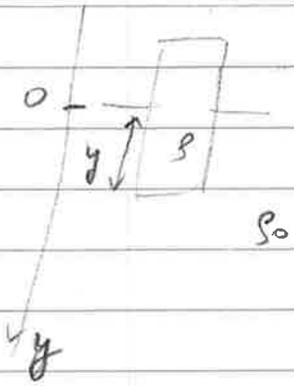
cylinder in fluid

(a)



$$F = F_{grav} + F_{buoy}$$

$$= mg - yA\rho_0 g$$

$$m = LA\rho$$

$$F = gA\left[L\rho - y\rho_0\right]$$

$$m\frac{d^2 y_{cm}}{dt^2} = m\frac{d^2 y}{dt^2} = LA\rho\frac{d^2 y}{dt^2} = gA\left[L\rho - y\rho_0\right]$$

$$\frac{d^2 y}{dt^2} = \frac{g}{L\rho}\left[L\rho - y\rho_0\right] = -\frac{g\rho_0}{L\rho}\left[y - \frac{\rho}{\rho_0}L\right]$$

$$\frac{d^2}{dt^2}\left[y - \frac{\rho}{\rho_0}L\right] = -\frac{g\rho_0}{L\rho}\left[y - \frac{\rho}{\rho_0}L\right]$$

equilibrium immersion

Solution

$$y = \frac{\rho}{\rho_0}L + A\cos(\omega t + \delta)$$

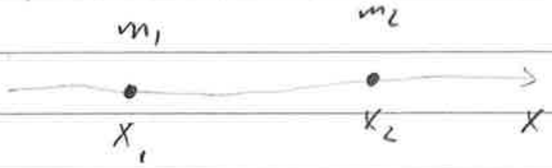
where $\omega_0 = \sqrt{\frac{g\rho_0}{L\rho}}$

$$\nu_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g\rho_0}{L\rho}}$$

(b) $\nu_0 = \frac{1}{2\pi}\sqrt{\frac{9.81\frac{m}{s^2} \times 1}{.03m \times 0.8}} \approx 3.2\ Hz$

2 masses

(a)



$$F_{12} = k(x_2 - x_1 - x_0)$$

points in $+x$ direction if x_2 much larger than x_1

$$F_{21} = -k(x_2 - x_1 - x_0)$$

points in $-x$ direction if x_2 much larger than x_1

$$m_1 \frac{d^2 x_1}{dt^2} = F_{12}$$

$$m_2 \frac{d^2 x_2}{dt^2} = F_{21} = -F_{12}$$

If we sum the eqs side by side, we get

$$\frac{d^2}{dt^2}(m_1 x_1 + m_2 x_2) = 0$$

which expresses the fact that the CM moves at constant velocity.

$$\frac{d^2}{dt^2} \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = 0 \qquad X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}.$$

$$\frac{d^2}{dt^2} X = 0 \quad \Rightarrow \quad \dot{X} = \text{const}$$

(b) To determine dynamics in $x_2 - x_1$, we write

$$\frac{d^2 x_1}{dt^2} = \frac{1}{m_1} F_{12}$$

and

$$\frac{d^2 x_2}{dt^2} = -\frac{1}{m_2} F_{12}$$

Subtracting first eq from the second, we get

$$\frac{d^2}{dt^2}(x_2 - x_1) = -\left(\frac{1}{m_2} + \frac{1}{m_1}\right) F_{12}$$

$$= -\left(\frac{1}{m_2} + \frac{1}{m_1}\right) k\,(x_2 - x_1 - x_0)$$

Further

$$\frac{d^2}{dt^2}(x_2 - x_1 - x_0) = -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(x_2 - x_1 - x_0)$$

Solution:

$$x_2 - x_1 = x_0 + A\cos(\omega_0 t + \delta)$$

where $\omega_0 = \sqrt{k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)} = \sqrt{\frac{k}{\mu}}$

with $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$ $\Rightarrow$ μ - reduced mass hw 1

$k = 4\,N/m \quad m = 0.015\,kg \quad A = .03\,m$

(a) $\nu_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 2.60\,Hz$

$T_0 = 1/\nu_0 = 0.385\,s$

(b) $E = \frac{1}{2}kA^2 = \frac{1}{2}\,4\,\frac{N}{m} \times (0.03\,m)^2 = 1.8\,mJ$

(c) $x = A\cos(\omega_0 t) \Rightarrow \dot{x} = A\omega_0 \sin(\omega_0 t)$

$v_{max} = A\omega_0 = 0.03\,m \times 2\pi \times 2.60\,Hz = 0.49\,m/s$

(d) $e^{-\beta t} = \frac{1}{2} \qquad \beta t = \ln 2$

$\beta = \frac{\ln 2}{t} = \frac{\ln 2}{8s} = 0.0866\,s^{-1}$

(e) $\nu_1 = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \beta^2} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}\sqrt{1 - \frac{\beta^2 m}{k}}$

$\simeq \frac{1}{2\pi}\sqrt{\frac{k}{m}}\cdot\left(1 - \frac{\beta^2 m}{2k}\right) = \nu_0 \times \left(1 - \frac{(.0866)^2 \times .015}{2 \cdot 8}\right)$

$= \nu_0 \times (1 - 7.0 \times 10^{-6}) \simeq \nu_0 = 2.60\,Hz$

$\frac{\nu_0 - \nu_1}{\nu_0} \simeq 7.0 \times 10^{-6}$ $\therefore$ small change

(f) $\tau_1 \simeq \tau_0 = 0.385\,s$

decrement $e^{-\beta\tau_1} = \exp(-.0866 \times 0.385)$

$= 0.967$

 problem

(a) $U = U_0 \cos^2\left(\frac{\pi x}{2\ell}\right)$

$$m\frac{d^2x}{dt^2} = F = -\frac{dU}{dx}$$

$$m\frac{d^2x}{dt^2} = +U_0\, 2 \times \frac{\pi}{2L} \sin\left(\frac{\pi x}{2\ell}\right)\cos\left(\frac{\pi x}{2\ell}\right)$$

$$= \frac{\pi U_0}{2\ell} \sin\left(\frac{\pi}{\ell} x\right)$$

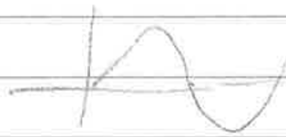
(b) Force vanishes for $x = n\ell$, where n is integer. Hower, every second case (n even) corresponds to potential maximum.

Only $x = (2j+1)\ell$ represent minima.

We can e.g. take the region of $x = \ell$, and write the force

$$F = \frac{\pi U_0}{2\ell} \sin\left(\frac{\pi}{\ell} x\right) = \frac{\pi U_0}{2\ell} \sin\left(\frac{\pi}{\ell}(x-\ell) + \pi\right)$$

$$= -\frac{\pi U_0}{2\ell} \sin\left(\frac{\pi}{\ell}(x-\ell)\right)$$



$$\simeq -\frac{\pi U_0}{2\ell} \times \frac{\pi}{\ell}(x-\ell) = -\frac{\pi^2 U_0}{2\ell^2}(x-\ell)$$

(c) $$m\frac{d^2x}{dt^2} = -\frac{\pi^2 U_0}{2\ell^2}(x-\ell)$$

$$\frac{d^2}{dt^2}(x-\ell) = -\frac{\pi^2 U_0}{2\ell^2 m}(x-\ell)$$

$$x = \ell + A\cos(\omega_0 t + \delta)$$

$$\omega_0 = \frac{\pi}{\ell}\sqrt{\frac{u_0}{2m}}$$

$$\tau_0 = \frac{2\pi}{\omega_0} = \frac{2\cancel{\pi}}{\cancel{\pi}}\,\ell\sqrt{\frac{2m}{u_0}} = 2\ell\sqrt{\frac{2m}{u_0}}$$

There is no dependence on amplitude A

(d) Percentage error

$$\left| \frac{\sin\left(\frac{\pi}{\ell}\,0.1\cdot\ell\right) - \frac{\pi}{\ell}\,0.1\ell}{\sin\left(\frac{\pi}{\ell}\,0.1\cdot\ell\right)} \right|$$

$$= \left| 1 - \frac{0.1\pi}{\sin(0.1\pi)} \right| = 1.7\%$$

Since the error in the force does not exceed 1.7%, I expect the error in the period to be at most of such magnitude and presumably less, as for $|x| < A$ the error is smaller.

τ needs to be an even function of A, so expanded τ will contain only even powers of (A/ℓ):

$$\tau = \tau_0 + c\,(A/\ell)^2 + d\,(A/\ell)^4 + \dots$$

With this, the anticipated error is of the order of $(A/\ell)^2 = (0.1)^2 = 1\%$ for the case in question.

Damped HO

(a) $\ddot{x} + 2\beta\dot{x} + w_0^2 x = 0$

$x = e^{-\beta t} f$

$\dot{x} = -\beta e^{-\beta t} f + e^{-\beta t}\dot{f}$

$\ddot{x} = \beta^2 e^{-\beta t} f - 2\beta e^{-\beta t}\dot{f} + e^{-\beta t}\ddot{f}$

$\beta^2 e^{-\beta t} f \cancel{- 2\beta e^{-\beta t}\dot{f}} + e^{-\beta t}\ddot{f}$

$-2\beta^2 e^{-\beta t} f \cancel{+ 2\beta e^{-\beta t}\dot{f}} + w_0^2 e^{-\beta t} f = 0$

$\ddot{f} + (w_0^2 - \beta^2) f = 0$

(b) $w_0 > \beta \Rightarrow f = A\cos(\omega_1 t - \delta)$

A, δ - constants $\omega_1 = \sqrt{w_0^2 - \beta^2}$

$w_0 < \beta \Rightarrow f = A\cosh(\omega_2 t - \delta)$

$A - \delta$ constants $\omega_2 = \sqrt{\beta^2 - w_0^2}$

$w_0 = \beta$ $\ddot{f} = 0 \Rightarrow f = A + Bt$

(c) $w_0 > \beta$ $x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$

$w_0 < \beta$ $x(t) = Ae^{-\beta t}\cosh(\omega_2 t - \delta)$

$w_0 = \beta$ $x(t) = (A + Bt)e^{-\beta t}$

$E(t)$ for damped HO

(a) $x = Ae^{-\beta t}\cos(\omega_1 t - \delta)$

$$\dot{x} = -\beta A e^{-\beta t}\cos(\omega_1 t - \delta)$$
$$- \omega_1 A e^{-\beta t}\sin(\omega_1 t - \delta)$$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$
$$= \frac{1}{2}m\beta^2A^2e^{-2\beta t}\cos^2(\omega_1 t - \delta)$$
$$+ \frac{1}{2}m\omega_1^2A^2e^{-2\beta t}\sin^2(\omega_1 t - \delta)$$
$$+ m\omega_1\beta A^2e^{-2\beta t}\sin(\omega_1 t - \delta)\cos(\omega_1 t - \delta)$$
$$+ \frac{1}{2}kA^2e^{-2\beta t}\cos^2(\omega_1 t - \delta)$$

$$\omega_1^2 = \frac{k}{m} - \beta^2$$

$$E = \frac{1}{2}m\beta^2A^2e^{-2\beta t}\left(\cos^2(\omega_1 t - \delta) - \sin^2(\omega_1 t - \delta)\right)$$
$$+ \frac{1}{2}kA^2e^{-2\beta t} + \frac{1}{2}m\omega_1\beta A^2e^{-2\beta t}\sin(2(\omega_1 t - \delta))$$

$$= \frac{1}{2}m\beta A^2e^{-2\beta t}\left[\beta\cos(2(\omega_1 t - \delta)) + \omega_1\sin(2(\omega_1 t - \delta))\right]$$
$$+ \frac{1}{2}kA^2e^{-2\beta t}$$

E(t) for damped HO

(b)

direct differentiation $\frac{dE}{dt}$

$$\frac{dE}{dt} = -2\beta E - 2\omega_1 \tfrac{1}{2} m \beta^2 A^2 e^{-2\beta t} \sin(2(\omega_1 t - \delta))$$
$$+ 2\omega_1^2 \tfrac{1}{2} m \beta A^2 e^{-2\beta t} \cos(2(\omega_1 t - \delta))$$

$$= -\beta k A^2 e^{-2\beta t}$$
$$- m \beta A^2 e^{-2\beta t} (\omega_1^2 - \beta^2) \cos(2(\omega_1 t - \delta))$$
$$- 2 m \omega_1 \beta^2 A^2 e^{-2\beta t} \sin(2(\omega_1 t - \delta))$$

$$= -\beta k A^2 e^{-2\beta t} - \beta A^2 e^{-2\beta t} (k - 2m\beta^2) \cos(2(\omega_1 t - \delta))$$
$$- 2 m \omega_1 \beta^2 A^2 e^{-2\beta t} \sin(2(\omega_1 t - \delta))$$

E(t) damped HO cont.

(c)

$$\beta = \frac{b}{2m}$$

$$\frac{dE}{dt} = -b\dot{x}^2$$

$$= -2m\beta\dot{x}^2 = -2m\beta\left(-\beta A e^{-\beta t}\cos(\omega_1 t-\delta) + \omega_1 A e^{-\beta t}\sin(\omega_1 t-\delta)\right)^2$$

$$= -2m\beta A^2 e^{-2\beta t}\left[-\beta\cos(\omega_1 t-\delta) + \omega_1 \sin(\omega_1 t-\delta)\right]^2$$

$$= -2m\beta A^2 e^{-2\beta t}\left\{\beta^2\cos^2(\omega_1 t-\delta) + \omega_1^2\sin^2(\omega_1 t-\delta) - 2\omega_1\beta\sin(\omega_1 t-\delta)\cos(\omega_1 t-\delta)\right\}$$

$$\cos^2\alpha = \frac{1}{2} + \cos^2\alpha - \frac{1}{2} = \frac{1}{2} + \frac{1}{2}(2\cos^2\alpha - 1)$$

$$= \frac{1}{2} + \frac{1}{2}\cos 2\alpha$$

$$\sin^2\alpha = 1 - \cos^2\alpha = \frac{1}{2} - \frac{1}{2}\cos 2\alpha$$

$$\frac{dE}{dt} = -2m\beta A^2 e^{-2\beta t}\left\{\frac{1}{2}\beta^2 + \frac{1}{2}\beta^2\cos(2(\omega_1 t-\delta)) + \frac{1}{2}\omega_1^2 - \frac{1}{2}\omega_1^2\cos(2(\omega_1 t-\delta)) - \omega_1\beta\sin(2(\omega_1 t-\delta))\right\}$$

$$= -k\beta A^2 e^{-2\beta t} + (k - 2m\beta)\beta A^2 e^{-2\beta t}\cos(2(\omega_1 t-\delta)) + 2m\omega_1\beta^2 A^2 e^{-2\beta t}\sin(2(\omega_1 t-\delta))$$

damped HO cont

(d) Under averaging over the period, $e^{-2\beta t}$ can be taken as constant when $\beta T_1 \ll 1$

On the other hand $\langle \cos(2(\omega_1 t - \delta)) \rangle = 0$

and $\langle \sin(2(\omega_1 t - \delta)) \rangle = 0$

With this

$$\langle E \rangle(t) = \frac{1}{2} k A^2 e^{-2\beta t}$$

(e) $$\left\langle \frac{dE}{dt} \right\rangle(t) = -\beta k A^2 e^{-2\beta t}$$

(f) $$D(t) = A e^{-\beta t} \qquad \frac{dD}{dt} = -\beta D(t)$$

$$\langle E \rangle(t) = \langle E \rangle(t=0)\, e^{-2\beta t}$$

$$\frac{d\langle E \rangle}{dt} = -2\beta \langle E \rangle(t)$$

Average energy decreases twice as fast as amplitude.