Hard-sphere scattering

$$\theta = \pi - 2\alpha \qquad d\theta = -2\,d\alpha$$

$$\sin\alpha = \frac{b}{R} \quad \Rightarrow \quad \cos\alpha\, d\alpha = \frac{db}{R}$$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

$$d\alpha = \frac{db}{R}\frac{1}{\cos\alpha} \qquad d\theta = -\frac{2\,db}{R}\frac{1}{\cos\alpha}$$

$$\left|\frac{d\theta}{db}\right| = \frac{2}{R\cos\alpha}$$

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta} \times \frac{R\cos\alpha}{2} = \frac{R\sin\alpha\, R\cos\alpha}{2\sin\theta}$$

$$= \frac{R^2 \sin 2\alpha}{4\sin\theta} \qquad \sin 2\alpha = \sin(\pi - \theta) = \sin\theta$$

$$\frac{d\sigma}{d\Omega} = \frac{R^2}{4}$$ — constant isotropic cross section

$$\sigma = \int_{4\pi} \frac{d\sigma}{d\Omega} = 4\pi \times \frac{R^2}{\cancel{4}} = \pi R^2$$

Rutherford

(a)

$$\frac{d\sigma}{d\Omega'} = \frac{1}{16}\left(\frac{kq_1\,q_2}{T'}\right)^2 \frac{1}{\sin^4(\theta'/2)} \qquad \sigma_{\theta'>\theta_0} = \int_{\theta'>\theta_0} \frac{d\sigma}{d\Omega'}\,d\Omega'$$

$$\sigma_{\theta>\theta_0} = \frac{1}{16}\left(\frac{kq_1q_2}{T'}\right)^2 \int \frac{d\Omega'}{\sin^4(\theta'/2)}$$

$$\int \frac{d\Omega'}{\sin^4(\theta'/2)} = 2\pi \int \frac{\sin\theta'\,d\theta'}{\sin^4(\theta'/2)} = 2\pi \int \frac{2\sin\frac{\theta'}{2}\cos\frac{\theta'}{2}\,d\theta'}{\sin^4\frac{\theta'}{2}} = 8\pi \int \frac{\cos\frac{\theta'}{2}\,d\frac{\theta'}{2}}{\sin^3\frac{\theta'}{2}}$$

$$= 8\pi \int \frac{\cos\alpha\,d\alpha}{\sin^3\alpha} = 8\pi \int \frac{d\sin\alpha}{\sin^3\alpha} = 8\pi \int \frac{dy}{y^3} = -\frac{4\pi}{y^2}\Big|$$

$\alpha = \frac{\theta'}{2}$ $\qquad y = \sin\alpha$

$$= -\frac{4\pi}{\sin^2\frac{\theta'}{2}}\Bigg|_{\theta_0}^{\pi} = -4\pi\left[1 - \frac{1}{\sin^2\frac{\theta_0}{2}}\right] = 4\pi\,\frac{1-\sin^2\frac{\theta_0}{2}}{\sin^2\frac{\theta_0}{2}}$$

$$= 4\pi\,\frac{\cos^2\frac{\theta_0}{2}}{\sin^2\frac{\theta_0}{2}} = 4\pi\,\mathrm{cotan}^2\frac{\theta_0}{2}$$

$$\sigma_{\theta>\theta_0} = \frac{\pi}{4}\left(\frac{kq_1q_2}{T'}\right)^2 \mathrm{cotan}^2\frac{\theta_0}{2}$$

(b)

$$k = \frac{1}{4\pi\varepsilon_0} \qquad \frac{kq_1q_2}{T'} = \frac{1}{4\pi\varepsilon_0}\frac{Z_\alpha e\, Z_{Au} e}{T'}$$

$$= \frac{e^2}{4\pi\varepsilon_0}\frac{Z_\alpha Z_{Au}}{T'} = 1.44\ \mathrm{MeV\cdot fm}\ \frac{2\times 79}{7.7\ \mathrm{MeV}} = 29.5\ \mathrm{fm}$$

$$\frac{\pi}{4}\times(29.5\ \mathrm{fm})^2 = 683\ \mathrm{fm}^2 = 6.83\ \mathrm{b}$$

I. $\theta_0 = 170^0$ $\sigma = 6.83\,b \cot^2 \frac{170}{2} = 0.052\,b$

$= 5.2 \times 10^{-30}\ m^2$

II. $\theta_0 = 90^0$ $\sigma = 6.8\ b = 6.8 \times 10^{-28}\ m^2$

III $\theta_0 = 10^0$ $\sigma = 892\ b = 8.9 \times 10^{-26}\ m^2$

Rocket in free space

$$v = u \ln \frac{m_0}{m}$$

$$p = mv = mu \ln \frac{m_0}{m}$$

$$\frac{\partial p}{\partial m} = u \ln \frac{m_0}{m} + \frac{mu}{m} = u \ln \frac{m_0}{m} + u$$

$$= u \left(\ln \frac{m_0}{m} + 1 \right) = 0$$

$$\ln \frac{m_0}{m} = -1 \quad \Rightarrow \quad \frac{m}{m_0} = \frac{1}{e}$$

Ariane

(a) $$v = -(m_0 - m)\frac{g}{\alpha} + u \ln \frac{m_0}{m}$$

$$\tau_0 = \frac{\alpha u}{m_0 g} \qquad \alpha = \frac{m_0 g \tau_0}{u}$$

$$v = -(m_0 - m)\frac{\cancel{g} u}{m_0 \cancel{g} \tau_0} + u \ln \frac{m_0}{m}$$

$$= -\left(1 - \frac{m}{m_0}\right)\frac{u}{\tau_0} + u \ln \left(\frac{m_0}{m}\right)$$

$$v = u \left\{ \ln \frac{m_0}{m} - \frac{1}{\tau_0}\left(1 - \frac{m}{m_0}\right)\right\}$$

(b) $$\frac{dv}{dm} = u\left\{-\frac{1}{m} + \frac{1}{m_0 \tau_0}\right\} \underset{\substack{\uparrow \\ m=m_0}}{\simeq} \frac{u}{m_0}\left\{\frac{1}{\tau_0} - 1\right\} < 1$$

condition for velocity to increase from zero as mass decreases

$$\frac{1}{\tau_0} < 1 \quad \Rightarrow \quad \underbrace{\tau_0 > 1}_{\text{condition for lift-off}}$$

Ariane continued

(c) $h = \int v\,dt$

$m = m_0 - \alpha t$

$dm = -\alpha\,dt$

τ absorbed into reversal of limits of integration

$$h = \frac{1}{\alpha}\int v\,dm$$

$$= \frac{u}{m_0 g \tau_0}\int_m^{m_0} dm \left\{ \ln\frac{m_0}{m} - \frac{1}{\tau_0}\left(1-\frac{m}{m_0}\right)\right\} u$$

$$= \frac{u^2}{g\tau_0}\int_{m/m_0}^{1} dx \left\{ -\ln x - \frac{1}{\tau_0}(1-x)\right\}$$

$$= \frac{u^2}{g\tau_0}\left\{ -x\ln x + x - \frac{1}{\tau_0}\left(x - \frac{x^2}{2}\right)\right\}$$

$$= \frac{u^2}{g\tau_0}\left\{ 1 - \frac{1}{2\tau_0} + \frac{m}{m_0}\ln\frac{m}{m_0} - \frac{m}{m_0} + \frac{1}{\tau_0}\frac{m}{m_0}\left(1-\frac{m}{2m_0}\right)\right\}$$

$$= \frac{u^2}{g\tau_0}\left\{ 1 - \frac{m}{m_0}\left(1+\ln\frac{m_0}{m}\right) - \frac{1}{2\tau_0}\left[1+\left(\frac{m}{m_0}\right)^2 - 2\left(\frac{m}{m_0}\right)\right]\right\}$$

$$= \frac{u^2}{g\tau_0}\left\{ 1 - \frac{m}{m_0}\left(1+\ln\frac{m_0}{m}\right) - \frac{1}{2\tau_0}\left(1-\frac{m}{m_0}\right)^2\right\}$$

Ariane cont.

(d)

$$\tau_o = \frac{\alpha u}{m_o g} = \frac{1.29 \times 10^7}{7.77 \times 10^5 \times 9.81} = 1.69$$

$$\frac{m_o}{m} = \frac{7.77 \times 10^5}{2.23 \times 10^5} = 3.48$$

$$h = \frac{(3010)^2}{9.81 \times 1.69} \left\{ 1 - \frac{1}{3.48}\left(1 + \ln 3.48\right) - \frac{1}{2 \times 1.69}\left(1 - \frac{1}{3.48}\right)^2 \right\}$$

$$= 111.5 \text{ km}$$

$$v = 3010 \times \left\{ \ln 3.48 - \frac{1}{1.69}\left(1 - \frac{1}{3.48}\right) \right\} = 2484 \frac{m}{s}$$

Rope over peg $\qquad \lambda = \frac{M}{L}$

(a)

$$F_{net} = \lambda x g - \lambda(L-x)g$$
$$= 2\lambda x g - \lambda L g$$
$$= \lambda g(2x - L) = 2\lambda g\left(x - \frac{L}{2}\right)$$

$$M\frac{d^2x}{dt^2} = F_{net}$$

$$\cancel{\lambda} L \frac{d^2}{dt^2}\left(x - \frac{L}{2}\right) = 2\cancel{\lambda} g\left(x - \frac{L}{2}\right)$$

$$\frac{d^2}{dt^2}\left(x - \frac{L}{2}\right) = \frac{2g}{L}\left(x - \frac{L}{2}\right)$$

Solution $\quad x - \frac{L}{2} = Ae^{\alpha t} + Be^{-\alpha t}$

where $\alpha = \sqrt{\frac{2g}{L}}$ $\qquad \equiv a\cosh(\alpha t) + b\sinh(\alpha t)$

(b) $t = 0 \qquad x_0 - \frac{L}{2} = a \qquad \dot{x} = b\alpha$

Since $\dot{x} = 0$ then $b = 0$

$$x = \frac{L}{2} + \left(x_0 - \frac{L}{2}\right)\cosh\left(\sqrt{\frac{2g}{L}}\,t\right)$$

If $x_0 > \frac{L}{2}$, then x grows, i.e. rope slides to the right.

If $x_0 < \frac{L}{2}$, then x drops, i.e. rope slides to the left.

rope over peg continued

for the fall

(c) $x = 0$ or $x = L$ depending on sign of $(x_0 - \frac{L}{2})$

$$x - \frac{L}{2} = (x_0 - \frac{L}{2}) \cosh(\ )$$

$$x - \frac{L}{2} = \pm \frac{L}{2}$$

Condition $\frac{L}{2} = |x_0 - \frac{L}{2}| \cosh\left(\sqrt{\frac{2g}{L}}\, t_f\right)$

$$t_f = \sqrt{\frac{L}{2g}} \cosh^{-1} \frac{L/2}{|x_0 - L/2|}$$

$$= \sqrt{\frac{L}{2g}} \cosh^{-1} \frac{1}{\left|\frac{2x_0}{L} - 1\right|}$$

Check: $x_0 = 0$ or $x_0 = L$, $\cosh^{-1} 1 = 0 \Rightarrow t_f = 0$

$|x_0 - \frac{L}{2}| \to 0$ $\quad t_f \to \infty$

mass on spring

(a) $F = -ky + mg$ ← force on mass + pointing down

In equilibrium $F = 0$

$0 = -ky_0 + mg \Rightarrow k = \frac{mg}{y_0}$

Otherwise

$$m\frac{d^2y}{dt^2} = F = -ky + mg = -ky + ky_0$$

$$= -k(y - y_0)$$

$$\frac{d^2y}{dt^2} = \frac{d^2}{dt^2}(y - y_0) = -\frac{k}{m}(y - y_0)$$

Solution $y - y_0 = A\cos(\omega_0 t + \delta) = A\cos(\omega_0 t)$

$\delta = 0$ from the condition of rest at $t = 0$

$\omega_0 = \sqrt{\frac{k}{m}}$

(b) $T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m y_0}{mg}} = 2\pi\sqrt{\frac{y_0}{g}}$