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PHY422/820: Classical Mechanics

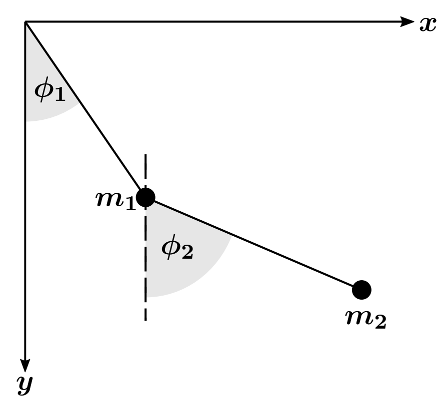
FS 2021

Homework #6 (Due: Oct 15)

October 12, 2021

Problem H11 – Double Pendulum

[15 Points] Consider a planar double pendulum with lengths l_1, l_2 and masses m_1, m_2 (see figure).



1. Construct the Lagrangian of the system and derive the equations of motion.

2. What are the normal frequencies for small angles ($\phi_1, \phi_2 \ll 1$)?
   HINT: Use the ansatz $\phi_k = \phi_{0,k}\exp(i\omega t)$ with identical ω for $k = 1, 2$, and determine ω such that the resulting system of equations has a non-trivial solution for $\phi_{0,1}, \phi_{0,2}$! Consider the energy conservation for oscillators to analyze whether terms containing $\dot{\phi}_i$ can be omitted or not.

Problem H12 – Geometry of Central-Force Trajectories

[15 Points] In general, an observed trajectory $\vec{r}(s)$ can result from a multitude of underlying forces. For central forces, however, we can derive a differential equation that determines the force law from $\vec{r}(s)$. Let us explore this in the following.

1. Use the equations of motion for a mass m in a central force field to prove the following differential equation:
$$f(r) = \frac{l^2}{mr^4}\left[\frac{d^2r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 - r\right], \tag{1}$$
   where
$$f(r) = -\frac{\partial}{\partial r}V(r)\,. \tag{2}$$

2. Use Eq. (1) to show that trajectories of the form
$$r(\phi) = \frac{\lambda}{1 + \epsilon\cos\phi} \tag{3}$$

with positive constants λ and ϵ are generated by a central force of the form

$$f(r) = -\frac{k}{r^2}\,, \quad k > 0\,. \tag{4}$$

How is k related to the other constants in the problem? Sketch or plot the trajectories for $\epsilon = 0$, $0 < \epsilon < 1$, $\epsilon = 1$, and $\epsilon > 1$.

3. Determine the force field that yields trajectories of the form

$$r(\phi) = \frac{r_0\sqrt{1-\epsilon^2}}{\sqrt{1+\epsilon\cos 2\phi}}\,. \tag{5}$$

Sketch or plot the trajectory for appropriate values for $0 < \epsilon < 1$. How is this trajectory distinct from (3), assuming that the meaning of ϵ is the same?

4. Compute the central force that makes the mass m move on the spiral trajectory

$$r(\phi) = r_0 \exp(-\phi)\,. \tag{6}$$