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PHY422/820: Classical Mechanics

FS 2021

Homework #13 (Due: Dec 3)

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Problem H26 – Change of Canonical Variables

[**10 Points, cf. Lemos 7.5**] The Lagrangian of a system with one degree of freedom is given by

$$L(q,\dot{q},t) = \frac{m}{2}\left(\dot{q}^2\cos^2\omega t - q\dot{q}\omega\sin 2\omega t - \omega^2 q^2\cos 2\omega t\right)\,. \tag{1}$$

1. Construct the Hamiltonian $H(p,q,t)$. Is it a constant of the motion?
2. Introduce a new variable $Q = q\cos\omega t$, and derive the associated Hamiltonian $H(P,Q)$. Is this new Hamiltonian conserved? What type of system does it describe?

Problem H27 – A Time-Dependent Lagrangian

[**15 Points**] Consider the following time-dependent Lagrangian for a system with a single degree of freedom:

$$L = e^{\beta t}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2\right)\,, \tag{2}$$

where $k > 0$ and β are constants.

1. Derive the Lagrange equation. What type of physical system does the Lagrangian describe?
2. Introduce the new variable $Q = e^{\beta t/2}q$, and rewrite the Lagrangian in terms of Q and $\dot{Q}$. Identify terms that are proportional to $Q\dot{Q}$ and argue that they can be dropped without changing the dynamics, leading to a new Lagrangian $L'(Q,\dot{Q})$.
3. Derive the Lagrange equation for Q. Transform the general solution of the equation of motion back to the original coordinate $q(t)$, and compare it with known solutions for the system under consideration.
4. Construct the Hamiltonians $H(p,q)$, $H(P,Q)$, and $H'(P',Q)$, where $P' = \frac{\partial L'}{\partial \dot{Q}}$. Which of them are conserved?
5. Express P' in terms of P and determine how $H(P,Q)$ and $H'(P',Q)$ are related.

Problem H28 – Hamiltonian in a Rotating Frame

[10 Points, cf. Lemos 7.15] The motion of a particle in a central potential $V(r)$ is described in a reference frame that rotates with a constant angular velocity ω with respect to an inertial frame. The center of the potential lies on the rotational axis.

1. Show that the conjugate momentum for $\vec{r}'$, the position vector in the rotating frame, is $\vec{p}' = m\left(\dot{\vec{r}}' + \vec{\omega} \times \vec{r}'\right)$.

2. Construct $H(\vec{p}', \vec{r}')$.

3. Show that $H(\vec{p}', \vec{r}')$ is conserved, but that it does not correspond to the total energy in the rotating frame, i.e., $H \neq \frac{\vec{p}'^2}{2m} + V(r')$ (where $r' = |\vec{r}'|$).

HINT: For the form of the Lagrangian in rotating frames, you can refer to our discussions of the three-body system or the rigid-body dynamics.