

# PHY422/820: Classical Mechanics

FS 2021

Final Exam / Subject Exam

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## Mechanics

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = Q_j^c + Q_j^{n.c.}, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$

$$S = \int_{t_1}^{t_2} dt L(q, \dot{q}, t), \quad \delta S = 0$$

$$\tilde{L}(q, \dot{q}, t, \vec{\lambda}) = L(q, \dot{q}, t) + \sum_{\alpha=1}^{N_h} \lambda_{\alpha} f_{\alpha}(q, t), \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \sum_{\alpha=1}^{N_h} \lambda_{\alpha} \frac{\partial f_{\alpha}}{\partial q_i} + \sum_{\beta=1}^{N_{n.h.}} \mu_{\beta} \frac{\partial g_{\beta}}{\partial \dot{q}_i}$$

$$q'_i(t') = q_i(t) + \epsilon \eta_i(t), \quad t' = t + \epsilon \tau(t), \quad L' = L + \frac{dF}{dt}$$

$$J = \sum_i \frac{\partial L}{\partial \dot{q}_i} (\dot{q}_i \tau - \eta_i) - L\tau + F = \text{const.}$$

$$V_{\text{eff}}(r) = \frac{l^2}{2mr^2} + V(r), \quad \vec{A} = \frac{\vec{p} \times \vec{l}}{m\kappa} - \frac{\vec{r}}{r}$$

$$\phi - \phi_0 = \pm \int_{r(\phi_0)}^{r(\phi)} dr' \frac{l}{r'^2 \sqrt{2m(E - V_{\text{eff}}(r'))}}$$

$$r(\phi) = \frac{\alpha}{1 + \epsilon \cos(\phi - \phi_0)}, \quad \alpha = \frac{l^2}{m\kappa}, \quad \epsilon = |\vec{A}|$$

$$I_{ab} = \int d^3r \rho(\vec{r}) (\vec{r}^2 \delta_{ab} - r_a r_b)$$

$$I_{ab} = I_{ab}^{\text{CoM}} + M (\vec{R}^2 \delta_{ab} - R_a R_b)$$

$$T_{\text{rot}} = \frac{1}{2} \vec{\omega} \cdot \mathbf{I} \cdot \vec{\omega}, \quad \vec{L} = \mathbf{I} \vec{\omega}$$

$$\vec{\omega} = \begin{pmatrix} \omega_{x'} \\ \omega_{y'} \\ \omega_{z'} \end{pmatrix} = \begin{pmatrix} \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi \\ \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi \\ \dot{\phi} \cos \theta + \dot{\psi} \end{pmatrix}$$

$$\left(\frac{d\vec{L}}{dt}\right)_{\Sigma} = \left(\frac{d}{dt}(\mathbf{I}\vec{\omega})\right)_{\Sigma'} + \vec{\omega} \times (\mathbf{I}\vec{\omega}) = \vec{N}$$

$$\frac{d}{dt}(A\omega_{x'}) + (C - B)\omega_{y'}\omega_{z'} = N_{x'}$$

$$\frac{d}{dt}(B\omega_{y'}) + (A - C)\omega_{z'}\omega_{x'} = N_{y'}$$

$$\frac{d}{dt}(C\omega_{z'}) + (B - A)\omega_{x'}\omega_{y'} = N_{z'}$$

$$L = \frac{1}{2}\dot{\vec{\eta}} \cdot \mathbf{T} \cdot \dot{\vec{\eta}} - \frac{1}{2}\vec{\eta} \cdot \mathbf{V} \cdot \vec{\eta}, \quad (\mathbf{V} - \omega^2\mathbf{T})\vec{\rho} = 0, \quad \det(\mathbf{V} - \omega^2\mathbf{T}) = 0$$

$$\mathbf{A} = (\vec{\rho}^{(1)} \quad \dots \quad \vec{\rho}^{(n)}), \quad \vec{\eta} = \mathbf{A}\vec{\zeta}, \quad \vec{\zeta} = \mathbf{A}^T\mathbf{T}\vec{\eta}, \quad \mathbf{A}^T\mathbf{T}\mathbf{A} = \mathbb{1}$$

$$(\vec{a}, \vec{b}) = \vec{a} \cdot \mathbf{T} \cdot \vec{b} = \sum_{kl} a_k T_{kl} b_l$$

$$p_j = \frac{\partial L}{\partial \dot{q}_j}, \quad H(q, p, t) = \sum_k p_k \dot{q}_k(q, p, t) - \tilde{L}(q, p, t)$$

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

$$\{f, g\} = \sum_k \left( \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q_k} \frac{\partial f}{\partial p_k} \right), \quad \frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$$

$$\{fg, h\} = \{f, h\}g + f\{g, h\}, \quad \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

$$\{q_j, q_k\} = \{p_j, p_k\} = 0, \quad \{q_j, p_k\} = \delta_{jk}$$

Function	Transformation			Simplest Case	
$F_1(q, Q, t)$	$p_i = \frac{\partial F_1}{\partial q_i}$	$P_i = -\frac{\partial F_1}{\partial Q_i}$	$K = H + \frac{\partial F_1}{\partial t}$	$F_1 = \sum_i q_i Q_i$	$Q_i = p_i, \quad P_i = -q_i$
$F_2(q, P, t)$	$p_i = \frac{\partial F_2}{\partial q_i}$	$Q_i = \frac{\partial F_2}{\partial P_i}$	$K = H + \frac{\partial F_2}{\partial t}$	$F_2 = \sum_i q_i P_i$	$Q_i = q_i, \quad P_i = p_i$
$F_3(p, Q, t)$	$P_i = -\frac{\partial F_3}{\partial Q_i}$	$q_i = -\frac{\partial F_3}{\partial p_i}$	$K = H + \frac{\partial F_3}{\partial t}$	$F_3 = \sum_i p_i Q_i$	$Q_i = -q_i, \quad P_i = -p_i$
$F_4(p, P, t)$	$q_i = -\frac{\partial F_4}{\partial p_i}$	$Q_i = \frac{\partial F_4}{\partial P_i}$	$K = H + \frac{\partial F_4}{\partial t}$	$F_4 = \sum_i p_i P_i$	$Q_i = p_i, \quad P_i = -q_i$

## Coordinate Systems

$$x = \rho \cos \phi = r \sin \theta \cos \phi, \quad y = \rho \sin \phi = r \sin \theta \sin \phi, \quad z = r \cos \theta$$

$$dV = \rho d\rho d\phi dz = r^2 \sin \theta dr d\theta d\phi = r^2 dr d\Omega$$

$$\nabla = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z} = \vec{e}_\rho \frac{\partial}{\partial \rho} + \vec{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \vec{e}_z \frac{\partial}{\partial z} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{e}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}$$

$$\nabla f(r) = f'(r) \frac{\vec{r}}{r}, \quad r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

## Rotations

$$\vec{r}' = \vec{r} \cos \phi + \vec{n}(\vec{n} \cdot \vec{r})(1 - \cos \phi) + (\vec{n} \times \vec{r}) \sin \phi$$

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix}, \quad R_y(\alpha) = \begin{pmatrix} \cos \alpha & 0 & \sin \alpha \\ 0 & 1 & 0 \\ -\sin \alpha & 0 & \cos \alpha \end{pmatrix}, \quad R_z(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{aligned} A = BCD &= \begin{pmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos \phi \cos \psi - \sin \phi \cos \theta \sin \psi & \sin \phi \cos \psi + \cos \phi \cos \theta \sin \psi & \sin \theta \sin \psi \\ -\cos \phi \sin \psi - \sin \phi \cos \theta \cos \psi & -\sin \phi \sin \psi + \cos \phi \cos \theta \cos \psi & \sin \theta \cos \psi \\ \sin \phi \sin \theta & -\cos \phi \sin \theta & \cos \theta \end{pmatrix} \end{aligned}$$

## Expansions

$$\frac{1}{(1+x)^n} = 1 - nx + \frac{1}{2}n(n+1)x^2 - \frac{1}{6}n(n+1)(n+2)x^3 + O(x^4)$$

$$\sqrt{1-x} = 1 - \frac{x}{2} + O(x^2), \quad x \ll 1$$

$$\sin x = x - \frac{1}{6}x^3 + O(x^5), \quad \cos x = 1 - \frac{1}{2}x^2 + O(x^4)$$

$$\ln 1+x = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^5)$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + O(x^5), \quad \sinh x = \frac{1}{2}(e^x - e^{-x}), \quad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

## Other Mathematical Formulas

### Linear Algebra

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123, 231, 312, \\ -1 & \text{if } ijk = 213, 321, 132, \\ 0 & \text{else.} \end{cases} \quad \sum_k \epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} M_{11} - a_{12} M_{12} = a_{11} a_{22} - a_{12} a_{21}$$

$$\det \mathbf{A} = \det \mathbf{A}^T, \quad \det \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}, \quad \det \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = a_1 \cdot \dots \cdot a_n$$

$$\vec{a} \times \vec{b} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}$$

### Properties of Trigonometric Functions

$$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x \quad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin(2x) = 2 \sin x \cos x \quad \cos(2x) = \cos^2 x - \sin^2 x$$

$$\int_0^{2\pi} dx \sin x = \int_0^{2\pi} dx \cos x = 0 \quad \int_0^{2\pi} dx \sin^2 x = \int_0^{2\pi} dx \cos^2 x = \pi$$

### Heaviside Step Function and Dirac Delta Function

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0, \\ \frac{1}{2} & \text{if } x = 0, \\ 0 & \text{if } x < 0. \end{cases} \quad \int_{-\infty}^{\infty} dx \theta(a-x) \theta(x-b) f(x) = \int_a^b dx f(x)$$

$$\int_{\alpha}^{\beta} dx f(x) \delta(x-x_0) = \begin{cases} f(x_0) & \text{if } x_0 \in (\alpha, \beta), \\ 0 & \text{else.} \end{cases}$$

$$\delta(ax) = \frac{1}{|a|} \delta(x) \quad \delta(g(x)) = \sum_k \frac{1}{|g'(x_k)|} \delta(x-x_k)$$

### Miscellaneous

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$