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PHY422/820: Classical Mechanics

FS 2021

Homework #8 (Due: Oct 29)

October 25, 2021

Problem H15 – Scattering off Constant Potentials

[**10 points**] A particle is scattering off the potential well

$$V(r) = \begin{cases} -V_0 & r \leq R\,, \\ 0 & r > R \end{cases}\,, \tag{1}$$

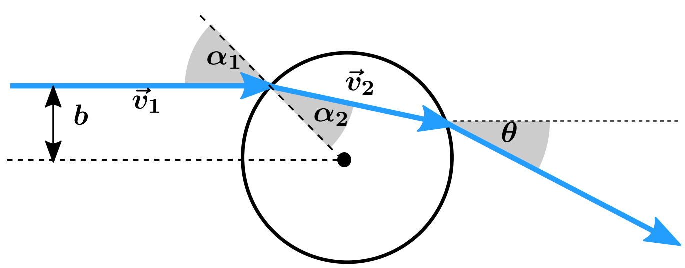
where V_0 and R are positive constants.

1. Show that

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{v_2}{v_1} = \sqrt{\frac{E+V_0}{E}} \equiv n\,. \tag{2}$$

This is identical to what we would find for the refraction of light rays at the boundary of two substances with relative index of refraction n.

Hint: Consider which quantities will be conserved or not conserved at the boundary of the potential region.



2. Show that the scattering angle θ is given by

$$\theta(b) = 2\left(\arcsin\frac{b}{R} - \arcsin\frac{b}{nR}\right)\,. \tag{3}$$

Problem H16 – Scattering From a Repulsive Inverse-Square Potential

[**10 points**] Show that the differential cross section for the repulsive inverse-square potential

$$V(r) = \frac{k}{r^2}\,, \quad k > 0\,, \tag{4}$$

is given by

$$\frac{d\sigma}{d\Omega} = \frac{k\pi^2}{E}\frac{\pi-\theta}{\theta^2\left(2\pi-\theta\right)^2\sin\theta}\,. \tag{5}$$

HINT: Determine the distance of closest approach from energy conservation, and use

$$\int du \frac{1}{\sqrt{1-\alpha^2 u^2}} = \frac{1}{\alpha} \arcsin(\alpha u) + c \tag{6}$$

(cf. hard-sphere scattering) or your favorite computer-algebra system.

Problem H17 – Inverse Scattering Problem

[10 points] In the following, we will use inverse-scattering techniques to extract the potential (4) from the differential cross section (5).

1. Determine $b(\theta)$ by rearranging and integrating

   $$\frac{d\sigma}{d\Omega} = -\frac{b}{\sin\theta} \frac{db}{d\theta} \,. \tag{7}$$

   How must the integration limits be chosen?

2. Invert $b(\theta)$ to obtain $\theta(b)$ and compute the function $T(y)$.

   NOTE: It is not necessary to work with the partial derivative trick discussed in the example in the lecture notes. Using the substitution from inverse Rutherford scattering, you should be able to cast the integral for $T(y)$ in the form

   $$T(y) = \int_0^\infty dx \left(\frac{1}{\sqrt{x^2+y^2}} - \frac{1}{\sqrt{x^2+y^2+a^2}} \right) . \tag{8}$$

   This can be solved using

   $$\int dx \frac{1}{\sqrt{x^2+y^2+a^2}} = \ln\left(x + \sqrt{x^2+y^2+a^2}\right) + C \,, \tag{9}$$

   which holds for $a \geq 0$.

3. Determine $r(y) = y \exp T(y)$ and extract the potential $V(r)$.