Continuum Shell Model for p-shell Nuclei

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I will describe two computer codes and their applications to continuum phenomena. The codes are bases on the recoil corrected continuum shell model (RCCSM), originally formulated by Philpott. The formalism allows one nucleon in the continuum, an effective $NN$ interaction of any form, structure as complicated as one wishes, and allows one to move relatively easily from one nucleus to the next, and yet be translationally invariant.

The RCCSM employs an $R$-matrix formalism for convenience in calculating wave functions and cross sections. Therefore, the first task is to solve the dynamical $R$-matrix equations of Lane and Robson. These equations have been written in the form

$$
\sum_{\lambda \neq I} \langle \lambda | H - E^| \lambda \rangle \sum_{\epsilon} \gamma_{\epsilon \lambda}^* (b_{\lambda \epsilon} - b_{\epsilon \lambda}^*) \gamma_{\epsilon \lambda} | \lambda \rangle = 0,
$$

where $H$ is the Hamiltonian,

$$
H = \sum_{i=1}^{A} p_i^2 / 2m - T_{c.m.} + \sum_{i<j}^{A} v_{ij} + \sum_{i<j<k}^{A} v_{ijk}.
$$

Equ. (1) may be regarded as an equation to determine the energies and wave functions of bound states or as an equation to determine the continuum wave function, given the energy.

The expansion basis for a channel wave function is the set of harmonic oscillator wave functions with oscillator size parameter, $\nu_0 = m o l / h$. The set $| \lambda \rangle$ consists of two types of states. The $\alpha$-states consist of a core state $| J, \alpha \rangle$, which is a nonspurious shell model state of the $A - 1$ system, coupled to a single particle state to a good total angular momentum, $[a_{n lj} \otimes | J, \alpha \rangle]$. The $\beta$-states are the nonspurious shell model states of the $A$-particle system $| J, \beta \rangle$. The core states and the $\beta$-states have negligible probability outside the channel radius, $a_c$. The $\alpha$-states can extend to the channel radius, and therefore, have non-zero reduced widths. The $\beta$-states can have energies above particle threshold and are sometimes referred to as bound states in the continuum. They eventually acquire widths due to their coupling to $\alpha$-states.

The RCCSM calculation begins by choosing the core states and the $\beta$-states that are considered important. These states could be generated by standard shell model codes. One then calculates the matrix elements of the Hamiltonian in Eq. (2) and the matrix elements of the unit operator (the overlap matrix) in the basis of $\alpha$-states. This basis must include all $n lj \tau$ such that $2n + l \leq \rho_{\text{max}}$, where $\rho_{\text{max}}$ is chosen to be large enough to provide a good representation of the
interior wave function. One must also calculate the matrix elements between $\alpha$-states and $\beta$-states. If one wishes to calculate transition rates, then the reduced matrix elements of the corresponding operator would be calculated, but the operator must be translationally invariant.

Matrix elements are then transformed to a channel spin basis, $|\alpha SnlJ_B\rangle$, and then transformed from the shell model coordinate, $\phi_{nl}(r_0)$, to the coordinate connecting the center of mass of the $A-1$ system to the last particle, $\Psi_{n}\hat{r}(r)$. This transformation is illustrated in Fig. 1. The transformations are given by

\[
\langle \text{int} | \beta J_B | \beta' J_B' \rangle_{\text{int}} = \langle \text{SM} | \beta J_B | \beta' J_B' \rangle_{\text{SM}},
\]

\[
\langle \text{int} | \alpha SnlJ_B | \beta' J_B' \rangle_{\text{int}} = (1 + A_{\text{c}}^{-1})^{(p/2)} \langle \text{SM} | \alpha SnlJ_B | \beta' J_B' \rangle_{\text{SM}},
\]

\[
\langle \text{int} | \alpha SnlJ_B | \alpha' Snl'J_{B'} \rangle_{\text{int}} = (1 + A_{\text{c}}^{-1})^{(p+\rho'/2)} \{ \langle \text{SM} | \alpha SnlJ_B | \alpha' Snl'J_{B'} \rangle_{\text{SM}}
\]

\[
- \sum_{n \ell m} \langle \text{int} J_{B}' | (\text{coeff}) \rangle \langle \text{int} | \alpha SnlJ_B | \alpha' Snl'J_{B'} \rangle_{\text{int}} \}
\]

where

\[
(\text{coeff}) = \langle \hat{J}_{B}' | \hat{J}_{B}' \hat{I}' \hat{l}' \sum_{NL \neq 0} W(S\hat{I}J_{B}L; \hat{J}_{B}L')W(S\hat{I}'J_{B}'L; \hat{J}_{B}'L')W(\hat{J}_{B}kLJ_{B}'; \hat{l}') \times \langle \hat{n}\hat{I}, NL, l | n\ell, 000, l \rangle \langle \hat{n}\hat{I}', NL, l' | n\ell', 00, l' \rangle.
\]

Fig. 1 The RCCSM coordinates. Relative to a fixed origin, $R_C$ locates the center of mass of the core, $R$ locates the center of mass of the composite system, and $r_0$ locates the continuum nucleon. Relative to the center of mass of the core, $r$ locates the continuum nucleon.
The angular brackets in Eq. (6) are the unequal-mass Talmi-Moshinsky brackets, $[J] = 2J + 1$, $\hat{J} = (2J + 1)^{1/2}$, $\rho = 2n + l$, $A_0 = A - 1$, and the reduced matrix elements are as defined in Ref. 1. Eq. (5) is a recursion relation for the intrinsic matrix elements in terms previously calculated intrinsic matrix elements of lesser energy. One may consider the single particle states $\phi_{nlj}$ and $\psi_{nlj}$ as creation operators since the antisymmetrizer operator is a scalar, acting on internal coordinates, and may be inserted where necessary. The transformation guarantees that the center of mass remains in a $0s$-state.

**The p-shell**

The development of the $p$-shell code$^4$ allows investigation of many reactions of astrophysical interest. Also, with the recent emphasis on rare isotopes, the codes can provide theoretical guidance in analyzing data and making predictions as to which reactions are feasible.

The $p$-shell codes include all $p$-shell states of the $A$-particle system and allow as many core states of the $A-1$ systems as one feels appropriate. Again any realistic two-body interaction is allowed. However, in addition, a Skryme interaction$^5$ of the form $v(r_1, r_2, r_3) = t_3 \delta(r_1-r_2) \delta(r_2-r_3)$ may also be included.

A channel wave function may be written in the form

$$\phi_{c^s}^f = \sum_c r^{-1} u_{c^s}^{(+)}(r) |\alpha'j'lj^fJ^f\rangle,$$

where $c'$ stands for $\alpha'j'lj^f$ and

$$u_{c}^{(+)}(r) \rightarrow (v_c / v_{c'})^{1/2} (I_c \delta_{cc'} - O_{c'} S_{cc'}).$$

for open channels and is proportional to a Whittaker function, $W_{-\eta, J+1/2}(2k_c r)$, for closed channels. Inside the channel radius both bound $\phi_{ls}$ and unbound $\phi_{c^s}$ may be expanded in oscillator wave functions as

$$\psi = \sum_{J_{\alpha(l\neq0)}} f_{nlj, \alpha} [a_{nlj}^+ \otimes \phi_{J^f}^f)]^{ls} + \sum_{\beta} d_{\beta} |\beta J^f\rangle.$$
\[ \langle J_A | \sum_{i} a_{i\tau}^+ a_{i\tau} \rangle^T, \langle J_A | \sum_{i} (a_{i\tau}^+ a_{i\tau})^T (a_{\tau}^+ a_{\tau})^T \rangle^T, \langle J_A | \sum_{i} a_{i\tau}^+ a_{i\tau} \rangle^T, \langle J_A | \sum_{i} (a_{i\tau}^+ a_{i\tau})^T (a_{\tau}^+ a_{\tau})^T \rangle^T \}\]

If the Skyrme interaction is included the one must also supply the three-body densities and two-particle overlaps,

\[ \langle J_A | \sum_{i} (a_{i\tau}^+ a_{i\tau})^T (a_{\tau}^+ a_{\tau})^T \rangle^T, \langle J_A | \sum_{i} (a_{i\tau}^+ a_{i\tau})^T (a_{\tau}^+ a_{\tau})^T \rangle^T \}\]

Nucleon scattering can be divided into two regimes, the direct reaction region and the resonance region. In the direct reaction region, one can explore the transition densities between states of the \( A-1 \) core. An example is the composite system \( ^7 \text{Be} \). Previously, calculations have been made by coupling one or two channels with potentials obtained by folding realistic interactions. With the RCCSM one can include as many channels as one wishes and solve the translationally invariant, microscopic problem with all exchanges included properly. A demonstration of the effects of channel coupling on medium energy nucleon scattering is shown in Fig. 2 for the \( ^6 \text{Li}(p,p)^6 \text{Li} \) reaction with the M3Y interaction along with the data of Ref. 7. The dashed line is for including only the \( ^6 \text{Li} \) ground state and the solid line includes all 15 \( ^6 \text{Li} \) and \( ^6 \text{Be} \) \( p \)-shell states, even though most are particle unstable. This figure differs at back angles from the equivalent figure in Ref. 4 due to correction of a sign error in a kinetic energy term. The channel coupling does improve the elastic cross section and analyzing power. However, the low

\[ \theta_p(\text{deg}) \]

\[ d\sigma/d\Omega \text{ (mb/sr)} \]

\[ \begin{array}{c}
\begin{array}{c}
E_p=49.75 \text{ MeV} \\
^6 \text{Li}(p,p)^6 \text{Li}
\end{array}
\end{array} \]

Fig. 2 Elastic scattering of protons on \( ^6 \text{Li} \). Data is from Ref. 39. Solid line is RCCSM calculation with M3Y and 15 core states. Dashed line had only \( ^6 \text{Li} \) ground state.
energy cross sections tend to favor only a few channels. Apparently the inclusion of the particle-unstable core states mimics the physical channels omitted from the calculations, such a nucleon knockout.

In Ref. 8 it was determined that calculations for nucleon scattering from the lithium isotopes could be improved by replacing the central components of M3Y with a density-dependent interaction. This cannot be done with the RCCSM because the interaction would not be translationally invariant. However, the effect can be obtained by including the Skyrme interaction mentioned at the beginning of this section. The effect is demonstrated in Fig. 3 where the \(^{7}\text{Li}(p,p)^{7}\text{Li}, 49.75\text{ MeV}\) cross section is plotted when including only the ground state. The dashed lines are for M3Y; the dot-dashed lines are for M3Y plus Skyrme with \(t_3 = -5000\ \text{MeV-fm}^6\); the solid lines are for M3Y plus Skyrme with \(t_3 = +5000\ \text{MeV-fm}^6\). The \(t_3 = +5000\ \text{MeV-fm}^6\) result which corresponds correctly to decreasing the interaction in the interior, provides the best agreement with the data of Ref. 7.

In the direct reaction region, one can test not only the transition densities, but also components of the effective interaction. The reaction \(^{6}\text{Li}(p,n)^{6}\text{Be}(0^+)\) is sensitive to the \(\sigma\sigma\tau\tau\) component of the interaction. In Fig. 4 one sees that the calculated cross section is too large. Since the \(B(M1)\) for \(1^+ \rightarrow 0^+\) in \(^{6}\text{Li}\) agrees with experiment, one concludes that the

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**Fig. 3** Elastic scattering of protons on \(^{7}\text{Li}\). Data is from Ref. 7. Dashed line is RCCSM calculation with M3Y and 1 core states. Solid line includes Skyrme with \(t_3 = +5000\ \text{MeV-fm}^6\). Dot-dashed line includes Skyrme with \(t_3 = -5000\ \text{MeV-fm}^6\).
Fig. 4. Cross section for the $^6\text{Li}(p,n)^6\text{Be}(0^+)$ reaction. Solid line is RCCSM calculation with 5 core states and M3Y.

The $\sigma\cdot\sigma\cdot\tau\cdot\tau$ component of M3Y is too large. This result persists throughout the shell.

In Fig. 5 one sees the cross section for $^7\text{Li}(p,n)^7\text{Be}(0^+)$ reaction. This reaction is sensitive to the $\tau\cdot\tau\cdot\tau$ component of the interaction. The solid line is a calculation that includes 6 core states and employs the M3Y interaction. The forward scattering cross section has a magnitude close to the experimental value, indicating that the $\tau\cdot\tau\cdot\tau$ component is approximately correct. This conclusion is reinforced by the dashes line which includes the Skyrme interaction.

Fig. 5. Cross section for the $^7\text{Li}(p,n)^7\text{Be}(3/2^-)$ reaction. Solid line is RCCSM calculation with 6 core states and M3Y. Dashes line includes Skyrme with $t_3 = +5000$ MeV-fm$^6$. 

\begin{align*}
\sigma\cdot\sigma\cdot\tau\cdot\tau & \text{ component of M3Y is too large. This result persists throughout the shell.}
\end{align*}

\begin{align*}
\text{In Fig. 5 one sees the cross section for }^7\text{Li}(p,n)^7\text{Be}(0^+). \text{ This reaction is sensitive to the } \tau\cdot\tau\cdot\tau \text{ component of the interaction. The solid line is a calculation that includes 6 core states and employs the M3Y interaction. The forward scattering cross section has a magnitude close to the experimental value, indicating that the } \tau\cdot\tau\cdot\tau \text{ component is approximately correct. This conclusion is reinforced by the dashes line which includes the Skyrme interaction.}
\end{align*}
An example of a calculation in the resonance region is one that has assisted in sorting out the structure in $^8$B. In Fig. 6 are plotted the $^7$Be $+p$, 148° excitation function and the data of Ref. 10. The calculation includes only the nucleon emission stable 3/2$^-$, 1/2$^-$, and 7/2$^-$ core states of $^7$Be. The theoretical curve is divided into its $J^G$ constituents. The calculation predicts 1$^+$, 3$^+$, and 2$^+$ resonances at $E^c.m. = 1.8$, 2.2, and 2.9 MeV, respectively. The overlapping resonances make it difficult to confirm the assignments from the data, but the RCCSM provides other information that can be used to test its prediction. Because a realistic interaction is employed, one has just as much confidence in the predictions for the inelastic cross section and the analyzing power. These are shown in Figs. 7a and 7b. A measurement of either would provide confirmation of the RCCSM spin assignments.

The General RCCSM

A set of codes have recently been constructed that will allow one to include any non-spurious states of the core and any non-spurious $\beta$-states. One must provide the one, two, and three-body densities for the core states, $\langle J_A\alpha | \{a^+_{i_s} a_{i_t}\}^t | J'_A\alpha \rangle$, $\langle J_A\alpha | \{(a^+_{i_s} a_{i_t})^t, (a_m a_n)^t\}_m^t | J'_A\alpha \rangle$ and $\langle J_A\alpha | \{(a^+_{i_s} a_{i_t})^t, a_{i_t}' \}^t | \{(a_m a_n)^t, a_{p_r}, a_{p_r}' \}^t | J'_A\alpha \rangle$. If
Fig. 7 (a) Inelastic excitation function for $^{7}\text{Be} + p$ to $^{7}\text{Be}(1/2^-)$ at $136^\circ$. Curves are from the calculation, and individual contributions are as marked. (b) Calculated analyzing power for $^{7}\text{Be} + p$ at $148^\circ$.

$\beta$-states are allowed, then one must provide the overlaps, $\langle J_B \beta | |J_A \alpha \rangle | |^s$,

$\langle J_B \beta | |[(a_i^+ a_i^+)_{i_j} J_{\alpha \tau} \alpha_i] | |J_A \alpha \rangle$, and $\langle J_B \beta | |[(a_i^+ a_i^+)_{i_j} J_{\alpha \tau} \alpha_i] | |J_A \alpha \rangle$. The codes presently allow only a two-body interaction.

The first application of this general RCCSM was to the $A = 4$ system, for which the $1p-1h$ approximation provides excellent agreement with almost all experiments involving one nucleon in the continuum. Two exceptions where elastic scattering above 60 MeV, and the convection current appeared to be slightly low at low energy and momentum transfer in $(e,e'p)$, while the Coulomb multipoles seemed too strong. These characteristics are demonstrated in a Photodisintegration calculation in Figs. 8 and 9 where the solid line represents the $1p-1h$ ($N = 0$) result for $^4\text{He}(\gamma,p)^3\text{H}$ and $^4\text{He}(p,p)^3\text{He}$. Also shown in these figures are the data of Refs. 12-17.

These two processes were investigated by expanding the model space for the $^3\text{He}$ and $^3\text{H}$ cores in an effort to determine if an improved core structure could provide better agreement with data. Calculations were performed in $N = 0, 2, 4$, and $6-\hbar\omega$ model spaces. These calculations are shown as solid, dotted, dashes, and dot-dashed lines, respectively. The calculations for $^4\text{He}(\gamma,p)^3\text{H}$ include only the dominate, spin-independent E1 contribution, but the upper set of lines in Fig. 8 is calculated with the E1 current operator, $Q_{10} = 2 \beta(i\hbar)p_z(t_3)$, and the lower set of lines is calculated with the standard Coulomb effective operator, $(Q_{10})_{\text{eff}} = e\zeta(t_3)$, obtained by applying the continuity equation to the matrix element of $p$. Here, $k = (E_f - E_i)/\hbar c$, $\beta = e\hbar/2mc$, and $t_3(p,n) = (-1/2, +1/2)$. The upper and lower sets of curves differ by much more than the few percent.
Fig. 8 The $^4$He ($\gamma,p$)$^3$H total cross section. Open circles, crosses, squares, diamonds, ×’s, and solid diamonds are the data of Refs. 42, 43, 44, 45, 46, and 47, respectively. Solid, dotted, dashes, and dot-dashed lines are for $N = 0, 2, 4,$ and 6, respectively. Upper curves are with Coulomb operator; lower curves are with transverse current operator.

one would expect from gauge invariant terms or exchange terms. However, the calculations with $Q_{10}$ are rising with $N$ and the calculations with $(Q_{10})_{\text{eff}}$ are falling with $N$, bringing the curves closer together and, hence, doing a better job of satisfying the continuity equation. This is good news for electron scattering calculations in that it shows that considerable progress toward current conservation can be made by improving the core wave functions. Also, given the disparity among data sets, one could say that the calculation is consistent with the data.

The calculated, medium energy elastic scattering cross sections are also improved as shown in Fig. 9. At 85 MeV the minimum at 120° begins to fill nicely as $N$ increases. Even better agreement can be obtained by switching to the Reid Soft Core $g$-matrix interaction of Ref. 18. This interaction gives low energy results which are very similar to M3Y, and as seen in Fig. 9 as the short-dashed line, provides better medium energy results. Therefore, part of the disagreement between the original $1p-1h$ calculation and the 85 MeV data was due to the simple structure and part due to the interaction.

Many applications for the general RCCSM lie ahead. It can be used to connect the large $h\omega$ shell model calculations for few particle systems to the continuum, and it will be ideal for core
states of light nuclei which require contributions from the $s$-$d$ shell or excitations out of the $0s$-shell.

![Graph of the 85 MeV cross section for $^3\text{He}(p,p)^3\text{He}$](image)

**Fig. 9** The 85 MeV cross section for $^3\text{He}(p,p)^3\text{He}$. Data are from Ref. 19. Solid, dashed, dotdashed, and dotted lines are for $N = 0, 4,$ and 6 from the M3Y potential and $N = 6$ for the effective potential derived from the Reid Soft Core.

**Conclusion**

Development of realistic continuum shell models will become increasingly important as more and better rare isotope beams become available. These models will provide a means of interpreting the continuum structure and provide predictions for future experiments. The ultimate goal is to have the continuum shell model on the same level of sophistication as the bound state shell model in the sense that complicated configurations may be included and realistic effective interactions, consistent with the model space are employed. The RCCSM appears to have made reasonable progress in achieving that parity.

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**References**
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