H. Hergert
Facility for Rare Isotope Beams
and Department of Physics & Astronomy

MICHIGAN STATE UNIVERSITY

PHY422/820: Classical Mechanics

FS 2021

Homework #7 (Due: Oct 22)

October 14, 2021

Problem H13 – Kepler Problem with Perturbation

[15 points; cf. problems G17, H12] Consider the potential

$$V(r) = -\frac{k}{r} - \frac{\alpha}{r^2} \tag{1}$$

with constants $k > 0$ and α, which corresponds to the traditional Kepler problem with an additional perturbing potential.

1. Construct the effective potential $V_{\text{eff}}(r)$. For which angular momenta l and energies E will we obtain bounded motion, i.e., orbits?

   HINT: Use your results from G17.

2. The trajectory $r(\phi)$ for a given potential can be obtained by solving the integral equation (cf. worksheet #7)

   $$\phi - \phi_0 = \int dr \frac{l}{r^2\sqrt{2m\left(E - V_{\text{eff}}(r)\right)}}\,, \tag{2}$$

   with integration constant ϕ_0. Show that

   $$r(\phi) = \frac{\lambda}{1 + \epsilon\cos\beta(\phi - \phi_0)}\,, \tag{3}$$

   where

   $$\beta^2 = 1 - \frac{2m\alpha}{l^2} > 0\,, \quad \lambda = \beta^2\frac{l^2}{mk}\,, \quad \epsilon = \sqrt{1 + \beta^2\frac{2l^2E}{mk^2}}\,. \tag{4}$$

   HINT: Use the substitution $u = 1/r$ and

   $$\int du \frac{1}{\sqrt{a + bu - u^2}} = -\arccos\frac{2u - b}{\sqrt{4a + b^2}} + c\,, \quad \text{if } 4a + b^2 > 0\,. \tag{5}$$

3. Which values of β correspond to attractive ($\alpha > 0$) and repulsive perturbations ($\alpha < 0$), respectively?

4. Show that orbits in this potential will be closed if β is rational. Why does this not contradict Bertrand's theorem?

5. Create and (briefly) discuss polar plots of the trajectories for $\epsilon = \frac{4}{5}$ and $\beta = \frac{1}{2}, \frac{1}{3}, \frac{4}{5}, \frac{2+\pi}{2\pi}, 3, \frac{3+8\pi}{3\pi}$. Use a sufficiently large range of values for ϕ to illustrate whether the orbits are closed or not.

Problem H14 – Laplace-Runge-Lenz Vector in Scattering

[15 points] For classical scattering off a repulsive potential $V(r) = \frac{k}{r}$ with $k > 0$ (e.g., Rutherford scattering), the relationship between the scattering angle θ and the impact parameter b is

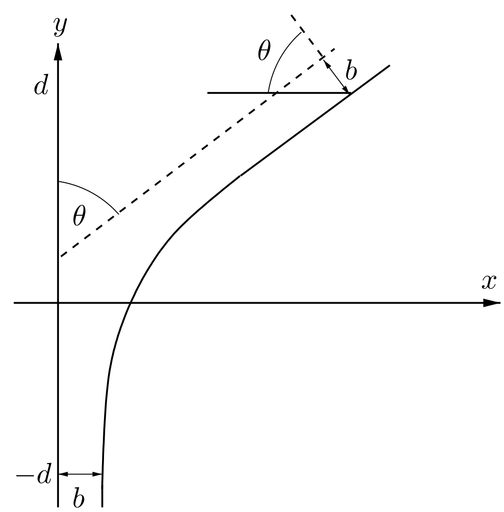
$$b(\theta) = \pm \frac{k}{2E} \cot \frac{\theta}{2} \,. \tag{6}$$

Validate this relationship using the conservation of the Laplace-Runge-Lenz vector by computing $\boldsymbol{A}$ for the incoming and outgoing particle(s), whose trajectories and velocities are

$$\begin{aligned} t \to -\infty : \quad & \boldsymbol{r}(-\infty) = (b, -d, 0)^T \,, \\ & \dot{\boldsymbol{r}} = (0, v_\infty, 0)^T \,, \end{aligned} \tag{7}$$

$$\begin{aligned} t \to \infty : \quad & \boldsymbol{r}(\infty) = (d \tan\theta + \frac{b}{\cos\theta}, d, 0)^T \,, \\ & \dot{\boldsymbol{r}} = (v_\infty \sin\theta, v_\infty \cos\theta, 0)^T \,, \end{aligned} \tag{8}$$

respectively. Here, we can assume that d is large at an appropriate stage of our calculation (see figure).



Hint: Use energy and angular momentum conservation. It is sufficient to consider a single component of $\boldsymbol{A}$.