Many-Body Problems with Low Momentum Nuclear Interactions

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Conventional Nuclear Many-Body Problem

\[ H = \sum_{i=1}^{A} T_i + \sum_{i<j}^{A} V_{ij} + \sum_{i<j<k}^{A} V_{ijk} \]

- fit Point-like non-relativistic nucleons
- \( V_{NN} \) models constrained to:
  - \( \delta_{NN}(E) \) (\( E_{lab} < 300-350 \) MeV) \( \chi^2 \sim 1 \)
  - \( E_D = -2.2246 \) MeV
- Calculate nuclear structure using MBT

But 2 complications immediately arise…
Handling the Hard Core

Conventional approach:
- re-sum $V_{NN}$ vertices into Brueckner $G$-matrices

\[ G(\omega) = V_{NN} + V_{NN} \frac{Q_{pp}}{\omega - H_0} G(\omega), \quad Q_{pp} = \sum_{n,n' > k_f} |nn'\rangle \langle nn'| \]

Annoying features of $G$
- $G$ parametrically depends on COM momentum
- $G$ is energy dependent $\Rightarrow$ “starting energies” (spectator dependence), additional self-consistency requirements
- $G$ must be re-calculated for different mass regions
- Double counting issues

Lots of work to tame strong high $k$ ($\sim$ GeV) components that are not constrained by low $E$ observables!
Model Dependence in Nuclear MBT

- **QCD non-perturbative at low energies:**
  
  No “unique” $V_{NN}$ in the low $E$ limit of QCD
  (no “coulomb’s law” as in electron many-body systems)

- **$V_{NN}$ models based on meson exchange/phenomenology**
  
  - Model Dependent at mid-to-short distances
  
  - form factors, treatment of the repulsive core
  
  - $2\pi$ physics (fictitious $\sigma$, dispersion theory, QFT)
  
  - Model Independent 1 $\pi$ tail

same deuteron/phase shifts $E_{lab} < 350$ MeV $\Rightarrow$ separation of scales
<table>
<thead>
<tr>
<th>model</th>
<th>$V_{\text{long}}$ (~2 fm)</th>
<th>$V_{\text{mid}}$ (~1 fm)</th>
<th>$V_{\text{short}}$ (~.5 fm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Paris</td>
<td>$1\pi$</td>
<td>$2\pi$ (dispersion theory)</td>
<td>OBE ($\omega$), sharp cutoff at ~ .8 fm</td>
</tr>
<tr>
<td>AV-18</td>
<td>$1\pi$</td>
<td>$1\pi$ “squared”</td>
<td>Woods-Saxon, exponential FF</td>
</tr>
<tr>
<td>Nijmegen 1</td>
<td>$1\pi$</td>
<td>Local OBE ($\sigma$)</td>
<td>Local OBE ($\delta, \rho, \omega$)</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>exponential FF</td>
</tr>
<tr>
<td>Nijmegen 2</td>
<td>$1\pi$</td>
<td>Non-local OBE ($\sigma$)</td>
<td>Non-local OBE ($\delta, \rho, \omega$)</td>
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<tr>
<td></td>
<td></td>
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<td>exponential FF</td>
</tr>
<tr>
<td>cd-Bonn</td>
<td>$1\pi$</td>
<td>Non-local OBE ($\sigma$)</td>
<td>Non-local OBE, (\delta, \rho, \omega) dipole FF</td>
</tr>
</tbody>
</table>
Very different k-space matrix elements

- Different short distance treatments
  - High k modes unconstrained by low energy NN data

- Gives model-dependent many-body results
  - In-medium nucleons offshell sensitive to high k modes
Low Energy Effective Theories

Main Idea: low E processes insensitive to short distance dynamics (separation of scales)

- Trade complicated (or unknown) dynamics for simple effective interactions
  - e.g., Effective Range Theory

\[
T(k) \approx \frac{1}{\frac{1}{a} - \frac{1}{2}r_0^2}
\]

- infinitely many V(r) models give the same (a, r_0) parameters!
Building Effective Theories

- separation of scales

\[ H = H_0 + V_L + V_H \]

\[ V_L = \text{unambiguous long wavelength interaction} \]
\[ \text{(e.g., } 1-\pi \text{ exchange in } V_{NN}) \]

\[ V_H = \text{unknown or complicated high } E \text{ dynamics} \]
\[ \text{(e.g., strong repulsion at small } r \text{ in } V_{NN}) \]

- **Step 1: Impose a cutoff } \Lambda \]

- } \Lambda \text{ has physical meaning} 
  (dividing line between } V_L \text{ and } V_H \)

- } keep } V_L \text{ explicit; replace } V_H \text{ w/something simpler}
- **Full theory:** (all states summed over)

  Schematically...

  \[
  \mathcal{T}_{fi} = V_{fi} + \sum_{n=0}^{\infty} \frac{V_{fn} V_{ni}}{E_i - E_n} + \cdots
  \]

- **Effective theory:**
  - only keep well-understood low energy states that mostly probe \( V_L \) (e.g., cutoff loop momenta)

  \[
  \mathcal{T}_{fi} = V_{fi}^{\text{eff}} + \sum_{n=0}^{\Lambda} \frac{V_{fn}^{\text{eff}} V_{ni}^{\text{eff}}}{E_i - E_n} + \cdots
  \]

  \[
  V^{\text{eff}} = V_L + \delta V_{ct}
  \]

  \( \delta V_{ct} \) corrects for truncating Hilbert space and excluding \( V_H \)

  mimics the effects of \( V_H \) on low \( E \) processes
- **Step 2: Impose RG Invariance**

  - Low E observables independent of $\Lambda$

    \[
    \frac{d}{d\Lambda} T_{fi} = 0 \Rightarrow \frac{d}{d\Lambda} V^{\text{eff}} = \beta[V^{\text{eff}}(\Lambda)]
    \]

  - $V^{\text{eff}}$ must run with $\Lambda$ to preserve low E physics

- **Step 3: Scale out high E modes via RG equation**

  - Bare $V$ as large $\Lambda_0$ initial condition

  - RG evolution ‘filters’ out detailed high E dynamics that are not important for low E processes

  - Encodes detail-independent effects of high E dynamics in $V^{\text{eff}}$ that are important for low E processes
General Form of $V^{\text{eff}}$

- $V_H$ couples to high-lying intermediate states
  - highly virtual intermediate states; propagate only for short distances
- Low momentum probes “see” smeared out contact interactions

$$V^{\text{eff}} = V_L + C_0 \frac{1}{\Lambda^2} \delta^8(r) + C_1 \frac{1}{\Lambda^4} \nabla^2 \delta^8(r) + C_2 \frac{1}{\Lambda^6} \nabla \cdot \delta^8(r) \nabla$$

- $\text{ALL}$ possible counterterms consistent with symmetries (e.g., rotational invariance)
- System-specific dynamics swept into $C$’s (operator structure independent of $V_H$)

‘Model Independent’ (i.e., no dynamics assumed)
“Bottom-Up” Approach (EFT)

- $V_H$ unknown or too hard to perform exact RG decimation (e.g., QCD)

- General form of $\delta V_{ct}$ still holds (Weinberg’s “folk theorem”)

- match the $C_{2n}$’s to low E data (EFT)

- truncate $\delta V_{ct}$ (power counting) to leading terms
  - residual $\Lambda$ dependence (limited accuracy in $k/\Lambda$)
  - Non-trivial power counting (unaturally large $a >> 1/m_\pi$)

- EFT treatments not as accurate as conventional $V_{NN}$
  - Need higher order to get same accuracy
Our Approach ("Top Down")

- Start from high-precision models of $V_{NN}$ based on meson exchange and phenomenology

- Perform an exact, non-perturbative RG decimation

$$V_{NN} \ [0 \leq k \leq \infty] \xrightarrow{} V_{\text{low-k}} \ [0 \leq k \leq \Lambda]$$

- $V_{\text{low-k}}$ contains 'all' counterterms
  - no power counting difficulties (no fitting to data)
  - no residual $\Lambda$ dependence

$V_{\text{low-k}}$ gives same NN properties as "high-precision" $V_{NN}$

- No ambiguous high $k$ components to introduce model dependence in many-body calculations!

- No hardcore = No Brueckner
Calculating $V_{\text{low-k}}$ by RG equations

**Full-space half-on-shell T-matrix:**

$$T(k', k) = V_{NN}(k', k) + \int_0^\infty \frac{V_{NN}(k', p)T(p, k)}{k^2 - p^2} p^2 dp$$

$$\tan \delta(k) = -kT(k, k)$$

**Low-k effective theory:**

$$T_{\text{low-k}}(k', k) = V_{\text{low-k}}(k', k) + \int_0^\Lambda \frac{V_{\text{low-k}}(k', p)T_{\text{low-k}}(p, k)}{k^2 - p^2} p^2 dp$$

$$T_{\text{low-k}}(k', k) = T(k', k) \ \forall \ (k,k') < \Lambda$$

Integrate down to $\Lambda \ll \Lambda_0$ to construct $V_{\text{low-k}}$
Solution of RGE ($V_{\text{low-k}}$) becomes $\sim$ independent of input $V_{\text{NN}}$ model for $\Lambda \sim 2.1$ fm$^{-1}$!
Similar RG evolution in other partial waves

$^1S_0$  

$^3S_1$-$^3D_1$

$V_{\text{low-k}}$ collapses onto one “universal” curve at $\Lambda \sim 2.1 \, \text{fm}^{-1}$
Form of \((\delta V_{ct} = V_{NN} - V_{low-k})\) generated by RG

- main effect is \(\approx\) constant shift + polynomial in \(k\) (in agreement with general RG expectations)
Other partial waves ($\Lambda = 2.1 \text{ fm}^{-1}$)

Note: $\Lambda \sim 2 \text{ fm}^{-1}$ describes NN scattering up to $\sim 350 \text{ MeV}$ lab energy.
Check of $V_{\text{low-k}}$ phase shifts
Phase equivalence and the “uniqueness” of $V_{\text{low-k}}$

- Deviations from universal $V_{\text{low-k}}$ curve most pronounced for older Paris/Bonn $V_{\text{NN}}$ models

- Fit to older PSA

Phase equivalence drives collapse, along with shared $\pi$ physics
Idaho $\delta(E)$ strongly deviates from the rest $E > 100$ MeV (similarly, Idaho $V_{\text{low } k}$ deviates above this scale)

Lowering $\Lambda \sim 1 \text{ fm}^{-1}$, “universal” $V_{\text{low- } k}$ curve restored.
Summarizing so far...

- $V_{\text{low-k}}$ renormalizes to a universal interaction for $\Lambda \leq 2.1 \text{ fm}^{-1}$
  - Corresponds to scale ($E_{\text{lab}} < 350 \text{ MeV}$) of phase equivalence
  - Preserves $\delta_{NN}$ and $B_d$ ( $\Lambda$ independence is exact)
  - Driven by phase equivalence and common pion tail
  - Gives “minimal” description of NN data
    - Short distance physics $\Rightarrow$ model-independent contact terms
  - $V_{\text{low-k}}$ can serve as unique microscopic input to many-body calculations (affords several technical simplifications)
    - “soft” core $\Rightarrow$ no more Brueckner resummations
    - energy-independent
Few-body results with $V_{\text{low-k}}$
(with A. Nogga and A. Schwenk)

• No huge/crazy 3-body forces induced by truncating to low k space

• Model dependence due to $V_{\text{NN}}$ greatly reduced
  • $\Delta B_3 \sim .4$ MeV (bare $V_{\text{NN}}$)
  • $\Delta B_3 \sim .15$ MeV ($V_{\text{low-k}} \Lambda = 2$ fm$^{-1}$)

• Can tune $\Lambda$ to give experimental $B_3$ w/out 3-body $V_{ijk}$ ($\Lambda \sim 2$)
  (caution: does not mean $V_{ijk} = 0$ !)

Should look at 4-body system!
- $B_4$ and $B_3$ linearly correlated when using phenomenological $V_{NN}$ models (Tjon-Line)
- 3-body $V_{NNN}$ break this linear correlation
Tjon-line slightly broken (~same degree as $V_{ijk}$)
- No crazy 3-body forces induced
- Still need $V_{ijk}$ to get $B_4$, even if tune $\Lambda$ to get $B_3$ exactly
Three-Nucleon Force

- $V_{\text{low } k} \sim \text{“Universal” } \Lambda < 2 \text{ fm}^{-1}$
- Chiral EFT also “low-momentum” theory ($\Lambda \sim 2-2.5 \text{ fm}^{-1}$)
- Numerical similarities between $V_{\text{low } k}$ and $V_{\text{EFT m.e.’s}}$

\[ V_{\text{low } k} \text{ effectively parameterizes } V_{\text{EFT}} + \text{H.O.T.} \]

- EFT Ideology: induced (low $k$) and omitted DOF ($\Delta$) 3NFs inseparable at low $E$’s

Absorb both effects by augmenting $V_{\text{low } k}$ with leading $\chi$-EFT 3N force
\( \chi \)-EFT 3N Force

**2\( \pi \)-exchange** (notation of Friar et. al. PRC 59,53)

\[
V_{3N_F}^{2\pi} = \sum_{i<j<k} \left( \frac{g_A}{2 F_\pi} \right)^2 \frac{\sigma_i \cdot \vec{q}_i \sigma_j \cdot \vec{q}_j}{(q_i^2 + m_\pi^2)(q_j^2 + m_\pi^2)} F_{i,j,k}^{\alpha \beta} \tau_i^\alpha \tau_j^\beta
\]

LERCs also appear in the 2N force

\[
F_{i,j,k}^{\alpha \beta} = \delta_{\alpha \beta} \left[ -\frac{4 c_1 m_\pi^2}{F_\pi^2} + \frac{2 c_2}{F_\pi^2} \sigma_i \cdot \vec{q}_i \right] + \frac{c_4}{F_\pi} \epsilon^{\alpha \beta \gamma} \tau_i^\gamma \sigma_i \cdot \vec{q}_i [\vec{q}_i \times \vec{q}_j]
\]

**1\( \pi \)-exchange**

\[
V_{3N_F}^{1\pi} = -\sum_{i<j<k} \left( \frac{g_A}{4 F_\pi^2} \right) \frac{c_D}{F_\pi^2 \Lambda_x} \frac{\sigma_j \cdot \vec{q}_j}{(q_j^2 + m_\pi^2)} (\tau_i \cdot \tau_j)(\sigma_i \cdot \vec{q}_i)
\]

**contact term**

\[
V_{3N_F}^c = \sum_{i<j<k} \frac{c_E}{F_\pi^4 \Lambda_x} (\tau_j \cdot \tau_k)
\]

Due to the antisymmetry of the 3N states, the number of independent LERCs in the 3NF terms at NNLO is reduced to 2!

- 2 free parameters (\( c_D \) and \( c_E \)) -> fit to \(^3\)H and \(^4\)He B.E.’s
- \( c_i \) taken from NN PSA implementing \( \chi \)-\( 2\pi \) piece (Rentmeester et.al., PRC67)
Determination of $c_E$ and $c_D$

- Use $A=3,4$ B.E.'s to fix the free constants $c_E(\Lambda)$ and $c_D(\Lambda)$

- Linear relation ($\Lambda < 2$ fm$^{-1}$)

- $\Lambda=3.0$ fm$^{-1}$ gives wrong $A=4$ B.E. ($\sim 500$ keV)
  (not too alarming since our identification of $V_{\text{low } k} \sim V_{\text{eff}}$ breaks down at larger $\Lambda$)

Guess 3NF can be treated in 1st order (verified numerically).
Expectation Values of different components

- Are the 3NF terms of “natural” size?

\[
\begin{align*}
\text{EFT: } & <V^3N> \sim (Q/\Lambda)^3 \quad <V^2N> \quad Q \sim m_\pi
\end{align*}
\]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\Lambda [\text{fm}^{-1}] & T & V_{\text{low }k} & c\text{-terms} & D\text{-term} & E\text{-term} \\
\hline
1.0 & 21.06 & -28.62 & 0.02 & 0.11 & -1.06 \\
1.3 & 25.71 & -34.14 & 0.01 & 1.39 & -1.46 \\
1.6 & 28.45 & -37.04 & -0.11 & 0.55 & -0.32 \\
1.9 & 30.25 & -38.66 & -0.48 & -0.50 & 0.90 \\
2.5(a) & 33.30 & -40.94 & -2.22 & -0.11 & 1.49 \\
2.5(b) & 33.51 & -41.29 & -2.26 & -1.42 & 2.97 \\
3.0(*) & 36.98 & -43.91 & -4.49 & -0.73 & 3.67 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\Lambda [\text{fm}^{-1}] & T & V_{\text{low }k} & c\text{-terms} & D\text{-term} & E\text{-term} \\
\hline
1.0 & 38.11 & -62.18 & 0.10 & 0.54 & -4.87 \\
1.3 & 50.14 & -78.86 & 0.19 & 8.08 & -7.83 \\
1.6 & 57.01 & -86.82 & -0.14 & 3.61 & -1.94 \\
1.9 & 60.84 & -89.50 & -1.83 & -3.48 & 5.68 \\
2.5(a) & 67.56 & -90.97 & -11.06 & -0.41 & 6.62 \\
2.5(b) & 68.03 & -92.86 & -11.22 & -8.67 & 16.45 \\
3.0(*) & 78.77 & -99.03 & -22.82 & -2.63 & 16.95 \\
\hline
\end{array}
\]

\[
<3\text{NF}> \sim 4\text{-}15\% \text{ of } <2\text{NF}> \text{ for } \Lambda \leq 3 \text{ fm}^{-1} \text{ in agreement with EFT expectations}
\]

Further confidence that our low-momentum theory
Doesn’t need crazy many-body forces
- No crazy effective operators induced by our truncation to low k
- $2N + 3N$ is less $\Lambda$ dependent (as expected!)
Worried about saturation?

- Saturates in HF (no BHF, jastrow, etc.)
- Holds for other $\Lambda$ too
- See a (relatively) satisfying insensitivity to cutoff
- Actual #’s aren’t too bad given the crude approximations
- Work in progress for “complete” 2nd+3rd order calculations
Conclusions

- **RG methods to eliminate model dependence of $V_{NN}$ models**
  - unique $V_{\text{low-}k}$ for $\Lambda \leq 2$ fm$^{-1}$
  - cutoff version of inverse scattering problem
    - (phase shifts/pion tail $\Rightarrow$ constrains $V_{\text{low-}k} (k,k)$

- **reproduces B.E. and $\delta_{NN}$ of “high precision” $V_{NN}$ models**
  - w/out ambiguous high $k$ components

- **“Unified” framework for S.M. calculation**
  - SAME input in different mass regions
  - no more “starting energies” etc.
  - $\sim$ independent of force model
  - No more Brueckner-Faddeev resummation

Re-do low E nuclear structure with $V_{\text{low-}k} + V_{ijk} (EFT)$ with less computational effort