energy must be for i) unbounded motion, ii) bounded motion between two turning points, and iii) circular motion.

(b) [2 pts] Is it possible for the particle's energy to be less than the minimum value of V_{eff}? Explain why or why not.

(c) [3 pts] Determine the radius of a circular orbit of the above particle, expressing it in terms of ℓ, m, c and k.

a)



(i) $E_1 \geq 0$ unbounded motion

(ii) $V_{eff}^{min} < E_2 < 0$ bounded between r_{min} + r_{max}

(iii) $E_3 = V_{eff}^{min}$ Circular motion at R

b) No. $E = \frac{1}{2}m\dot{r}^2 + V_{eff} \Rightarrow \frac{1}{2}m\dot{r}^2 = E - V_{eff}$

$\Rightarrow$ if $E < V_{eff}^{min}$, then $\frac{1}{2}m\dot{r}^2 < 0$ impossible

c) $\frac{dV_{eff}}{dr} = 0 = -\frac{2}{r^3}\left(\frac{\ell^2+2mc}{2m}\right) + \frac{k}{r^2}$

$$\Rightarrow \boxed{r = \frac{\frac{\ell^2+2mc}{m}}{k} = \frac{\ell^2+2mc}{mk} = \frac{\frac{\ell^2}{m}+2c}{k}}$$