H. Hergert
Facility for Rare Isotope Beams
and Department of Physics & Astronomy

PHY422/820: Classical Mechanics

FS 2021

Homework #14 (Due: Dec 10)

November 29, 2021

Problem H29 – Poisson Brackets for Laplace-Runge-Lenz Vector

[**15 Points**] The Hamiltonian of the Kepler problem is given by

$$H = \frac{\vec{p}^2}{2m} - \frac{k}{r}\,, \quad k > 0\,. \tag{1}$$

Compute the Poisson brackets $\{l_i, H\}$ und $\{A_i, H\}$, to show that the angular momentum and the Laplace-Runge-Lenz vector

$$\vec{A} = \frac{\vec{p} \times \vec{l}}{mk} - \frac{\vec{r}}{r} \tag{2}$$

are conserved.

Hint: Start by proving

$$\{f(r), p_i\} = \frac{\partial f}{\partial r}\frac{x_i}{r}\,, \tag{3}$$

and use the properties of the Poisson brackets.

Problem H30 – Complex Transformations

[**15 Points**] The Hamiltonian of a harmonic oscillator with a single degree of freedom is given by

$$H(q, p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2\,. \tag{4}$$

Hamilton's equations can be decoupled by introducing the new variables

$$a = \sqrt{\frac{m\omega}{2}}q + \frac{ip}{\sqrt{2m\omega}}\,, \quad a^* = \sqrt{\frac{m\omega}{2}}q - \frac{ip}{\sqrt{2m\omega}}\,. \tag{5}$$

1. Show that this transformation is not canonical.
2. Construct the Hamiltonian in the new variables.

3. Evaluate the Jacobian of the transformation and show that

$$\{f,g\}_{(a,a^*)} = \frac{\partial f}{\partial a}\frac{\partial g}{\partial a^*} - \frac{\partial g}{\partial a}\frac{\partial f}{\partial a^*} = i\{f,g\}_{(q,p)} \tag{6}$$

4. Derive the dynamical equations for a and a^* by performing the change of variables in Hamilton's equations, as well as using the algebraic approach with the new Poisson bracket.

5. Solve the equations of motion and determine $q(t), p(t)$.