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MICHIGAN STATE UNIVERSITY

PHY422/820: Classical Mechanics

FS 2021

Homework #1 (Due: Sep 10)

September 2, 2021

Problem H1 – Useful Identities in Cylindrical and Spherical Coordinates

[**10 Points**] Here, we want to prove some useful identities in cylindrical and spherical coordinates (cf. worksheet #1).

1. Show that
$$\dot{\vec{r}} = \dot{\rho}\vec{e}_\rho + \rho\dot{\phi}\vec{e}_\phi + \dot{z}\vec{e}_z = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + (r\sin\theta)\dot{\phi}\vec{e}_\phi\,. \tag{1}$$

2. Using Eqs. (1) and the properties of the basis vectors, show that the kinetic energy for a particle of mass m is given by
$$T = \frac{1}{2}m\dot{\vec{r}}^2 = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + (r^2\sin^2\theta)\dot{\phi}^2\right)\,. \tag{2}$$
No Cartesian coordinates allowed!

3. Show that the force field generated by a spherically symmetric potential is of the form
$$\vec{F}(\vec{r}) = -\vec{\nabla}V(r) = -\frac{\partial V}{\partial r}\vec{e}_r\,. \tag{3}$$

Problem H2 – Conservative Forces

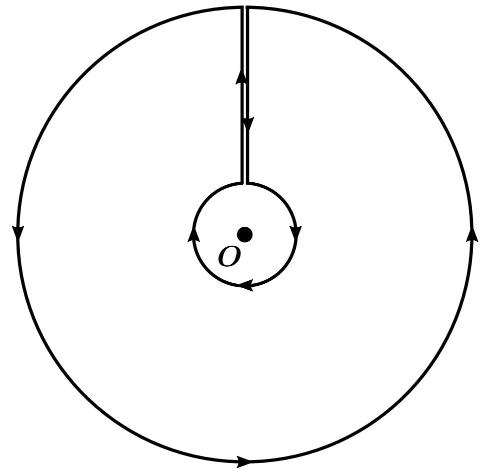
[**10 Points**] Consider the force field
$$\vec{F}(\vec{r}) = V_0\frac{e^{-(r-b)/a}}{a(1+e^{-(r-b)/a})^2}\vec{e}_r\,,\quad r = |\vec{r}|,\, a, b = \text{const.}\,. \tag{4}$$

1. Compute $\vec{\nabla}\times\vec{F}(\vec{r})$ to show that the force is conservative, and determine the underlying potential $V(\vec{r})$.

2. Explicitly calculate the work that is required to move a mass m from the origin to the point $\vec{r} = (x, y, z)$: *(i)* along a straight line, and *(ii)* from the origin to $(0, 0, r)$, where $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$, and from there along a great circle to $\vec{r}$.

   Hint: Exploit the properties of the unit vectors before you start computing integrals!

Problem H3 – Force Fields with Singularities



[10 Points] For force fields with singularities, we must be careful when we want to argue that the force is conservative based on the properties of $\vec{\nabla} \times \vec{F}(\vec{r})$. Consider, for example, the forces

$$\vec{F}_1(\vec{r}) = \frac{a_0}{\rho}\vec{e}_\phi\,, \quad a_0 = \text{const.}\,, \tag{5}$$

in cylinder coordinates (ρ, ϕ, z), and the central force

$$\vec{F}_2(\vec{r}) = -\frac{k}{r^2}\vec{e}_r\,, \quad k > 0\,, \tag{6}$$

in spherical coordinates.

1. Calculate $\vec{\nabla} \times \vec{F}_1(\vec{r})$ and $\vec{\nabla} \times \vec{F}_2(\vec{r})$!

2. For both forces, compute the work for moving a mass m on a circle with radius ϵ in the xy-plane whose center is the origin. What do you find for $\epsilon \to 0$?

3. What is the work if the mass is moved along the closed contour that excludes the origin, as shown in the figure?

   Hint: Use your results for the integrals along a circle from the previous part, and mind the directions in which the circles and linear segments of the contour are traversed.

Formulas

$$\vec{\nabla} \times \vec{A} = \frac{1}{\rho}(\partial_\phi A_z - \rho\partial_z A_\phi)\vec{e}_\rho + (\partial_z A_\rho - \partial_\rho A_z)\vec{e}_\phi + \frac{1}{\rho}(\partial_\rho(\rho A_\phi) - \partial_\phi A_\rho)\vec{e}_z \tag{7}$$

$$\begin{aligned}\vec{\nabla} \times \vec{A} = {}& \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(A_\phi \sin\theta\right) - \frac{\partial A_\theta}{\partial\phi}\right)\vec{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA_\phi\right)\right)\vec{e}_\theta \\ & + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right)\vec{e}_\phi \end{aligned}\tag{8}$$