Solution Key

PHY321 Homework Set 2

1. [5 pts] Consider the forces from the previous homework set, $\vec{F}^A(\vec{r})$ and $\vec{F}^B(\vec{r})$, acting on a particle. The force components depend on position $\vec{r}$ of the particle according to

$$F_x^A = F_x^B = y^2\,, \quad F_y^A = 2xy\,, \quad F_y^B = xy\,, \quad F_z^A = F_z^B = 0\,,$$

where the force components are in newtons and the coordinates are in meters.

   (a) Calculate the *curl* of these two forces.

   (b) What can you say about the nature of these two forces? If one or both of these forces are conservative, determine the associated potential energy.

2. [10 pts] Coordinates of a particle moving in the x-y plane change with time t according to:

$$x(t) = 2\,\mathrm{m} + 2\,\mathrm{m/s}\cdot t\,, \qquad y(t) = 3\,\mathrm{m} + 4\,\mathrm{m/s^3}\cdot t^3\,,$$

where t is in seconds and coordinates are in meters.

   (a) Obtain the equation of the trajectory $y = y(x)$ for the particle. Draw the trajectory within the region $0 \le x \le 4\,\mathrm{m}$ and $-1\,\mathrm{m} \le y \le 7\,\mathrm{m}$.

   (b) Obtain the velocity components $v_x(t)$ and $v_y(t)$ as well as the magnitude of the velocity $v(t)$. For $t = -0.5$, 0 and $0.5\,\mathrm{s}$, mark particle locations on the trajectory and the velocity vectors as arrows.

   (c) Obtain the components of the acceleration $a_x(t)$ and $a_y(t)$ as well as the magnitude of the acceleration $a(t)$. Indicate the acceleration vectors with arrows by the trajectory, for $t = -0.5$, 0 and $0.5\,\mathrm{s}$.

   (d) Obtain the components of the acceleration along $a_\parallel(t)$ and perpendicular $a_\perp(t)$ to the trajectory. Hint: Using $\vec{v}$, construct unit vectors tangential $\hat{u}_\parallel = \vec{v}/v$ and perpendicular $\hat{u}_\perp$. The parallel and perpendicular components of the acceleration may be then found from $a_\parallel = \vec{a}\cdot\hat{u}_\parallel$ and $a_\perp = \vec{a}\cdot\hat{u}_\perp$.

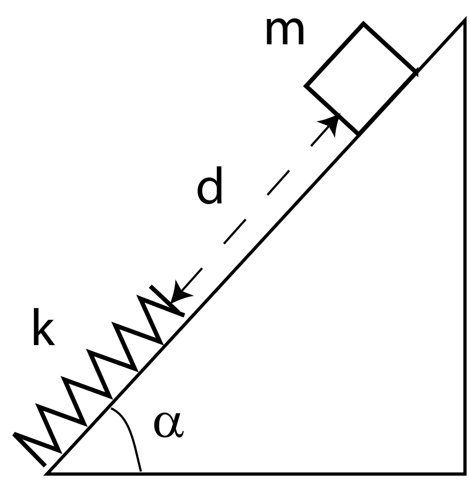
   (e) Discuss the behavior of the components $a_\parallel$ and $a_\perp$ with time in the context of the shape of the trajectory.

3. [10 pts] A package of mass $m = 3.0\,\mathrm{kg}$ is released on an incline at an angle of $\alpha = 50°$ from the horizontal. At the moment of release the package is distance $d = 2.5\,\mathrm{m}$ away from a long spring with spring constant $k = 150\,\mathrm{N/m}$ attached at the bottom of the incline. The coefficient of static friction between the package and the incline is $\mu_S = 0.45$ and the coefficient of kinetic friction is $\mu_k = 0.21$. The mass of the spring is negligible.

   (a) What is the speed of the package just before it reaches the spring?

   (b) What is the maximum compression of the spring? Take into account that the package needs to travel the distance by which the spring is compressed.

   (c) Decide whether the package rebounds after the maximal compression of the spring. If it were to rebound, determine how close the package would get to its original position. Describe in words the fate of the package from the release until permanent stop.



4. [5 pts] A ski jumper travels down a slope and leaves the ski track moving in the horizontal direction with a speed of $28\,\mathrm{m/s}$. The landing incline falls off with a slope of $32°$.

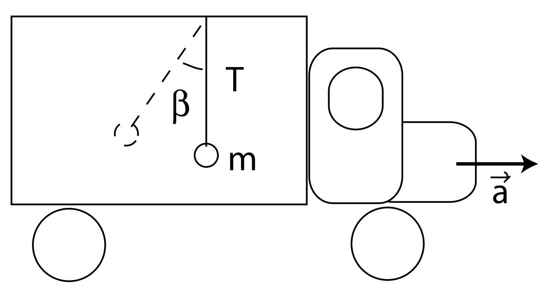
   (a) How long is the ski jumper air borne? Ignore effects of air resistance.

   (b) How far does the jumper land along the incline?

5. [5 pts] Solve this problem by employing the *noninertial* reference frame of the vehicle. A small toy of mass m hangs from a thread inside a vehicle, see the figure.

   (a) Find the equilibrium angle β of the thread relative to the vertical when the vehicle is accelerating forward at a constant acceleration a.

   (b) At what minimal acceleration a is the thread going to break if the thread can withstand the tensions only up to T_c?



6. [5 pts] A race track has a curve banked at an angle $\theta = 40°$ degrees with respect to the horizontal. The radius of the curve (looking down from directly above) is $R = 50\,\mathrm{m}$.

   (a) If the race track is icy so that the tires slide without friction, at what exact speed must the car go around the curve so as not to slide up or down the track? Hint: This problem can be simplified by considering a noninertial frame that is instantaneously comoving and accelerating with the car, though unlike the car with axes that have a fixed orientation in space. (You will learn about reference frames with rotating coordinate axes in CM2.)

   (b) On a dry day, the coefficient of friction between the tires and the track is $\mu_s = 0.50$. What are the minimum and maximum speeds at which the car could go around the curve without sliding up or down the track?

1.) $\vec{F} = y^2\,\hat{\imath} + \alpha x y\,\hat{\jmath}$ where $\alpha = \begin{cases} 2 & \text{force A} \\ 1 & \text{force B} \end{cases}$

a) $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix}$

$$= \hat{\imath}\left(\cancelto{0}{\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y}\right) + \hat{\jmath}\left(\cancelto{0}{\frac{\partial}{\partial z}F_x - \frac{\partial}{\partial x}F_z}\right) + \hat{k}\left(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right)$$

$$= (\alpha y - 2y)\,\hat{k} \qquad = y(\alpha - 2)\,\hat{k}$$

$$\boxed{\begin{aligned} \therefore\ \vec{\nabla}\times\vec{F}^A &= 0 \\ \vec{\nabla}\times\vec{F}^B &= -y\,\hat{k} \end{aligned}}$$

b) $\vec{F}^A$ is conservative since a necessary and sufficient condition for $\vec{F} = -\vec{\nabla}U$ is that $\vec{\nabla}\times\vec{F} = 0$. In contrast, $\vec{F}^B$ is not conservative. We find U^A from

(i) $\frac{\partial}{\partial x}U = -y^2$ and (ii) $\frac{\partial}{\partial y}U = -2xy$

(i) $\Rightarrow U(x,y) = -y^2 x + f(y)$. Plugging into (ii) $\Rightarrow -2\cancel{y}x + f'(y) = -\cancel{2xy}$

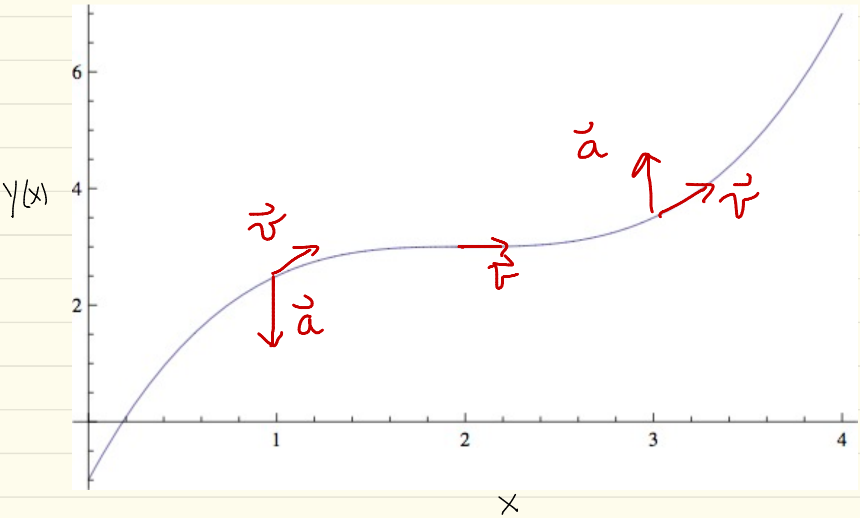
$$\Rightarrow f(y) = C \quad \text{(constant)}$$

$$\boxed{\Rightarrow U = -y^2 x + C}$$

2.) $x = 2 + 2t$ and $y = 3 + 4t^3$

a) from $x = 2+2t \Rightarrow t = \left(\frac{x-2}{2}\right)$. Plugging in to y gives

$$\boxed{y(x) = 3 + \frac{1}{2}(x-2)^3}$$



b) $v_x = \dot{x} = 2$ (for all t)

$v_y = \dot{y} = 12t^2 \Rightarrow v_y(-.5) = 3,\ v_y(0) = 0,\ v_y(+.5) = 3$

and $v(t) = \sqrt{v_x^2 + v_y^2} = \sqrt{4 + 144t^4} = 2\sqrt{1 + 36t^4}$

2c) $a_x = 0, \quad a_y = 24t = \begin{cases} -12 & t = -.5 \\ 0 & t = 0 \\ 12 & t = +.5 \end{cases}$

$$\hat{u}_{\parallel} = \frac{\vec{v}}{v} \Rightarrow \hat{u}_{\parallel}^{x} = \frac{v_x}{v} = \frac{1}{\sqrt{1+36t^4}}$$

$$\hat{u}_{\parallel}^{y} = \frac{v_y}{v} = \frac{6t^2}{\sqrt{1+36t^4}}$$

Clearly, $\hat{u}_{\perp} = -u_{\parallel}^{y}\hat{\imath} + u_{\parallel}^{x}\hat{\jmath}$ since $\hat{u}_{\perp} \cdot \hat{u}_{\parallel} = -u_{\parallel}^{y}u_{\parallel}^{x} + u_{\parallel}^{x}u_{\parallel}^{y} = 0$

$$\boxed{\begin{aligned} \therefore\ a_{\parallel} &= \vec{a} \cdot \hat{u}_{\parallel} = a_x u_{\parallel}^{x} + a_y u_{\parallel}^{y} = \frac{144t^3}{\sqrt{1+36t^4}} \\ a_{\perp} &= \vec{a} \cdot \hat{u}_{\perp} = a_x u_{\perp}^{x} + a_y u_{\perp}^{y} = -a_x u_{\parallel}^{y} + a_y u_{\parallel}^{x} = \frac{24t}{\sqrt{1+36t^4}} \end{aligned}}$$

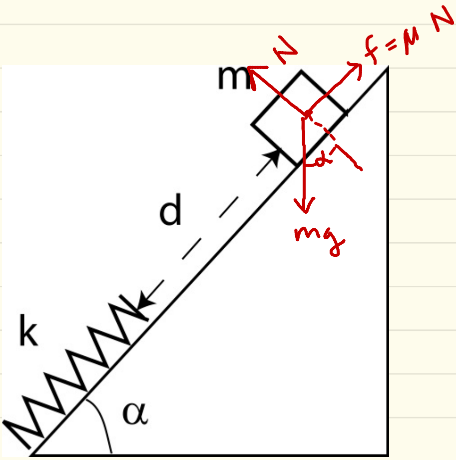
2d) as $t \to \pm\infty$, $a_{\perp} \to 0$ and $a_{\parallel} \to 24t$. Otherwise, $a_{\parallel,\perp} < 0$ for $t < 0$ and $a_{\parallel,\perp} > 0$ for $t > 0$.

Trajectory curves down for $t < 0$ consistent with $a_{\perp} < 0$. For $t > 0$, it curves up consistent with $a_{\perp} > 0$. Curvature changes at $t = 0$, which is consistent with $a_{\perp} = 0$.

3) A) energy conservation $\Rightarrow$ (1 = block at top of incline, 2 = block just above spring)

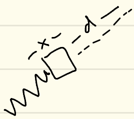
$$T_2 - \cancel{T_1}^{\,0} = U_1^g - \cancel{U_2^g}^{\,0} + W_{12}^{fric}$$

$$\frac{1}{2}mv^2 = mgd\sin\alpha - \mu_k mgd\cos\alpha$$



$$\boxed{\begin{aligned} &\Rightarrow v = \sqrt{2gd(\sin\alpha - \mu_k\cos\alpha)} \\ &= \sqrt{2\cdot 9.8\tfrac{m}{s^2}\cdot 2.5m\,(\sin 50^\circ - .21\cos 50^\circ)} \\ &= 5.56\ \tfrac{m}{s} \end{aligned}}$$

B) let x = max. compression of spring



again, energy conservation gives (1 = block at top, 2 = block just above spring, 3 = block when spring maximally compressed)

$$\cancel{T_3}^{\,0} - \cancel{T_1}^{\,0} = \int_1^3 \vec{F}\cdot d\vec{r} = \int_1^3 (\vec{F}_g + \vec{f})\cdot d\vec{r} + \int_2^3 \vec{F}_s\cdot d\vec{r}$$

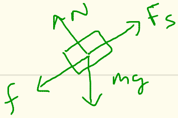
$$0 = mg(d+x)\sin\alpha - \mu_k mg(d+x)\cos\alpha - \frac{1}{2}kx^2 \qquad *$$

$$\Rightarrow \frac{1}{2}kx^2 + mg(\mu_k\cos\alpha - \sin\alpha)x + mg(\mu_k\cos\alpha - \sin\alpha)d = 0$$

* Plugging in #'s this becomes:

$$\boxed{\begin{aligned} &x = \frac{18.6 \pm \sqrt{(18.6)^2 + 4\times 75\times 46.4}}{2\times 75} \\ &\therefore\ x = \frac{18.6 + 119.4}{150} = .92\ m \end{aligned}}$$

3c) at max. spring compression,



$$F_s = kx = 150 \times .92 = 138 \text{ N}$$

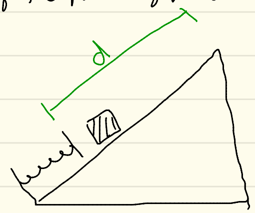
$$mg\sin\alpha = 22.5 \text{ N}$$

max friction force

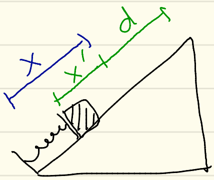
* since $(F_s - mg\sin\alpha) = 138 - 22.5 = 115.N > \mu_s N = \mu_s mg\cos\alpha = 8.5 \text{ N}$

∴ The box will rebound.

* Now, we must determine if the position of the block after it rebounds is such that either A) the spring is fully relaxed:



or B) the spring is still partially compressed



* Let's assume B) holds (i.e., $x, x' > 0$ and $x > x'$) and see if things are consistent. Applying energy conservation (state 1 = block sits at fully compressed spring, state 2 = block at rebounded position $(d+x')$ down the ramp)

$$(\cancel{T_2}^{\,0} - \cancel{T_1}^{\,0}) + (U_2^g - \cancel{U_1^g}^{\,0}) + (U_2^s - U_1^s) = W_{12}^{fric} \Rightarrow mg(x-x')\sin\alpha + \tfrac{1}{2}kx'^2 - \tfrac{1}{2}kx^2 = -\mu_k mg(x-x')\cos\alpha$$

* Solving for x':

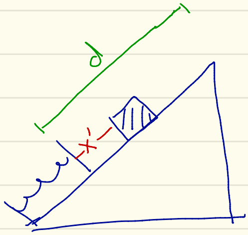
$$-\tfrac{1}{2}k(x^2 - x'^2) = -mg(x-x')[\sin\alpha + \mu_k\cos\alpha]$$

$$\Rightarrow \tfrac{k}{2}(x+x') = mg[\sin\alpha + \mu_k\cos\alpha]$$

$$\Rightarrow \frac{k}{2}(x+x') = mg\left[\sin\alpha + \mu_k \cos\alpha\right]$$

$$x' = \frac{2mg}{k}\left[\sin\alpha + \mu_k\cos\alpha\right] - x \qquad = -.57\,m \text{ upon plugging in \#'s.}$$

However, this violates the assumption that $x' > 0$ (and $< x$). Therefore, this implies the rebounded position looks like case A)



$$(\cancelto{0}{T_2} - \cancelto{0}{T_1}) + (U_2^g - \cancelto{0}{U_1^g}) + (\cancelto{0}{U_2^s} - U_1^s) = W_{12}^{fric}$$

$$\Rightarrow mg(x'+x)\sin\alpha - \tfrac{1}{2}kx^2 = -\mu_k mg(x'+x)\cos\alpha$$

$$\Rightarrow x' mg(\sin\alpha + \mu_k\cos\alpha) = \tfrac{1}{2}kx^2 - mgx(\sin\alpha + \mu_k\cos\alpha)$$

$$x' = \frac{\frac{1}{2}kx^2}{mg(\sin\alpha + \mu_k\cos\alpha)} - x$$

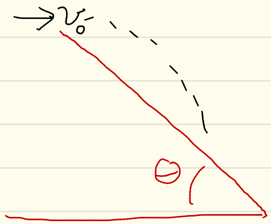
$$= \frac{(75\,\frac{N}{m})(.92\,m)^2}{(9.8)(3.0)N\cdot(\sin 50^\circ + .21\cos 50^\circ)} - .92\,m$$

$$= \frac{63.48\,\cancel{N}\cdot m}{26.4894\,\cancel{N}} - .92\,m$$

$\boxed{x' = 1.48\ m}$ $\Rightarrow$ distance of closest approach $= d - x'$

$= 1.02\ m$

4)



$v_0 = 28\ m/s$

$\theta = 32^\circ$

$\ddot{x} = 0 \Rightarrow x(t) = v_0 t$ (take origin at launch point)

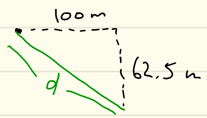
$\ddot{y} = -g \Rightarrow \dot{y}(t) = -gt + \cancel{\dot{y}(0)}^{\,0} \Rightarrow y(t) = -\frac{gt^2}{2}$

<u>Need $y(x)$:</u> $t = \frac{x(t)}{v_0} \Rightarrow y = -\frac{gx^2}{2v_0^2}$

$\tan\theta = \frac{y}{x} = -\frac{gx}{2v_0^2} \Rightarrow x = -\frac{2v_0^2}{g}\tan(180^\circ - 32^\circ) = -\frac{2(28)^2}{9.8}(-.625)$

$\Rightarrow x = 100\ m$

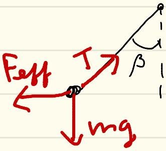
$\Rightarrow y = 100\,m \times \tan(180-32) = -62.5$



$\boxed{d = \sqrt{100^2 + (62.5)^2} \cong 118\ m \text{ down the incline.}}$

5.)

a)



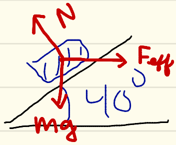
$$\tan\beta = \frac{F_{eff}}{mg} = \frac{\cancel{m}a}{\cancel{m}g} \Rightarrow \boxed{\beta = \arctan\left(\frac{a}{g}\right)}$$

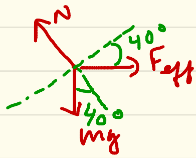
b) $T_c^2 = T_{c,x}^2 + T_{c,y}^2 = F_{eff}^2 + m^2g^2 = m^2a_c^2 + m^2g^2$

$$\Rightarrow a_c^2 = \frac{T_c^2 - m^2g^2}{m^2}$$

$$\boxed{\Rightarrow a_c = \sqrt{\frac{T_c^2}{m^2} - g^2}}$$

6.)





a) to stay @ constant height, need

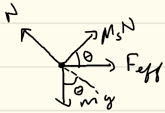
$$mg\sin 40^\circ = F_{eff}\cos 40^\circ = ma\cos 40^\circ$$

$$* \text{ but } a = \frac{v^2}{R}$$

$$\boxed{\Rightarrow Rg\tan 40^\circ = v^2 \Rightarrow v = \sqrt{(50)(9.8)(.839)} = 20.3 \ \frac{m}{s}}$$

6b) Now $\mu_s = .5$

Case 1: Min. v to keep from sliding down



then $N = mg\cos\theta + F_{eff}\sin\theta$

$$= mg\cos\theta + \frac{mv^2}{R}\sin\theta$$

and condition for not sliding down is

$$mg\sin\theta = \mu_s N + F_{eff}\cos\theta$$

$$mg\sin\theta = \mu_s mg\cos\theta + \mu_s \frac{mv^2}{R}\sin\theta + \frac{mv^2}{R}\cos\theta$$

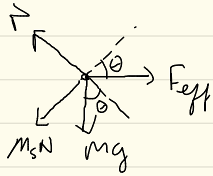
$$\Rightarrow \frac{\cancel{m}v^2}{R}\left[\sin\theta + \cos\theta\right] = \cancel{m}g(\sin\theta - \mu_s\cos\theta)$$

$$\Rightarrow v_{min} = \sqrt{\frac{Rg(\sin\theta - \mu_s\cos\theta)}{(\sin\theta + \cos\theta)}}$$

$$= \sqrt{\frac{(50)(9.8)[\sin 40 - .5\cos 40]}{(\sin 40 + \cos 40)}}$$

$$\boxed{\Rightarrow v_{min} = 10.8\ \frac{m}{s}}$$

<u>Case 2</u>: find v_{max} such that car doesn't slide <u>up</u>



$$N = mg\cos\theta + F_{eff}\sin\theta$$
$$= mg\cos\theta + \frac{mv^2}{R}\sin\theta$$

and condition to <u>not</u> slide <u>upwards</u> is

$$\mu_s N + mg\sin\theta = F_{eff}\cos\theta$$

$$\Rightarrow \frac{mv^2}{R}\cos\theta - \mu_s mg\cos\theta - \mu_s\frac{mv^2}{R}\sin\theta = mg\sin\theta$$

$$\frac{\cancel{m}v^2}{R}\left[\cos\theta - \mu_s\sin\theta\right] = \cancel{m}g\left[\sin\theta + \mu_s\cos\theta\right]$$

$$\boxed{\Rightarrow v_{max} = \sqrt{\frac{Rg\left[\sin\theta + \mu_s\cos\theta\right]}{\left[\cos\theta - \mu_s\sin\theta\right]}} = 33.6\ \frac{m}{s}}$$