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PHY422/820: Classical Mechanics

FS 2021

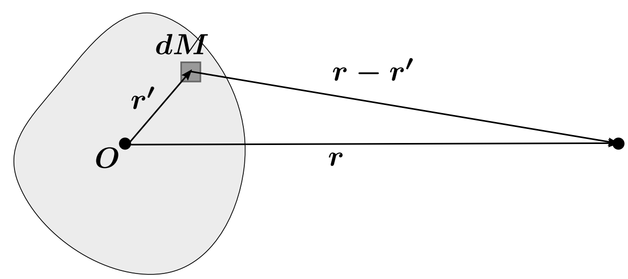
Homework #11 (Due: Nov 19)

November 21, 2021

Problem H21 – Gravitational Potential of Extended Objects

[15 points] The gravitational potential between a mass m at the point $\boldsymbol{r}$ and a general mass distribution $\rho(\boldsymbol{r})$ can be obtained from

$$V(\boldsymbol{r}) = -Gm\int d^3r'\,\frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} \tag{1}$$



(see figure, where $dM = \rho dV$ at a given point $\boldsymbol{r}'$).

1. Show that for $|\boldsymbol{r}| \gg |\boldsymbol{r}'|$, we can perform a **multipole expansion** of the potential,

$$V(\boldsymbol{r}) = -Gm\left(\frac{M}{r} + \frac{\boldsymbol{d}\cdot\boldsymbol{r}}{r^3} + \frac{1}{2}\frac{\boldsymbol{r}\cdot\boldsymbol{Q}\cdot\boldsymbol{r}}{r^5} + \ldots\ldots\right) \tag{2}$$

   where the **mass dipole moment** is defined as

$$\boldsymbol{d} = \int d^3r\,\rho(\boldsymbol{r})\boldsymbol{r}\,. \tag{3}$$

   and the (Cartesian) **mass quadrupole tensor** $\boldsymbol{Q}$ is defined componentwise as

$$Q_{ij} = \int d^3r\,\rho(\boldsymbol{r})\left(3r_ir_j - \boldsymbol{r}^2\delta_{ij}\right)\,. \tag{4}$$

   Hint: Perform a Taylor expansion of the integrand around $\boldsymbol{r}' = 0$. Evaluate the required partial derivatives in Cartesian coordinates.

2. How is $\boldsymbol{d}$ related to the center of mass of the mass distribution? What happens if we switch to the center-of-mass frame?

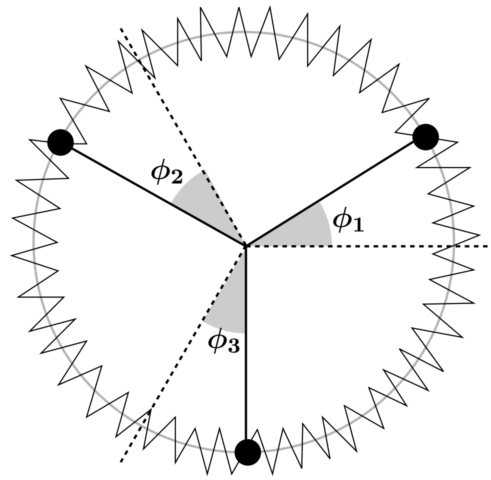
3. Show that the quadrupole tensor is related to the moment-of-inertia tensor by

$$Q_{ij} = -\left(3I_{ij} - (\mathrm{tr}\,\boldsymbol{I})\delta_{ij}\right)\,. \tag{5}$$

   What happens if all principal moments of inertia are identical?

Problem H22 – Coupled Oscillators on a Circle

[15 Points] Consider three identical masses m that can move on a circular track of radius R (see figure). Each of the masses is coupled to its neighbors by identical springs with constant k. In static equilibrium, the three masses will form an equilateral triangle, and the length of the springs will be $\frac{2\pi R}{3}$.



1. Show that the Lagnrangian can be expressed (up to an irrelevant constant) directly in terms of the *displacements from equilibrium* ϕ_i as

$$L = \frac{1}{2}mR^2\left(\dot{\phi}_1^2 + \dot{\phi}_2^2 + \dot{\phi}_3^2\right) - \frac{1}{2}kR^2\left[(\phi_2 - \phi_1)^2 + (\phi_3 - \phi_2)^2 + (\phi_1 - \phi_3)^2\right]. \tag{6}$$

   Hint: Make the ansatz $q_i(t) = R(\phi_{i0} + \phi_i(t))$, where the ϕ_{i0} indicate the absolute angles in static equilibrium.

2. Derive the equations of motion for the angles ϕ_i.

3. Determine the normal modes, i.e., characteristic frequencies and vectors (vectors do not need to be normalized). Interpret your solutions.