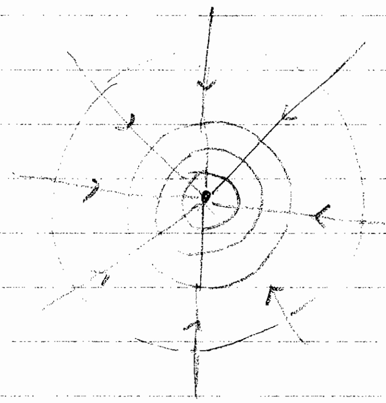
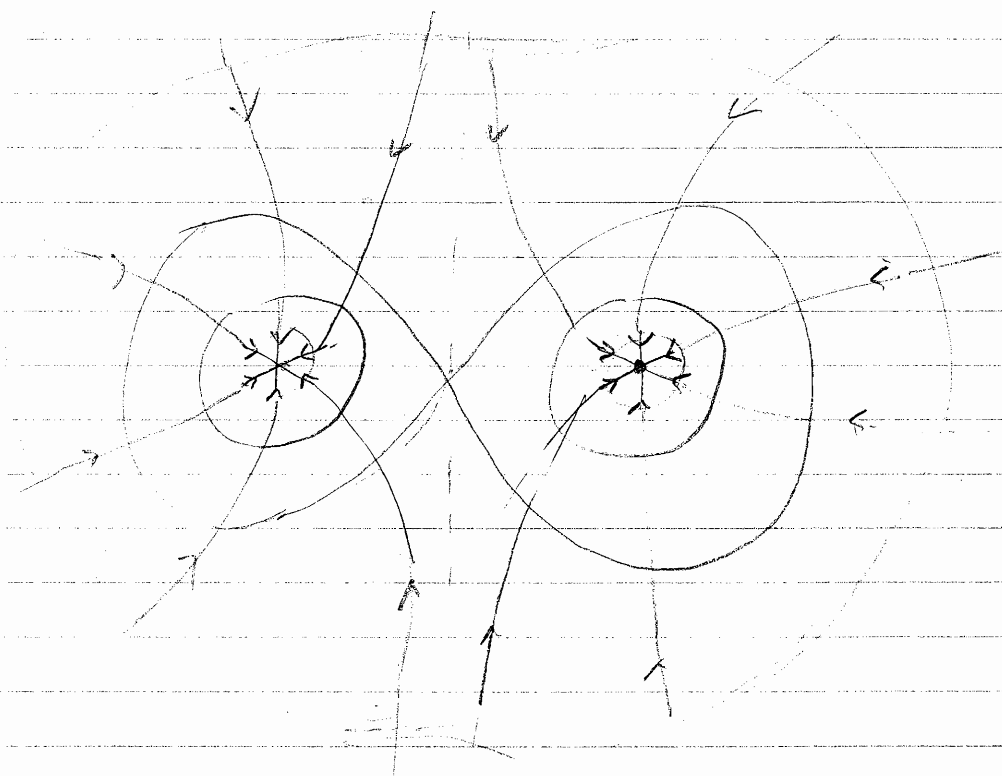
Lines & surfaces

(a) $g(r) = -G\frac{M}{r^2}$ $\Phi(r) = -\frac{GM}{r}$

$-n\Phi_0 = -\frac{GM}{r} \Rightarrow r = \frac{r_0}{n}$



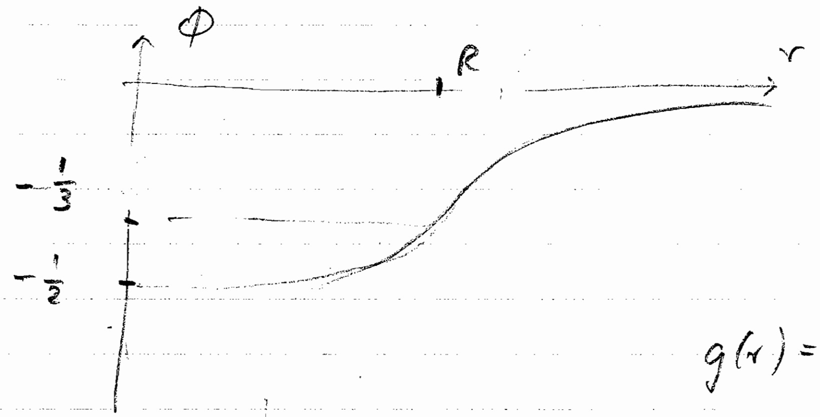
(b)



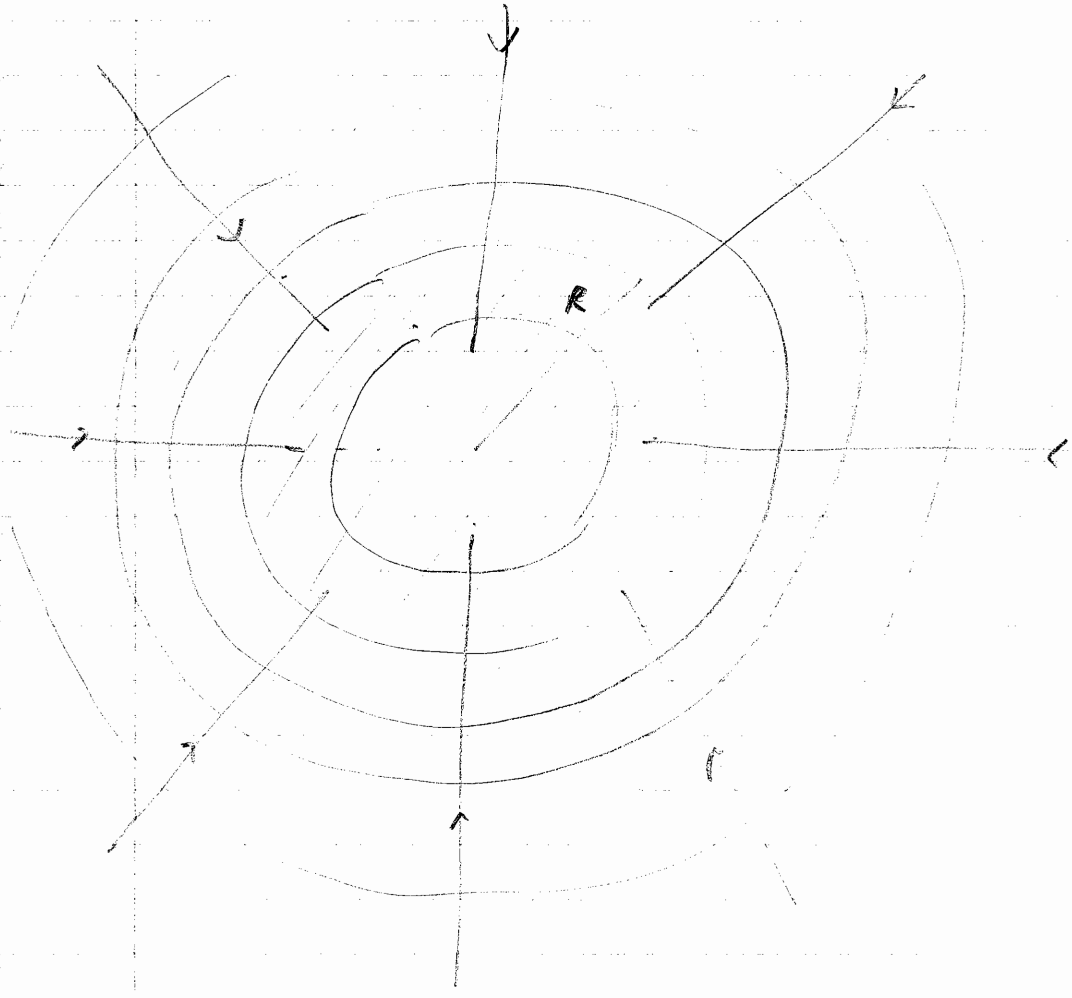
Lines & surfaces cont.

(c) $\phi(r) = -4\pi\rho G\left(\frac{R^2}{2} - \frac{r^2}{6}\right) \qquad r \le R$

$\phi(r) = -G\,\frac{4\pi}{3}R^3\,\frac{1}{r}$



$$g(r) = \begin{cases} -\frac{4\pi}{3}\rho G R & r \le R \\ -\frac{4}{3}\pi\rho G R^3 \frac{1}{r^2} & r > R \end{cases}$$



From g to ρ

(a) 'Gauss' law

$$4\pi r^2 g(r) = -4\pi G \cdot 4\pi \int_0^r r'^2 \rho(r')\, dr'$$

$$r^2 g(r) = -4\pi G \int_0^r r'^2 \rho(r')\, dr'$$

Both sides differentiated with respect to r

$$\frac{d}{dr}(r^2 g) = -4\pi G r^2 \rho(r)$$

(b) $g(r) = -g_0$

$$-2 r g_0 = -4\pi G r^2 \rho(r)$$

$$\Rightarrow \rho(r) = \frac{g_0}{2\pi G}\frac{1}{r}$$

(c)

$$M = 4\pi \int_0^R r^2 \frac{g_0}{2\pi G}\frac{1}{r}\, dr = \frac{2g_0}{G}\frac{1}{2}R^2 = \frac{g_0 R^2}{G}$$

(d) $$g = -\frac{GM}{r^2} = -\frac{G g_0 R^2}{G r^2} = -\frac{g_0 R^2}{r^2}$$

When $r \to R$, then $g \to -g_0 \Rightarrow$ continuous

Escape velocity

$$0 = \frac{m\upsilon^2}{2} - \frac{G\cancel{m}M}{R} \Rightarrow \upsilon = \sqrt{\frac{2GM}{R}}$$

$G = 6.673 \times 10^{-11}\ Nm^2/kg^2$

(a) $R_E = 6.38 \times 10^6\ m$ $\quad M = 5.97 \times 10^{24}\ kg$

$\upsilon = 11{,}175\ m/s$ Earth

(b) $R_M = 1.74 \times 10^6\ m$ $\quad M = 7.35 \times 10^{22}\ kg$ Moon

$\upsilon = 2{,}374\ m/s$

(c) $R_M = 3.40 \times 10^6\ m$ $\quad M = 6.42 \times 10^{23}\ kg$ Mars

$\upsilon = 5{,}020\ m/s$

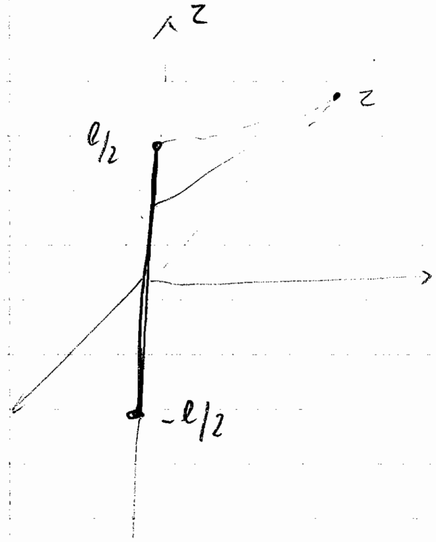
(d) $R_J = 7.15 \times 10^7\ m$ $\quad M = 1.90 \times 10^{27}\ kg$ Jupiter

$\upsilon = 59{,}550\ m/s$

(e) $\upsilon = \sqrt{2G\left(\frac{M_E}{R_E} + \frac{M_S}{R_{E\text{-}S}}\right)} = 43{,}591\ m/s$

$M_S = 1.99 \times 10^{30}\ kg$ $\quad R_{E\text{-}S} = 1.496 \times 10^{11}\ m$

Rod



(a) $$\phi(r, z) = -\frac{GM}{l} \int_{-l/2}^{l/2} dz' \, \frac{1}{\sqrt{r^2 + (z'-z)^2}}$$

$$= -\frac{GM}{l} \ln\left(z' - z + \sqrt{(z'-z)^2 + r^2} \right) \Bigg|_{-l/2}^{l/2}$$

$$= -\frac{GM}{l} \ln \frac{\sqrt{\left(\frac{l}{2} - z\right)^2 + r^2} + \frac{l}{2} - z}{\sqrt{\left(\frac{l}{2} + z\right)^2 + r^2} - \frac{l}{2} - z}$$

Rod cont

$\ell \to 0$

$$\phi \simeq -\frac{GM}{\ell}\ln\frac{\sqrt{r^2+z^2-\ell z}+\frac{\ell}{2}-z}{\sqrt{r^2+z^2+\ell z}-\frac{\ell}{2}-z}$$

$$\simeq -\frac{GM}{\ell}\ln\frac{\sqrt{r^2+z^2}\left(1-\frac{\ell z}{2(r^2+z^2)}\right)+\frac{\ell}{2}-z}{\sqrt{r^2+z^2}\left(1+\frac{\ell z}{2(r^2+z^2)}\right)-\frac{\ell}{2}-z}$$

$$= -\frac{GM}{\ell}\ln\frac{\sqrt{r^2+z^2}-z-\frac{\ell z}{2\sqrt{r^2+z^2}}+\frac{\ell}{2}}{\sqrt{r^2+z^2}-z+\frac{\ell z}{2\sqrt{r^2+z^2}}-\frac{\ell}{2}}$$

$$= -\frac{GM}{\ell}\ln\frac{1+\frac{\ell}{2\sqrt{r^2+z^2}-z}\left(1-\frac{z}{\sqrt{r^2+z^2}}\right)}{1-\frac{\ell}{2\sqrt{r^2+z^2}-z}\left(1-\frac{z}{\sqrt{r^2+z^2}}\right)}$$

$$\simeq -\frac{GM}{\ell}\ln\frac{1+\frac{\ell}{2\sqrt{r^2+z^2}}}{1-\frac{\ell}{2\sqrt{r^2+z^2}}}$$

$$\simeq -\frac{GM}{\ell}\frac{\ell}{\sqrt{r^2+z^2}} = -\frac{GM}{\sqrt{r^2+z^2}}$$ OK

Rod cont

(c) $r \to 0$

$$\phi \simeq -\frac{GM}{\ell} \ln \frac{\left(\frac{\ell}{2}-z\right)\left(1+\frac{r^2}{2\left(\frac{\ell}{2}-z\right)^2}\right)+\frac{\ell}{2}-z}{\left(\frac{\ell}{2}+z\right)\left(1+\frac{r^2}{2\left(\frac{\ell}{2}+z\right)^2}\right)-\frac{\ell}{2}-z}$$

$$= -\frac{GM}{\ell} \ln \left\{ \frac{\frac{\ell}{2}-z}{\frac{\ell}{2}+z} \quad \frac{2+\frac{r^2}{2\left(\frac{\ell}{2}-z\right)^2}}{\frac{r^2}{2\left(\frac{\ell}{2}+z\right)^2}} \right\}$$

$$\simeq -\frac{GM}{\ell} \ln \frac{4\left(\frac{\ell^2}{4}-z^2\right)}{r^2}$$

$$= +\frac{2GM}{\ell} \ln r - \frac{GM}{\ell} \ln \left[4\left(\frac{\ell^2}{4}-z^2\right)\right]$$

$$= +\frac{2GM}{\ell} \ln r + \text{const}$$

1 Earth w/hole

(a) $$\cancel{4\pi} r^2 g = -\cancel{4\pi} G \cancel{\tfrac{4}{3}\pi} r^3 \times \frac{M}{\cancel{\tfrac{4}{3}\pi} R^3} \qquad r<R$$

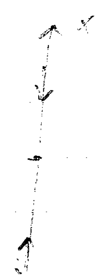
$$g = -\frac{GM}{R^3} r = -g_0 \frac{r}{R}$$

$$\cancel{4\pi} r^2 g = -\cancel{4\pi} G M$$

$$g = \frac{-GM}{r^2} = -\frac{GM}{R^2}\frac{R^2}{r^2} = -g_0 \frac{R^2}{r^2} \qquad r>R$$

$g_0 = 9.81\,m/s^2$

(b) $$\cancel{m}\ddot{x} = -g_0 \frac{x}{R}\cancel{m}$$ works for $x>0$ & $x<0$



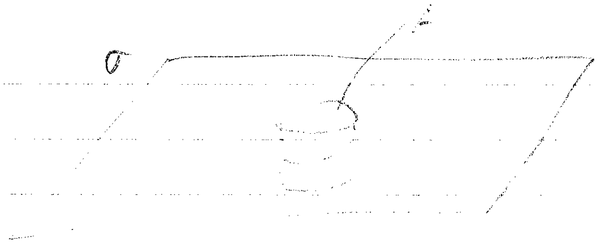
$$\ddot{x} = -\frac{g_0}{R} x$$

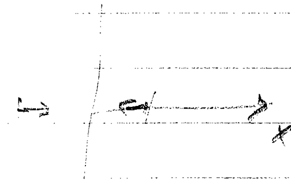
$$\omega_0^2 = \frac{g_0}{R}$$

$$\omega_0 = \sqrt{\frac{g_0}{R}}$$

(c) $$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{R}{g_0}} = 2\pi\sqrt{\frac{6.38\times10^6}{9.81}} = 5067\,s$$

$= 84\,min\ 27s = 1h\ 24min\ 27s$





$$2\cancel{A}g = 4\pi G\cancel{A}\sigma$$

(a) $g = 2\pi G\sigma$ ← constant value

(b) $F = gM = 2\pi G\sigma M$

(c) same in magnitude $F = 2\pi G\sigma M$