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PHY422/820: Classical Mechanics

FS 2021
Homework #9 (Due: Nov 5)

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Problem H18 – The Group of Rotations $SO(3)$

[10 points] Rotations in $\mathbb{R}^3$ can be represented by **special orthogonal** 3×3 **matrices**, which have the following properties:

$$\det \boldsymbol{R} = 1\,, \qquad \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{R}^T\boldsymbol{R} = \mathbb{1}\,, \qquad \boldsymbol{R}^T = \boldsymbol{R}^{-1}\,. \tag{1}$$

Show that these matrices form a group by proving that the following axioms are satisfied:

1. The product $\boldsymbol{R}_3 = \boldsymbol{R}_1\boldsymbol{R}_2$ of two rotation matrices $\boldsymbol{R}_1, \boldsymbol{R}_2 \in SO(3)$ is also a rotation matrix, $\boldsymbol{R}_3 \in SO(3)$.
2. There exists a **neutral element** $\boldsymbol{E} \in SO(3)$ such that $\boldsymbol{E}\boldsymbol{R} = \boldsymbol{R}\boldsymbol{E} = \boldsymbol{R}$ for all $\boldsymbol{R} \in SO(3)$.
3. For each $\boldsymbol{R} \in SO(3)$ there exists an **inverse element** $\boldsymbol{R}^{-1} \in SO(3)$ which satisfies $\boldsymbol{R}^{-1}\boldsymbol{R} = \boldsymbol{R}\boldsymbol{R}^{-1} = \boldsymbol{E}\,$.

Now we relax the condition on the determinant and consider the more general group of orthogonal matrices $O(3)$.

4. Show that the determinant of a real orthogonal matrix can only be $\det \boldsymbol{O} = \pm 1$.
5. Show that the orthogonal matrices with $\det \boldsymbol{O} = -1$ do *not* form a group.

HINT:

$$\det \boldsymbol{A}\boldsymbol{B} = \det \boldsymbol{A}\,\det \boldsymbol{B}\,, \quad \det \boldsymbol{A}^{-1} = \frac{1}{\det \boldsymbol{A}}\,, \quad \det \boldsymbol{A}^T = \det \boldsymbol{A}$$

Problem H19 – Infinitesimal Rotations and SO(3) Generators

[10 points] A counter-clockwise rotation by an angle ϕ around the axis $\vec{n}$ can be expressed in vector form as

$$\vec{r}' = \vec{r}\cos\phi + \vec{n}(\vec{n}\cdot\vec{r})(1-\cos\phi) + (\vec{n}\times\vec{r})\sin\phi\,. \tag{2}$$

1. Show that for infinitesimal angles
$$\vec{r}' = \vec{r} + (\epsilon\vec{n}) \times \vec{r} = (\mathbb{1} + \boldsymbol{\epsilon})\,\vec{r}\,, \tag{3}$$
where we have defined
$$\boldsymbol{\epsilon} \equiv \Phi(\epsilon\vec{n})\,, \tag{4}$$
with Φ as defined in problem G23.

2. Use the mapping Φ between vectors and antisymmetric matrices to show that
$$\boldsymbol{\epsilon} = \epsilon n_x\,\boldsymbol{L}_x + \epsilon n_y\,\boldsymbol{L}_y + \epsilon n_z\,\boldsymbol{L}_z\,, \tag{5}$$
where
$$\boldsymbol{L}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}\,, \quad \boldsymbol{L}_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\,, \quad \boldsymbol{L}_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \tag{6}$$
are the so-called **generators** of infinitesimal rotations.

3. Show that the generators satisfy
$$\left[\boldsymbol{L}_x, \boldsymbol{L}_y\right] = \boldsymbol{L}_z\,, \quad \left[\boldsymbol{L}_y, \boldsymbol{L}_z\right] = \boldsymbol{L}_x\,, \quad \left[\boldsymbol{L}_z, \boldsymbol{L}_x\right] = \boldsymbol{L}_y\,, \tag{7}$$
where the commutator is defined as
$$\left[\boldsymbol{A}, \boldsymbol{B}\right] = \boldsymbol{A}\boldsymbol{B} - \boldsymbol{B}\boldsymbol{A}\,. \tag{8}$$

4. The generators can be used to construct arbitrary antisymmetric matrices. Show that the matrix exponential of any antisymmetric matrix is a rotation matrix, i.e.,
$$\left(e^{\boldsymbol{A}}\right)^T = \left(e^{\boldsymbol{A}}\right)^{-1}\,, \quad \det e^{\boldsymbol{A}} = 1\,. \tag{9}$$
The matrix exponential is defined by the series
$$e^{\boldsymbol{A}} = \sum_{k=0}^{\infty} \frac{1}{k!}\boldsymbol{A}^k\,. \tag{10}$$

Problem H20 – Tensors

[10 points] A **rank-n tensor** on $\mathbb{R}^3$ is an $n-$tuple of real numbers which has the following behavior under rotations:
$$T'_{i_1\cdots i_n} = \sum_{j_1,\cdots,j_n=1}^{3} R_{i_1 j_1}\cdots R_{i_n j_n} T_{j_1\cdots j_n}\,, \quad i_k, j_k = 1,\ldots,3\,, \tag{11}$$
where $R \in SO(3)$. In the special case of a scalar, i.e., a rank-0 tensor, this implies $T' = T$ (there are no indices to transform).

1. Show through an explicit calculation that the scalar product of two arbitrary vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$ is invariant under rotations. Interpret this result geometrically.

2. Show that the moment of inertia tensor

$$I_{ij} = \int d^3r \, \rho(\vec{r}) \left(\vec{r}^2 \delta_{ij} - r_i r_j\right) \tag{12}$$

is a rank-2 tensor in the sense of Eq. (11).

Hint: Prove that d^3r is a scalar by considering how the volume element is related to the Cartesian unit vectors.