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The elusive d'Alembert-Lagrange dynamics of nonholonomic systems

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While the d'Alembert-Lagrange principle has been widely used to derive equations of state for dynamical systems under holonomic (geometric) and non-integrable linear-velocity (kinematic) constraints, its application to general kinematic constraints with a general velocity and acceleration-dependence has remained elusive, mainly because there is no clear method, whereby the set of linear conditions that restrict the virtual displacements can be easily extracted from the equations of constraint. We show how this limitation can be resolved by requiring that the states displaced by the variation are compatible with the kinematic constraints. A set of linear auxiliary conditions on the displacements is established and adjoined to the d'Alembert-Lagrange equation via Lagrange's multipliers to yield the equations of state. As a consequence, new transpositional relations satisfied by the velocity and acceleration displacements are also established. The theory is tested for a quadratic velocity constraint and for a nonholonomic penny rolling and turning upright on an inclined plane.
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I. INTRODUCTION

There has been growing interest in the analysis of nonholonomic systems.[1–5] Recent developments in robotics, intelligent transportation systems, cruise controls, sensors, feedback control, servomechanisms, and other advanced technologies make possible interesting systems operating under nonlinear velocity and acceleration constraints.[6–9] A simple example[5] is a car moving on a road of varying slope, with a cruise control that maintains the speed constant. Graduate texts[10–19] on analytical dynamics primarily deal with the fundamental d'Alembert-Lagrange principle, which involves virtual displacements $\delta\mathbf{r}$ to the particle's position $\mathbf{r}(t)$ with constraints held fixed during the displacement and which yields the familiar Lagrange's equations of state. Although Lagrange[20] designed it with only geometric constraints in mind, the d'Alembert-Lagrange principle can also be applied[10–19] to kinematic rolling constraints that are linear in the velocity, whose existence Lagrange did not anticipate. Lagrange assumed that independent coordinates could always be chosen for any system once the constraints were included. Although Euler had already studied[21] the small-oscillation dynamics of a rolling rigid body moving without slipping on a horizontal plane, Hertz, by coining the term "nonholonomic," was the first to highlight the essential difference between geometric (holonomic) constraints on the configuration and non-integrable kinematic (nonholonomic) constraints that directly restrict the velocities/accelerations of the state.[22] Gauss[23] had already provided a very different principle of least constraint[11–14,24] based on virtual displacements to the acceleration alone, keeping the state $(\mathbf{r}, \dot{\mathbf{r}})$ fixed at time t. The Gauss principle was later realized[25–27] to be applicable to both holonomic and nonholonomic systems and resulted in the Gibbs–Appell equations,[12,24–28] which, in turn have been shown to lead to Lagrange's equations of state for nonholonomic systems.[29,30]

Standard texts[10–19] confine their discussion of nonholonomic systems to linear-velocity constraints. Direct application of the d'Alembert-Lagrange principle to general velocity and acceleration constraints has remained elusive until recently,[30] because of the difficulty of extracting the conditions restricting the displacements $\delta\mathbf{r}$ from the equations of constraint.

In this paper, we outline how the d'Alembert-Lagrange principle can successfully treat nonholonomic systems under general velocity and acceleration constraints. The theory will be tested for a true non-integrable quadratic velocity constraint and for the interesting and instructive example of the nonholonomic penny, which rolls and turns upright on an inclined plane. The full solution, which has not been available, will be treated in detail and yields beautiful illustrations of the various orbits. The theory is presented at a level accessible for instructors and graduate students of classical dynamics.

II. d'ALEMBERT-LAGRANGE PRINCIPLE, EXISTING APPLICATIONS, AND PROBLEM

The classical state specified by the representative point $q(t) = \{q_j\}$ and $\dot{q}(t) = \{\dot{q}_j\}$ in the state space of a system at time t of N-particles with Lagrangian L and generalized coordinates q_j $(j = 1, 2, \ldots, n = 3N)$ is, in principle, determined by the solution of

$$L_j \equiv \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right] = Q_j^{\mathrm{NP}} + Q_j^{\mathrm{C}}, \tag{1}$$

obtained by setting the Lagrangian derivative L_j equal to the known applied non-potential forces Q_j^{NP} plus the unknown forces Q_j^{C}, which constrain the system. The Lagrangian $L(q, \dot{q}, t)$ in Eq. (1) is unconstrained, because it is written in terms of the $2n$ generalized coordinates q_j and velocities $\dot{q}_j$ for the unconstrained system. Because the constraint forces Q_j^{C} are generally unknown, Eq. (1) cannot be solved, except under the special circumstance when the Q_j^{C} are "ideal," that is, when the summed virtual work $Q_j^{\mathrm{C}}\delta q_j$ done in the virtual displacements $\delta q_j(t)$ from the unknown physical configuration $q(t)$ vanishes. The constrained system then evolves with time so that the summed projections

$$(L_j - Q_j^{\mathrm{NP}})\delta q_j = Q_j^{\mathrm{C}}\delta q_j = 0, \qquad (\mathrm{d'Alembert-Lagrange principle}), \tag{2}$$

onto δq_j along the q-surface are zero. The summation convention for repeated indices j is adopted.

Equation (2) is the d'Alembert-Lagrange equation, a fundamental principle of analytical dynamics established by Lagrange[20] and based on the J. Bernoulli principle of virtual work in statics and the d'Alembert principle for a single rigid body. The coefficient $(L_j - Q_j^{\mathrm{NP}})$ of δq_j is the projection $(m_i\ddot{\mathbf{r}}_i - \mathbf{F}_i).\ (\partial\mathbf{r}_i/\partial q_j)$ of Newton's equations summed over all N particles at positions $\mathbf{r}_i$ onto the various tangent vectors $\hat{q}_j \equiv \partial\mathbf{r}_i/\partial q_j$ along the direction of increasing q_j on the multi-surface $q = \{q_j\}$. The Newtonian equivalent of Eq. (2) is $(m_i\ddot{\mathbf{r}}_i - \mathbf{F}_i)\cdot\delta\mathbf{r}_i = 0$, where the forces $\mathbf{F}_i$ exclude the constraint forces. Equation (2), is therefore limited to these "workless" ideal constraints Q_j^{C}, and applies to a wide class of problems that can be solved without direct knowledge of the forces actuating the constraints. The $n = 3N$ values of δq_j are, in general, not all independent of each other but are linked by conditions restricting the displacements, so that the coefficient $(L_j - Q_j^{\mathrm{NP}})$ of each δq_j in Eq. (2) cannot be arbitrarily set to zero.

Although Eq. (2) is, in principle, valid for all ideal constraints, its application has been limited to geometric and linear-velocity constraints, because the relations restricting the displacements δq_j are easy to determine in the linear form required for adjoining them to the already linear set in Eq. (2) via Lagrange's multiplier method. The d'Alembert-Lagrange theory[10–19] has therefore been traditionally confined to dynamical systems with c holonomic (geometric) constraints

$$f_k(q,t) = 0, \quad (k = 1,2,\ldots,c), \tag{3}$$

which can be written in velocity form as

$$\dot{f}_k(q,t) = \left(\frac{\partial f_k}{\partial q_j}\right)\dot{q}_j + \frac{\partial f_k}{\partial t} = 0, \quad (k = 1,2,\ldots,c), \tag{4}$$

and those with c nonholonomic (kinematic) constraints

$$\dot{\theta}_k(q,t) = g_k^{(1)}(\dot{q},q,t) = A_{kj}(q,t)\dot{q}_j + B_k(q,t) = 0, \tag{5}$$

with only a linear-velocity dependence. Because virtual displacements δq coincide[18] with possible displacements dq in the limit of frozen constraints when $(\partial f_k/\partial t)dt = 0$ and $(\partial\theta_k/\partial t)dt = 0$, they therefore satisfy the linear set of conditions,

$$\delta f_k = \left(\frac{\partial f_k}{\partial q_j}\right)\delta q_j = 0 \tag{6a}$$

$$\delta\theta_k = \left(\frac{\partial g_k^{(1)}}{\partial \dot{q}_j}\right)\delta q_j = 0 \tag{6b}$$

for holonomic constraints and linear-velocity constraints, respectively. The frozen constraint short-cut derivation of Eq. (6) follows from the basic definition of virtual displacements to the state in Sec. IV. The c constraints in Eq. (3) may be used *ab initio* to reduce the number of coordinates to a set of $m = n - c$ independent degrees of freedom and L to a reduced Lagrangian L' based on the free coordinates $(q_i, i = 1,2,\ldots,m)$. Then Eq. (2) yields $L_i' = Q_i^{\mathrm{NP}}$. Alternatively, the displacement conditions in Eq. (6) may often be used to reduce Eq. (2) to a sum over only independent displacements, whose coefficients may then be set to zero. The general procedure, however, is to adjoin the sets of auxiliary conditions, Eqs. (6), via c Lagrange's multipliers λ_k to Eq. (2), where the δq_j is effectively regarded as all independent, so as to provide the standard equations of state[10–19]

$$(L_j - Q_j^{\mathrm{NP}}) = Q_j^{\mathrm{C}} = \lambda_k\left(\frac{\partial f_k}{\partial q_j}\right), \tag{7a}$$

$$(L_j - Q_j^{\mathrm{NP}}) = Q_j^{\mathrm{C}} = \lambda_k\left(\frac{\partial g_k^{(1)}}{\partial \dot{q}_j}\right), \tag{7b}$$

to be solved in conjunction with Eqs. (3) or (5), respectively. On noting that the coefficients of $\ddot{q}_j$ in $\ddot{f}_k$ and $\ddot{\theta}_k$ are the coefficients of δq_j and λ_k in Eqs. (6) and (7), respectively, it is tempting to suggest[31] for general velocity constraints g_k that the acceleration coefficient $(\partial g_k/\partial\dot{q}_j)$ in $\dot{g}_k$ be taken correspondingly as the δq_j coefficient of the nonholonomic conditions. Then Eqs. (6b) and (7b) hold with $g_k^{(1)}$ replaced by g_k. The argument is however axiomatic, requires explicit proof and cannot cover general acceleration constraints.

The commutation rule $\delta\dot{q}_j = (\delta q_j)'$, where $\dot{q}_j = dq_j/dt$ and $(\delta q_j)' = d(\delta q_j)/dt$, is traditionally accepted for the calculation of the velocity displacements $\delta\dot{q}_j$ in Lagrangian dynamics. Under this rule and displacement conditions, Eqs. (6), we can show (Sec. IV) that $\delta\dot{f}_k = 0$ and $\delta g_k^{(1)} \neq 0$, which imply that the displaced state $(q + \delta q, \dot{q} + \delta\dot{q})$ is compatible (possible) with geometric constraints Eq. (3) but is *not* compatible with velocity constraints, Eq. (5).[3,14,16]

Application of Eq. (2) to nonholonomic systems under constraints with a general dependence on velocity and acceleration has so far remained elusive because the displacement conditions to be adjoined to Eq. (2) prove impossible to determine from basic procedures, while the conventional commutation rule remains in operation.

We shall show how the needed displacement conditions can be obtained from the property of possible displaced states, with the result that Eq. (2) may be applied to general kinematic constraints. As a consequence, new transpositional rules that relate the $\delta\dot{q}_j$ to $(\delta q_j)'$ for velocity constraints and the $\delta\ddot{q}_j$ to $(\delta\dot{q}_j)'$ for acceleration constraints are established. Under these rules, the virtual displacements in Eq. (2) can now be taken to be compatible with the nonholonomic constraints.

III. HOMOGENEOUS VELOCITY CONSTRAINTS: EQUATIONS OF STATE

Before addressing whether or not d'Alembert-Lagrange principle is capable of covering general kinematic constraints, consider first the simpler case of how the d'Alembert-Lagrange principle can be applied to velocity constraints $g_k^{(p)}$ that are homogeneous to degree p in the velocities $\dot{q}_j$. An example is the Benenti problem[32] in which two identical rods move on a plane under no external forces in such a way that the rods and the velocities of the midpoints remain parallel. The constraint may be expressed as

$$g_1^{(2)} = \dot{x}_1\dot{y}_2 - \dot{x}_2\dot{y}_1 = 0, \tag{8}$$

which is non-integrable and quadratic in the velocity. Another example is the Appell–Hamel problem.[19,27] On differentiating the property $g_k^{(p)}(\alpha\dot{q},q,t) = \alpha^p g_k^{(p)}(\dot{q},q,t)$ of general homogeneous functions with respect to α and setting $\alpha = 1$, we have

$$\left(\frac{\partial g_k^{(p)}}{\partial \dot{q}_j}\right)\dot{q}_j = p g_k^{(p)} = 0, \tag{9}$$

which is the Euler theorem for functions homogeneous in $\dot{q}_j$. The set of linear conditions,

$$\left(\frac{\partial g_k^{(p)}}{\partial \dot{q}_j}\right)\delta q_j = 0, \tag{10}$$

on the displacements readily follows in the linear form required for adjoining to Eq. (2). When Eq. (10) is adjoined to Eq. (2), the δq_j in effect is regarded as all independent to give the $j = 1, 2, \ldots, n$ equations of state

$$L_j = Q_j^{\mathrm{NP}} + \lambda_k \left(\frac{\partial g_k^{(p)}}{\partial \dot{q}_j}\right), \tag{11}$$

and the forces $Q_j^{\mathrm{C}} = \lambda_k(\partial g_k^{(p)}/\partial \dot{q}_j)$ actuating the homogenous velocity constraints. By using Hertz' principle of least curvature (which is a geometrical version[11] of Gauss' principle of least constraint), Rund also derived Eq. (11) but only after a lengthy geometrical analysis.[33]

Cases: (a) For exactly integrable constraints, $g_k^{(1)} = \dot{f}_k$ and $\partial \dot{f}_k/\partial \dot{q}_j = \partial f_k/\partial q_j$. In this case, Eqs. (10) and (11) reproduce Eqs. (6a) and (7a) for holonomic systems. (b) Eq. (5) with $B_k = 0$ and Eqs. (6b) and (7b) are simply the $(p = 1)$ case of Eqs. (9)–(11). (c) The solution of Eq. (11) under the Benenti constraint Eq. (8) reveals that $\lambda_1 = 0$. The force of constraint is therefore zero, and the motion of the two particles given parallel velocities initially is free, in accord with intuition. (d) The displacement condition Eq. (10) will also agree with that derived below for general velocity constraints.

IV. GENERAL VELOCITY CONSTRAINTS

Direct application of the d'Alembert-Lagrange principle in Eq. (2) to systems under nonlinear kinematic constraints,

$$g_k(\dot{q}, q, t) = 0 \quad (k = 1, 2, \ldots, c), \tag{12}$$

with a general velocity-dependence has remained elusive because the traditional procedures used to obtain the displacement conditions Eqs. (6) and (10) for holonomic, linear-velocity and homogeneous velocity constraints were not viable.

It is sometimes thought that a virtual displacement takes place instantaneously at a frozen time t. Then its time derivative $(\delta q_j)'$ will not exist. This misconception is resolved as follows. From the infinity of possible velocity sets $\{\dot{q}_{j1}\}, \{\dot{q}_{j2}\}, \ldots$ which satisfy the constraint Eq. (12), there is only one set $\{\dot{q}_j\}$ that is realized in the actual motion as determined by the equations of state. Possible position and velocity displacements $dq_j = \dot{q}_j dt$, $dq_{j1} = \dot{q}_{j1} dt$, $d\dot{q}_j = \ddot{q}_j dt$, and $d\dot{q}_{j1} = \ddot{q}_{j1} dt$ between dynamically possible adjacent states during interval dt therefore satisfy

$$dg_k = \left(\frac{\partial g_k}{\partial \dot{q}_j}\right) d\dot{q}_j + \left(\frac{\partial g_k}{\partial q_j}\right) dq_j + \left(\frac{\partial g_k}{\partial t}\right) dt = 0. \tag{13}$$

The virtual displacements δq_j and $\delta \dot{q}_j$ to position and velocity are defined as the differences

$$\delta q_j = dq_{j1} - dq_j = (\dot{q}_{j1} - \dot{q}_j)\, dt, \tag{14a}$$

$$\delta \dot{q}_j = d\dot{q}_{j1} - d\dot{q}_j = (\ddot{q}_{j1} - \ddot{q}_j)\, dt, \tag{14b}$$

of two possible displacements in position and velocity, respectively, during interval dt. With the aid of Eq. (13), the displacements therefore satisfy

$$\delta g_k = \left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\delta \dot{q}_j + \left(\frac{\partial g_k}{\partial q_j}\right)\delta q_j = 0, \tag{15}$$

the condition for possible virtually displaced states. Comparison of Eqs. (13) and (15) shows that virtual displacements δq_j and $\delta \dot{q}_j$ then coincide with possible displacements under frozen constraints $(\partial g_k/\partial t)dt = 0$ and may be regarded in effect as displacements between two simultaneous possible states. The appropriate shortcut is that δq_j is taken not with time frozen but with the constraints frozen. The displacement conditions Eq. (6) are recovered upon using the basic definition Eq. (14a) in df_k and $d\theta_k$, respectively.

In terms of the Lagrangian derivative,

$$g_{kj} \equiv \left(\frac{\partial g_k}{\partial \dot{q}_j}\right)' - \frac{\partial g_k}{\partial q_j}, \tag{16}$$

of the constraint in Eq. (12), Eq. (15) can be recast as the transpositional relation

$$\delta g_k - \left(\frac{\partial g_k}{\partial \dot{q}_j}\,\delta q_j\right)' = \left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\left[\delta \dot{q}_j - (\delta q_j)'\right] - g_{kj}\,\delta q_j, \tag{17}$$

derived without any condition imposed on the function g_k. In Sec. IV C, Eq. (17) is reduced to a new transpositional relation, which provides the time derivative $(\delta q_j)'$ appropriate to general velocity constraints. However, we first note the following important consequences of Eq. (17).

A. Deductions

(1) For exactly integrable velocity constraints, we have $g_k = \dot{f}_k(q, t)$. The Lagrangian derivative $\dot{f}_{kj}$ vanishes because $(\partial \dot{f}_k/\partial \dot{q}_j)' = (\partial f_k/\partial q_j)' = (\partial \dot{f}_k/\partial q_j)$. Then Eq. (17) reduces to the transpositional rule

$$\delta \dot{f}_k - \frac{d}{dt}(\delta f_k) = \left(\frac{\partial f_k}{\partial q_j}\right)\left[\delta \dot{q}_j - \frac{d}{dt}(\delta q_j)\right]. \tag{18}$$

The known condition $\delta f_k = 0$ of Eq. (6a) on the displacements and the condition $\delta \dot{f}_k = 0$ for possible displaced states $(q + \delta q, \dot{q} + \delta \dot{q})$ show that the commutation rule,

$$\delta \dot{q}_j = \frac{d}{dt}(\delta q_j) \quad \text{(Traditional commutation rule)}, \tag{19}$$

is satisfied for exactly integrable constraints. Otherwise, Eq. (19) can be independently proven[30] from first principles for all dependent and independent coordinates of holonomic systems so that the combination of Eqs. (6a) and (19) implies $\delta \dot{f}_k = 0$ for possible displaced states.

(2) Under condition Eq. (6b) for linear-velocity constraints and commutation rule Eq. (19), Eq. (17) yields $\delta g_k^{(1)} = -g_{kj}^{(1)}\delta q_j$. The displaced states are, therefore, not possible, unless $g_{kj}^{(1)}$ or the sum $g_{kj}^{(1)}\delta q_j$ vanish. But $g_{kj}^{(1)} = 0$ is satisfied only by exactly integrable constraints[3,14,16] $g_k = \dot{f}_k(q,t)$, which do not require an integrating factor. And the sum $g_{kj}^{(1)}\delta q_j = 0$ is satisfied only by integrable constraints,[30] which require an integrating factor. While Eq. (19) is in operation, possible displaced states are realised only for integrable constraints.

(3) The traditional commutation rule Eq. (19) is therefore inconsistent with possible displaced states in non-holonomic systems. Because constrained variational principles rely on possible variational paths, they cannot be constructed with validity for ideal nonholonomic systems. If displaced states were (mistakenly) taken as possible under the commutation rule, the δq_j would satisfy

$$\delta g_k = \left(\frac{\partial g_k}{\partial \dot{q}_j}\,\delta q_j\right)' - g_{kj}\delta q_j = 0. \tag{20}$$

Now apply the constrained Hamilton's least-action principle $\delta\int_{t_1}^{t_2}(L-\mu_k g_k)\,dt = 0$ or, equivalently, the condition Eq. (20) adjoined to Hamilton's integral principle, $\int_{t_1}^{t_2} L_j\,\delta q_j\,dt = 0$. The following equations of state:

$$L_j = \dot{\mu}_k\left(\frac{\partial g_k}{\partial \dot{q}_j}\right) + \mu_k g_{kj}, \tag{21}$$

are then obtained.[34–38] Equations (21), first proposed by Ray[34] but then retracted,[34] were later re-discovered[37,38] as the vakonomic equations (of the variational axiomatic kind) or the variational nonholonomic equations.[2] They remain axiomatic without basic theoretical justification[1,3,15,30,34–36] and do not reproduce either the correct state Eqs. (7) for linear and homogeneous velocity constraints or Eq. (29) obtained below for general g_k. Different solutions are obtained[38] for the "vakonomic" and "nonholonomic" ice-skaters on an inclined plane. The essential reason for failure of Eq. (21) is that Eq. (19) and $\delta g_k = 0$ can never be simultaneously satisfied for non-integrable constraints. Also, Eq. (20) does not yield the correct conditions Eqs. (6b) and (10) or Eq. (28) derived below for general g_k. The physical state of a nonholonomic system does not result from a stationary value of the constrained action.

B. Displacement conditions and equations of state

A desirable property in analytical dynamics is that the δq_j-variations result in possible dynamically displaced states. Instead of using Eq. (12) directly for the velocity constraint, we note that use of its linear-acceleration form,

$$\dot{g}_k = \left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\ddot{q}_j + \left(\frac{\partial g_k}{\partial q_j}\right)\dot{q}_j + \frac{\partial g_k}{\partial t} = 0, \tag{22}$$

automatically guarantees possible displaced states, because it leads directly to the correct (tangency) condition

$$\delta g_k = \left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\delta\dot{q}_j + \left(\frac{\partial g_k}{\partial q_j}\right)\delta q_j = \nabla_Q g_k\cdot\delta \mathrm{Q} = 0, \tag{23}$$

for possible states. Because $\nabla_Q\, g_k$ is normal to g_k, the displacement δQ of the representative point $Q = (q,\dot{q})$ in state space is tangential to the g_k-surface and the displaced state lies on the manifold of velocity constraints g_k. Partition the n states with $j = 1,2,\ldots,n$ into m-independent states $(q_i,\dot{q}_i)$, where $i = 1,2,\ldots,m$ and c-dependent states $(\eta_d,\dot{\eta}_d)$, where $\eta_d = q_{m+d}$ and $d = 1,2,\ldots,c$, so that Eq. (22) decomposes into

$$\dot{g}_k = G_{kd}\ddot{\eta}_d + \left[\left(\frac{\partial g_k}{\partial \dot{q}_i}\right)\ddot{q}_i + \left(\frac{\partial g_k}{\partial q_j}\right)\dot{q}_j + \frac{\partial g_k}{\partial t}\right] = 0, \tag{24}$$

where $G_{kd}(q,\dot{q},\eta,\dot{\eta},t) = \partial g_k/\partial\dot{\eta}_d$ are the elements of the matrix $G = \{G_{kd}\}$, assumed to be positive definite (invertible). The solutions of Eq. (24) for the dependent accelerations are therefore

$$\ddot{\eta}_d = -\tilde{G}_{dr}\left[\left(\frac{\partial g_r}{\partial \dot{q}_i}\right)\ddot{q}_i + \left(\frac{\partial g_r}{\partial q_j}\right)\dot{q}_j + \frac{\partial g_r}{\partial t}\right], \tag{25}$$

where the elements $\tilde{G}_{dr}$ of the matrix $\tilde{G}$, the inverse of G, satisfy $G_{kd}\tilde{G}_{dr} = \delta_{kr}$, with $k,r,d = 1,2,\ldots,c$. Although the coordinate function $\eta_d = \eta_d(q_1,q_2,\ldots,q_m,t)$ is unknown for non-integrable Eq. (12), the dependent displacements

$$\delta\eta_d = \left(\frac{\partial \eta_d}{\partial q_i}\right)\delta q_i = \left(\frac{\partial \dot{\eta}_d}{\partial \dot{q}_i}\right)\delta q_i = \left(\frac{\partial \ddot{\eta}_d}{\partial \ddot{q}_i}\right)\delta q_i, \qquad (i = 1,2,\ldots,m), \tag{26}$$

can be now obtained in terms of the independent δq_i from Eq. (25) to give

$$\delta\eta_d = -\tilde{G}_{dr}\left(\frac{\partial g_r}{\partial \dot{q}_i}\right)\delta q_i. \tag{27}$$

Multiplication by G_{kd}, followed by a d-summation, yields the relation

$$\left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\delta q_j \equiv \left(\frac{\partial g_k}{\partial \dot{q}_i}\right)\delta q_i + \left(\frac{\partial g_k}{\partial \dot{\eta}_d}\right)\delta\eta_d = 0, \qquad (j = 1,2,\ldots,n), \tag{28}$$

where η_d reverts back to its original q_{m+d}. In geometrical terms, the tangency condition Eqs. (22) or (23) for possible displaced states provides the auxiliary conditions, Eq. (28), on the displacements under the general velocity constraints in Eq. (12). Equation (28), when applied to exactly integrable constraints $g_k = \dot{f}_k = 0$, provides the original displacement condition $(\partial \dot{f}_k/\partial\dot{q}_j)\delta q_j = (\partial f_k/\partial q_j)\delta q_j = 0$, in agreement with Eq. (6a). Equation (28) also covers Eqs. (6b) and (10) obtained via different procedures.

On adjoining the required set of linear restrictions, Eq. (28), on the displacements to the d'Alembert-Lagrange principle, Eq. (2), the δq_j is effectively regarded as all free, so that

$$L_j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{\mathrm{NP}} + \lambda_k\left(\frac{\partial g_k}{\partial \dot{q}_j}\right) \qquad \text{(nonholonomic equation of state)}, \tag{29}$$

are the equations of state for nonholonomic systems under the general velocity constraints in Eq. (12). Equation (29) is identical with the equation of state derived previously[29,30] from the application of Gauss' principle to the general velocity constraints in Eq. (12). It covers all the previous equations of state. The conditions in Eq. (28) on δq_j confirm that the ideal constraint forces do no combined virtual work, $Q_j^{\mathrm{C}}\delta q_j = \lambda_k(\partial g_k/\partial \dot{q}_j)\delta q_j = 0$.

C. New transpositional relation for velocity constraints

Because Eqs. (23) and (28) are each zero, the quantity

$$\delta g_k - \left(\frac{\partial g_k}{\partial \dot{q}_j}\,\delta q_j\right)' = 0 \quad (k = 1, 2, \ldots, c), \tag{30}$$

is also zero. Thus the basic relation, Eq. (17), provides the set of c transpositional rules,

$$\left(\frac{\partial g_k}{\partial \dot{q}_j}\right)\left[\delta \dot{q}_j - (\delta q_j)'\right] = g_{kj}\delta q_j. \tag{31}$$

The time derivative $(\delta q_j)'$ therefore exists and Eq. (31) shows how it is obtained from δq_j and $\delta \dot{q}_j$ which, in turn, are linked via Eq. (23). Equation (30) guarantees possible displaced states.

For integrable constraints, $g_{kj}\delta q_j$ vanishes[30] and Eq. (31) reduces to the traditional commutation rule, Eq. (19). For non-integrable constraints, the transpositional relation in Eq. (31), in contrast to the commutation rule, now guarantees displaced states compatible with the constraints. Even though the displaced states are now possible, it has been shown[30] that Eq. (31) also prevents the valid construction of a constrained Hamilton principle for non-integrable constraints. The relation $\delta g_k^{(1)} = -g_{kj}^{(1)}\delta q_j \neq 0$ associated with the traditional commutation rule, Eq. (19) offers the same conclusion.[1,3,15,30–36]

Because Eq. (31) is a set of only $c = (n - m)$ equations for the $n = (m + c)$ unknown $\delta \dot{q}_j$, we are at liberty to specify that the commutation relation Eq. (19) is obeyed by the m-independent velocity displacements $\delta \dot{q}_i$. Then Eq. (31) is reduced to the set of c equations

$$G_{kd}\left[\delta \dot{q}_d - (\delta q_d)'\right] = g_{kj}\delta q_j \quad G_{kd} = \left(\frac{\partial g_k}{\partial \dot{q}_d}\right),$$
$$d = m+1, m+2, \ldots, n, \tag{32}$$

for the c dependent velocity displacements, which are therefore given by the solution

$$\delta \dot{q}_i - (\delta q_i)' = 0, (i = 1, 2, \ldots, m) \tag{33a}$$

$$\delta \dot{q}_d - (\delta q_d)' = \tilde{G}_{dk} g_{kj}\delta q_j, (d = m+1, m+2, \ldots, n), \tag{33b}$$

where the elements $\tilde{G}_{dk}$ of the $(c \times c)$ inverse matrix $\tilde{G}$ satisfy $\tilde{G}_{dk}G_{kj} = \delta_{dj}$. These subrules in Eq. (33) based on Eq. (31) show how to evaluate the independent and dependent derivatives $(\delta q_j)'$ from δq_j and $\delta \dot{q}_j$. A geometrical interpretation of these rules provides further insight.[30]

V. GENERAL ACCELERATION CONSTRAINTS

As for the case of general velocity constraints, Eq. (12), direct application of the d'Alembert-Lagrange principle, Eq. (2), to systems under nonlinear kinematic constraints

$$h_k(\ddot{q}, \dot{q}, q, t) = 0 \quad (k = 1, 2, \ldots, c), \tag{34}$$

with general acceleration-dependence, has also remained elusive because the traditional methods for holonomic and linear-velocity constraints cannot be implemented. The change in the acceleration constraints due to the δq-displacement is

$$\delta h_k = \left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)\delta \ddot{q}_j + \left(\frac{\partial h_k}{\partial \dot{q}_j}\right)\delta \dot{q}_j + \left(\frac{\partial h_k}{\partial q_j}\right)\delta q_j. \tag{35}$$

We have

$$\left(\frac{\partial h_k}{\partial \ddot{q}_j}\,\delta q_j\right)'' = \left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)(\delta q_j)'' + 2\left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)'(\delta q_j)' + \left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)''\delta q_j, \tag{36}$$

which, with the aid of Eq. (35), provides the basic transpositional relation,

$$\delta h_k - \left(\frac{\partial h_k}{\partial \ddot{q}_j}\,\delta q_j\right)'' = \left(\frac{\partial h_k}{\partial \dot{q}_j}\right)(\delta \dot{q}_j - (\delta q_j)') + \left(\frac{\partial h_k}{\partial \ddot{q}_j}\right) \times \left(\delta \ddot{q}_j - \left(\delta q_j\right)''\right) - \Delta h_k, \tag{37}$$

for acceleration constraints, where the end term is

$$\Delta h_k = \left[2\left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)' - \left(\frac{\partial h_k}{\partial \dot{q}_j}\right)\right](\delta q_j)' + \left[\left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)'' - \left(\frac{\partial h_k}{\partial q_j}\right)\right]\delta q_j. \tag{38}$$

The meaning of Eq. (37), which is analogous to Eq. (17) for velocity constraints, is made apparent for exact constraints $h_k = \dot{g}_k$ when it is found that Δh_k reduces to $(g_{kj}\,\delta q_j)'$. Then Eq. (37) is simply

$$\delta \dot{g}_k - \left(\frac{\partial \dot{g}_k}{\partial \ddot{q}_j}\,\delta q_j\right)'' = \left(\frac{\partial \dot{g}_k}{\partial \dot{q}_j}\right)(\delta \dot{q}_j - (\delta q_j)') + \left(\frac{\partial g_k}{\partial \dot{q}_j}\right)(\delta \ddot{q}_j - (\delta q_j)'') - \left(g_{kj}\delta q_j\right)', \tag{39}$$

which is a higher-order version of Eq. (17). With the aid of the identity,

$$\frac{\partial \dot{g}_k}{\partial \dot{q}_j} = \left(\frac{\partial g_k}{\partial \dot{q}_j}\right)' + \frac{\partial g_k}{\partial q_j}, \tag{40}$$

we can also show that Eq. (39) minus the time derivative of Eq. (17) provides the transpositional relation

$$\delta \dot{g}_k - (\delta g_k)' = \left(\frac{\partial g_k}{\partial q_j}\right)(\delta \dot{q}_j - (\delta q_j)') + \left(\frac{\partial g_k}{\partial \dot{q}_j}\right)(\delta \ddot{q}_j - (\delta \dot{q}_j)'), \tag{41}$$

which is the analogue of Eq. (18) for holonomic constraints. Equations (17), (37), and (39) and Eqs. (18) and (41) are members of two families[30] of basic transpositional relations, valid for the functions f_k, g_k, and h_k without any conditions imposed. Other family members have recently been obtained.[30]

A. Displacement conditions and equations of state

In a similar fashion to Eq. (22), use of the time derivative

$$\dot{h}_k = \left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)\dddot{q}_j + \left(\frac{\partial h_k}{\partial \dot{q}_j}\right)\ddot{q}_j + \left(\frac{\partial h_k}{\partial q_j}\right)\dot{q}_j + \frac{\partial h_k}{\partial t} = 0, \tag{42}$$

instead of the acceleration constraint, Eq. (34), automatically guarantees possible displaced states, because it leads directly to the correct condition

$$\delta h_k = \left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)\delta\ddot{q}_j + \left(\frac{\partial h_k}{\partial \dot{q}_j}\right)\delta\dot{q}_j + \left(\frac{\partial h_k}{\partial q_j}\right)\delta q_j$$
$$= \nabla_Q\, h_k \cdot \delta Q = 0, \tag{43}$$

for possible states $Q = (q, \dot{q}, \ddot{q})$. In geometrical terms, Eqs. (42) or (43) expresses the tangency condition that the displacement δQ of the representative point Q is tangential to the h_k-surface and the displaced state therefore lies on the acceleration constraint manifold h_k. Partition the n states with $j = 1, 2, \ldots, n$ into m-independent states $(q_i, \dot{q}_i)$, where $i = 1, 2, \ldots, m$ and c-dependent states $(\eta_d, \dot{\eta}_d)$, where $\eta_d = q_{m+d}$ and $d = 1, 2, \ldots, c$. The Eq. (42) decomposes into

$$\dot{h}_k = H_{kd}\,\dddot{\eta}_d + \left[\left(\frac{\partial h_k}{\partial \ddot{q}_i}\right)\dddot{q}_i + \left(\frac{\partial h_k}{\partial \dot{q}_j}\right)\ddot{q}_j + \left(\frac{\partial h_k}{\partial q_j}\right)\dot{q}_j + \frac{\partial h_k}{\partial t}\right] = 0, \tag{44}$$

where $H_{kd}(q, \dot{q}, \ddot{q}, \eta, \dot{\eta}, \ddot{\eta}, t) = (\partial h_k / \partial \ddot{\eta}_d)$ are the elements of the matrix $H = \{H_{kd}\}$, assumed to be positive definite (invertible). The solutions $\dddot{\eta}_d$ of Eq. (44) are therefore

$$\dddot{\eta}_d = -\tilde{H}_{dr}\left[\left(\frac{\partial h_r}{\partial \ddot{q}_i}\right)\dddot{q}_i + \left(\frac{\partial h_r}{\partial \dot{q}_j}\right)\ddot{q}_j + \left(\frac{\partial h_r}{\partial q_j}\right)\dot{q}_j + \frac{\partial h_r}{\partial t}\right], \tag{45}$$

where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, and where the elements $\tilde{H}_{dr}$ of matrix $\tilde{H}$, the inverse of matrix $H = \{H_{kd}\}$, satisfy $H_{kd}\tilde{H}_{dr} = \delta_{kr}$ with $d, r = 1, 2, \ldots, c$. Although the coordinate function $\eta_d = \eta_d(q, t)$ is unknown for the non-integrable Eq. (34), the dependent displacements

$$\delta\eta_d = \left(\frac{\partial \eta_d}{\partial q_i}\right)\delta q_i = \left(\frac{\partial\, \dddot{\eta}_d}{\partial\, \dddot{q}_i}\right)\delta q_i, \tag{46}$$

may now be obtained in terms of the independent δq_i from Eq. (45) to give

$$\delta\eta_d = -\tilde{H}_{dr}\left(\frac{\partial h_r}{\partial \ddot{q}_i}\right)\delta q_i. \tag{47}$$

Multiplication by H_{kd}, followed by a d-summation, yields the relation

$$\left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)\delta q_j \equiv \left(\frac{\partial h_k}{\partial \ddot{q}_i}\right)\delta q_i + \left(\frac{\partial h_k}{\partial \ddot{\eta}_d}\right)\delta\eta_d = 0,$$
$$(j = 1, 2, \ldots, n), \tag{48}$$

where η_d reverts back to its original q_{m+d}. Equation (48) obtained from the tangency condition, Eq. (43) is the required set of linear conditions on the displacements to be adjoined to the d'Alembert-Lagrange principle in Eq. (2) for nonholonomic systems under general acceleration constraints in Eq. (34). On adjoining Eq. (48) to Eq. (2), the δq_j are effectively regarded as all free, so that

$$L_j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^{\mathrm{NP}} + \lambda_k\left(\frac{\partial h_k}{\partial \ddot{q}_j}\right),$$
$$\text{(nonholonomic equation of state)} \tag{49}$$

are the equations of state for nonholonomic systems under the general acceleration constraints in Eq. (34). Equation (49) is identical to the equation of state derived[30] from Gauss' principle and, with the aid of Eq. (22), it covers the previous result in Eq. (29) for velocity constraints and all the other equations of state. The restrictions Eq. (48) on δq_j ensure that the constraint forces do no combined virtual work, $Q^{\mathrm{C}}\delta q_j = \lambda_k(\partial h_k/\partial \ddot{q}_j)\delta q_j = 0$.

B. New transpositional relation for acceleration constraints

Because Eqs. (43) and (48) are each zero, the quantity

$$\delta h_k - \left(\frac{\partial h_k}{\partial \ddot{q}_j}\,\delta q_j\right)'' = 0, \tag{50}$$

is also zero. The basic expression, Eq. (37), then provides the $k = 1, 2, \ldots, c$ transpositional relations

$$\left(\frac{\partial h_k}{\partial \dot{q}_j}\right)(\delta\dot{q}_j - (\delta q_j)') + \left(\frac{\partial h_k}{\partial \ddot{q}_j}\right)(\delta\ddot{q}_j - (\delta q_j)'') = \Delta h_k, \tag{51}$$

for acceleration constraints. Because of Eqs. (43) and (48), use of Eq. (51) implies that the displaced states are all possible. Subrules analogous to Eq. (33) for velocity constraints can be similarly deduced from Eq. (51) by taking $\delta\dot{q}_j = (\delta q_j)'$ for all n velocity displacements and $\delta\ddot{q}_i = (\delta q_i)''$ only for the m independent acceleration displacements with $i = 1, 2, \ldots, m$. The c-dependent acceleration displacements then satisfy

$$\delta\ddot{q}_d - (\delta q_d)'' = \tilde{H}_{dk}\Delta h_k, \quad H_{kd} = \left(\frac{\partial h_k}{\partial \ddot{q}_d}\right)$$
$$(d = m+1, m+2, \ldots, n), \tag{52}$$

where $\tilde{H}_{dk}H_{kr} = \delta_{dr}$ is satisfied by elements of the $(c \times c)$ matrix $\tilde{H}$, the inverse of H.

VI. TEST CASE: THE NONHOLONOMIC PENNY

The theory we have developed has been tested by providing the correct physical solution of the Benenti problem under the non-integrable quadratic velocity constraint in Eq. (8). Consider the solution of the nonholonomic penny obtained from the d'Alembert-Lagrange principle in Eq. (2) and from Eq. (29) for general velocity constraints. In the rolling and turning of a penny/thin disk along a two-dimensional inclined plane, illustrated in Fig. 1, the penny of mass M, radius R, and center of mass at (x, y, z) is initially placed

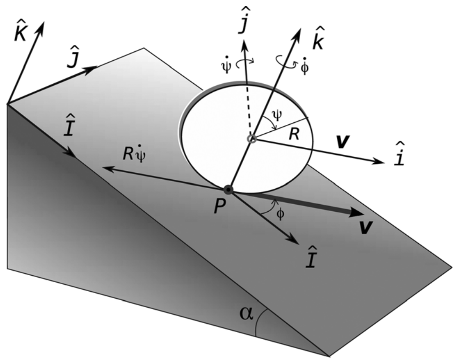

Fig. 1. The penny rolls upright while turning on an inclined plane of angle α. The directions of the space-fixed axes are $\hat{I}$, $\hat{J}$, and $\hat{K}$, as indicated. The disk rolls along the plane with angular velocity $\dot{\psi}\hat{j}$ about the symmetry axis $\hat{j}(t)$, which turns with constant angular velocity $\dot{\phi}\hat{k}$ about the fixed figure axis $\hat{k}$. The center of mass has velocity $\mathbf{v}(t) = [R\dot{\psi}(t)]\hat{i}(t)$. The point of contact P is instantaneously at rest and provides the nonholonomic constraint Eqs. (56).

upright at (x_0, y_0) and given both an initial velocity v_0 to begin rolling with angular speed $\dot{\psi}_0 = v_0/R$ about the $\hat{j}$-axis of axial symmetry, and an angular speed $\dot{\phi}_0 = \omega$ for turning about the fixed figure axis $\hat{k}$. The penny is constrained to remain upright $\hat{k} = \hat{K}$ so that its center of mass coordinate $z = R$. The Lagrangian, in terms of the four generalized coordinates (x, y, ψ, ϕ), is

$$L = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_2\dot{\psi}^2 + \frac{1}{2}I_3\dot{\phi}^2 + Mgx\sin\alpha, \qquad (53)$$

where $I_2 = \beta MR^2$ with $\beta = 1/2$, and I_3 are the moments of inertia of the body about the symmetry and figure axes $\hat{j}$ and $\hat{k}$, respectively. The penny's angular velocity is $\mathbf{\Omega} = (\dot{\psi}\hat{j} + \dot{\phi}\hat{k})$ so that the instantaneous velocity $\mathbf{v}_P$ of the point P of contact is $\mathbf{v}_P = \mathbf{v} + \mathbf{\Omega} \times (-R\hat{k})$, where $\mathbf{v}$ is the center of mass velocity. The condition for rolling without slipping is therefore

$$\mathbf{v}_P = \dot{x}\hat{I} + \dot{y}\hat{J} - (R\dot{\psi})\hat{i} = 0. \qquad (54)$$

The components of $\mathbf{v}_P$ along the fixed directions $\hat{I}$ and $\hat{J}$ are

$$G_1 = \dot{x} - R\dot{\psi}\cos\phi = 0 \qquad (55a)$$

$$G_2 = \dot{y} - R\dot{\psi}\sin\phi = 0, \qquad (55b)$$

and are

$$g_1 = \dot{x}\cos\phi + \dot{y}\sin\phi - R\dot{\psi} = 0 \qquad (56a)$$

$$g_2 = \dot{x}\sin\phi - \dot{y}\cos\phi = 0, \qquad (56b)$$

along the rotating directions $\hat{i}(t) = (\hat{I}\cos\phi + \hat{J}\sin\phi)$ and $\hat{j}(t) = (-\hat{I}\sin\phi + \hat{J}\cos\phi)$ shown in Fig. 1. Equations (55) and (56) are non-integrable linear-velocity constraints. Equations (56a) and (56b) represent the rolling and knife-edge (skater) constraints, respectively, where $\mathbf{v}$ remains directed along the axis $\hat{i}$ knife-edge. The $2n - c = 6$ initial conditions required for solution are $(x_0, y_0, \dot{\psi}_0 = v_0/R,\ \dot{\phi}_0 = \omega$, and $\psi_0 = 0, \phi_0 = 0)$. Any of the constraints may be replaced by the $(\hat{i}, \hat{j})$-components

$$\dot{g}_1 = (\ddot{x}\cos\phi + \ddot{y}\sin\phi) - R\ddot{\psi} = 0 \qquad (57a)$$

$$\dot{g}_2 = (\ddot{y}\cos\phi - \ddot{x}\sin\phi) - R\dot{\psi}\dot{\phi} = 0 \qquad (57b)$$

of acceleration of the point P of contact. These will later prove useful in the calculation of the forces of constraint in Sec. VI A 1. Application of Eq. (49) for acceleration constraints to Eq. (57) yields results identical with those obtained from Eq. (29) applied to Eq. (56), as expected, because the displacement conditions Eqs. (28) and (48) coincide for linear acceleration constraints.

Application of Eq. (29) to the homogeneous quadratic velocity constraint

$$g_1^{(2)} = (g_1)^2 + (g_2)^2 = \dot{x}^2 + \dot{y}^2 - R^2\dot{\psi}^2 = 0, \qquad (58)$$

also yields results identical to those for the linear-velocity case. Equation (58) is, however, not a true quadratic velocity constraint because the tangency condition $\dot{g}_1^{(2)} = 0$ reduces to the original conditions $\dot{g}_{1,2} = 0$ used to establish the displacement conditions in Eq. (28). In contrast, the Benenti constraint, Eq. (8), is locally written and cannot be reduced to a simpler (linear-velocity) form.

A. Direct application of the d'Alembert-Lagrange principle: Constraints embedded

The d'Alembert-Lagrange principle in Eq. (2) yields

$$L_j\delta q_j = L_x\delta x + L_y\delta y + L_\psi\delta\psi + L_\phi\delta\phi = 0. \qquad (59)$$

The constraints in Eq. (55) can be readily embedded within Eq. (59) by expressing the dependent displacements as $\delta x = (R\cos\phi)\delta\psi$ and $\delta y = (R\sin\phi)\delta\psi$. Then Eq. (59) reduces to

$$[(R\cos\phi)L_x + (R\sin\phi)L_y + L_\psi]\delta\psi + L_\phi\delta\phi = 0, \qquad (60)$$

where $\delta\psi$ and $\delta\phi$ are independent and arbitrary. The displaced states are possible provided that the velocity displacements obey the subrules, Eq. (33a), for the independent displacements $(\delta\psi, \delta\phi)$, and Eq. (33b), which provides

$$\left[\delta\dot{x} - \frac{d}{dt}(\delta x)\right] = R\sin\phi\left(\dot{\phi}\delta\psi - \dot{\psi}\delta\phi\right) \qquad (61a)$$

$$\left[\delta\dot{y} - \frac{d}{dt}(\delta y)\right] = \cos\phi\left(\dot{\psi}\delta\phi - \dot{\phi}\delta\psi\right) \qquad (61b)$$

for the dependent displacements δx, δy. Although linear-velocity constraints in general cannot be embedded, both L_x and L_y for a linear potential are not functions of the dependent coordinates (x, y) so that embedding is possible. On calculating L_j, Eq. (60) with the aid of Eq. (57a) yields the equations of state

$$(1 + \beta)R\ddot{\psi} = (g\sin\alpha)\cos\phi \qquad (62a)$$

$$I_2\ddot{\phi} = 0 \qquad (62b)$$

for the nonholonomic penny. The solution of Eq. (62) is that the penny continues to turn counterclockwise with constant angular velocity $\dot{\phi}\hat{k} = \omega\hat{k}$, and the center of mass has velocity

$$\mathbf{v}(t) = R\dot{\psi}(t)\hat{i} = (v_0 + 4a\omega\sin\omega t)\hat{i} \qquad (63)$$

along $\hat{i}$, where the radius

$$a = \left[\frac{g \sin \alpha}{4\omega^2 (1+\beta)}\right] = \frac{g_d}{4\omega^2} \tag{64}$$

is determined by $g_d = g \sin \alpha/(1+\beta)$, the gravitational downhill component $g \sin \alpha$ offset by the uphill frictional component $\beta g \sin \alpha/(1+\beta)$ required for rolling downhill. Because $d\hat{i}/dt = \omega \hat{j}$, Eq. (63) provides the acceleration

$$\dot{\mathbf{v}} = (4\omega^2 a \cos \omega t)\, \hat{i} + \omega^2 (R' + 4a \sin \omega t)\, \hat{j}, \tag{65}$$

where the radius

$$R' = \frac{v_0}{\omega}, \tag{66}$$

is established by the initial conditions (v_0, ω). The constraints in Eq. (55) furnish, with the aid of Eq. (63), the $(\hat{I}, \hat{J})$-components

$$\dot{x} = v_0 \cos \omega t + 2a\omega \sin 2\omega t, \tag{67a}$$

$$\dot{y} = v_0 \sin \omega t + 2a\omega(1 - \cos 2\omega t), \tag{67b}$$

of the velocity. The speed $\dot{x}$ directly down the plane is purely oscillatory and averages to zero over the range $0 \le t \le 2\pi/\omega$, while the speed $\dot{y}$ across the plane averages to $\langle \dot{y} \rangle = 2a\omega$. Equation (67) yields $dy/dx = \tan \omega t$ for the gradient, in compliance with the knife-edge condition (56b). The (x, y) coordinates of the contact point P (or the center of mass) and the distance $s = R\psi$ covered between (x_0, y_0) and (x, y) for $v \ge 0$ are

$$x(t) - x_0 = a(1 - \cos 2\omega t) + R' \sin \omega t, \tag{68a}$$

$$y(t) - y_0 = a(2\omega t - \sin 2\omega t) + R'(1 - \cos \omega t), \tag{68b}$$

$$s(t) = v_0 t + 4a(1 - \cos \omega t). \tag{68c}$$

Limits: (a) For motion on a horizontal plane, $a = 0$, so that

$$x(t) - x_0 = \left(\frac{v_0}{\omega}\right) \sin \omega t \tag{69a}$$

$$y(t) - y_0 = \left(\frac{v_0}{\omega}\right)(1 - \cos \omega t). \tag{69b}$$

The penny traces out the fixed circular path

$$(x - x_0)^2 + \left[y - (y_0 + \frac{v_0}{\omega})\right]^2 = \left(\frac{v_0}{\omega}\right)^2, \tag{70}$$

of fixed radius $R' = (v_0/\omega)$ at fixed speed v_0 with fixed center at $(x_0, y_0 + v_0/\omega)$, a standard result.[17] Friction provides the required centripetal force $Mv_0^2/R' = Mv_0\omega$ toward the fixed center.

(b) For zero initial speeds on an inclined plane, $R' = 0$, and Eq. (68) reduces to the parametric equations for a cycloid, which is the path of a point on the rim of a circle of radius a which is rolling on the straight line $x = x_0$.

(c) For either spin-less motion $\omega = 0$ or the limit $t \ll 2\pi/\omega$ for non-zero ω, Eq. (68) reduces to $x(t) - x_0 = s(t) = v_0 t + \frac{1}{2} g_d t^2$ and $y(t) = y_0$, as expected for rectilinear motion under constant acceleration g_d right down the plane. More general and interesting orbits involve a mixture of the cases (a) and (b) and are discussed in Sec. VI B.

1. The constraints: Frictional force and applied torque

The components (F_i, F_j) of the frictional constraint force along $\hat{i}$ and $\hat{j}$ may be determined either via Lagrange's equations for adjoined constraints (Sec. VI B), or, more simply, by comparing the solution Eq. (65) for the acceleration obtained from constraints embedded in the d'Alembert-Lagrange principle with Newton's equation

$$\begin{aligned} M\dot{\mathbf{v}} = {} & (Mg \sin \alpha \cos \omega t + F_i)\, \hat{i} \\ & + (-Mg \sin \alpha \sin \omega t + F_j)\, \hat{j}. \end{aligned} \tag{71}$$

This comparison provides both components

$$F_i(t) = -\left(\frac{\beta}{1+\beta}\right) Mg \sin \alpha \cos \omega t, \tag{72a}$$

$$F_j(t) = (Mg \sin \alpha) \sin \omega t + M\omega(v_0 + 4a\omega \sin \omega t), \tag{72b}$$

$$= M\omega[v_0 + 4a(2+\beta)\omega \sin \omega t]. \tag{72c}$$

The frictional component F_i acting at P is directed opposite to the rolling motion along $\hat{i}$ and generates the torque along the direction $\hat{j}$ required for rolling along the perturbed cycloid. It is oscillatory and averages to zero over the period $T = 2\pi/\omega$. Rolling rather than sliding always occurs, if the coefficient of friction with the plane is greater than $\beta(1+\beta)^{-1} \tan \alpha$. The transverse frictional component F_j at P is also oscillatory with an average of $\langle F_j \rangle = M\omega v_0$. The radius of curvature of the (x, y)-trajectory is

$$\rho(t) \equiv \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{(\dot{x}\ddot{y} - \ddot{x}\dot{y})} = (R' + 4a \sin \omega t) = \left(\frac{v}{\omega}\right), \tag{73}$$

with the result that (72c) may be re-expressed as

$$F_j(t) = \frac{Mv^2(t)}{\rho(t)} + (Mg \sin \alpha) \sin \omega t. \tag{74}$$

The frictional component F_j at P provides the required centripetal (inward) force $(Mv^2/\rho) = M\omega v$ for curved motion and offsets the center of mass gravitational component along $-\hat{j}$. Equation (72) may also be determined by comparing the acceleration constraints, Eq. (57a), supplemented by (62a), and Eqs. (57b) with (71), thereby highlighting the value of utilizing constraint equations expressed in acceleration form.

In addition to supplying the centripetal force, the frictional component, F_j, also generates a torque $(RF_j)\hat{i}$ about the center of mass, which will cause the penny to fall flat on its face. A supporting counter-balancing torque N_a must therefore be applied along $\hat{i}$ to ensure that the disk remains upright and can be determined as follows. The angular momentum about the center of mass is $\mathbf{L} = (I_2\dot{\psi})\hat{j} + (I_3\dot{\phi})\hat{k}$. Because $d\hat{j}/dt = -\omega\hat{i}$, the torque-angular momentum rule yields

$$\begin{aligned} \dot{\mathbf{L}} &= -(I_2\dot{\psi}\dot{\phi})\hat{i} + (I_2\ddot{\psi})\hat{j} + (I_3\ddot{\phi})\hat{k} \\ &= (N_a + RF_j)\hat{i} - (RF_i)\hat{j}, \end{aligned} \tag{75}$$

where $F_{i,j}$ are the frictional components given in Eq. (72). Hence, $I_2\ddot{\psi} = RF_i$ and $I_3\ddot{\phi} = 0$, in expected agreement with Eq. (62), supplemented by (72a). Also, $N_a = -(I_2\dot{\psi}\dot{\phi} + RF_j)$ is the torque applied about the center of mass to keep the penny upright. Then

$$N_a\,\hat{i} = -M(1+\beta)\omega R(v_0 + 8a\omega\sin\omega t)\hat{i} = -2bF_a\,\hat{i} \quad (76)$$

is the applied oscillatory torque with the average value $-M(1+\beta)\omega R v_0$. This torque, directed along $-\hat{i}$ opposite to the motion, may be supplied via a force couple $F_a\hat{j}$ and $-F_a\hat{j}$ acting, respectively, at fixed points $(R\pm b)\hat{k}$ on the penny.

B. Equations of state with adjoined constraints and physical motion

When the constraints are expressed as Eqs. (56b) and (58), they can be adjoined to Eq. (2) via the method developed in Sec. IV. From Eq. (29) for general velocity constraints, the equations of state are

$$M\ddot{x} = Mg\sin\alpha + 2\lambda_1\dot{x} + \lambda_2\sin\phi = Mg\sin\alpha + F_x \quad (77a)$$

$$M\ddot{y} = 2\lambda_1\dot{y} - \lambda_2\cos\phi = F_y \quad (77b)$$

$$I_2\ddot{\psi} = -2\lambda_1 R^2\dot{\psi} = N_j \quad (77c)$$

$$I_3\ddot{\phi} = 0, \quad (77d)$$

where $F_{x,y}$ are the frictional components along the fixed directions $\hat{I}$ and $\hat{J}$ and $N_j = RF_i$ is the frictional torque along $\hat{j}$ required for rolling. The solution of Eq. (77) reproduces the orbit, Eq. (68), and provides the multipliers $\lambda_{1,2}$, from which the frictional components $F_i = (F_x\cos\phi + F_y\sin\phi) = 2\lambda_1\dot{x}$ and $F_j = (-F_x\sin\phi + F_y\cos\phi) = -\lambda_2$ along $\hat{i}$ and $\hat{j}$ are directly determined. They are in agreement with those in Eq. (72). On introducing the inclination angle θ that the symmetry axis $\hat{j}$ of the penny makes with the fixed axis $\hat{K}$, a more complicated Lagrangian L involving five generalized coordinates $(x, y, \psi, \phi, \theta)$ was constructed. The resulting five equations of state with the constraints Eqs. (56b), (58), and $\theta = \pi/2$ adjoined by the multipliers $\lambda_{1,2,3}$ reproduce Eq. (77) together with the additional equation $(I_2\dot{\psi}\dot{\phi}) = -\lambda_3 = -(N_a + RF_j)$, where λ_3 is the torque in the direction $\hat{j}$. The torque N_a applied to keep the penny upright agrees with Eq. (76) obtained from the Newtonian analysis.

The virtual work performed by the constraints is

$$\begin{aligned} Q_j^{\mathrm{C}}\delta q_j &= 2\lambda_1(\dot{x}\delta x + \dot{y}\delta y - R^2\dot{\psi}\delta\psi) \\ &\quad + \lambda_2(\sin\phi\,\delta x - \cos\phi\,\delta y), \end{aligned} \quad (78)$$

which, with the aid of Eqs. (56b) and (58) reduces to zero, as required for ideal constraints.

The orbit and physical motion. The orbit of the contact point P is given by Eq. (68) which, in terms of the turning angle $\phi = \omega t$, has the parametric form

$$x(\phi) = a(1-\cos 2\phi) + R'\sin\phi \quad (79a)$$

$$y(\phi) = a(2\phi - \sin 2\phi) + R'(1-\cos\phi) \quad (79b)$$

$$s(\phi,\phi_1) = R'(\phi - \phi_1) + 4a(\cos\phi_1 - \cos\phi), \quad (79c)$$

with respect to an origin centered at the initial starting point (x_0, y_0). For turning and rolling motion about the penny's figure and symmetry axes $\hat{k}$ and $\hat{j}$, the orbits will vary in size and shape according to the parameters $a = g_d/4\omega^2$ and $R' = v_0/\omega$ established by the initial conditions. When $R' = 0$ and $a > 0$, the orbit is a cycloid. When $a = 0$, the motion is on a horizontal plane and the orbit for non-zero v_0 is the circle of Eq. (70) with fixed radius $R' = v_0/\omega$. As R' is increased from zero, the paths for non-zero a range from cycloids perturbed by additional circular motion to circles perturbed by cycloidal motion. The orbit may also be represented by

$$(x-a)^2 + [y - (R' + 2a\phi)]^2 = R'^2 + 2aR'\sin\phi + a^2 \quad (80)$$

which is a path of a point of the rim of a circle, whose center $(a, R' + 2a\phi)$ moves along $x = a$ at constant speed $\dot{y} = 2a\omega$ and whose radius varies between $|R' - a|$ and $(R' + a)$. When viewed in a frame moving with speed $\dot{y} = 2a\omega$, the orbit convolutes to the closed orbit

$$(x-a)^2 + (y-R')^2 = R'^2 + 2aR'\sin\phi + a^2. \quad (81)$$

The general orbit Eq. (79) can also be expressed in terms of the path-length (68c) as

$$[(4a + R'\phi) - s]^2 = 8a[(2a + R'\sin\phi) - x]. \quad (82)$$

Equations (79)–(82) facilitate analysis of the featured orbits.

The following five cases, each characterized by an increase in the initial velocity v_0, emerge naturally and are illustrated in Figs. 2 and 3 for various values of $R'/a = v_0/(\omega a)$. Equation (63) shows that the rolling motion can be forward or backward when $R' < 4a$ and that it is only forward for $R' \geq 4a$. By obeying the knife-edge condition, (56b), the gradients $dy/dx = \tan\phi$ remain the same for all orbits at a fixed φ. The patterns for all cases have a period of $\phi = 2\pi$. Animations of the motion along each trajectory are also presented in the online publication.

Case 1. $R' = 0$, that is, $v_0 = 0$. The penny rolls from rest down the hill, constantly turning counterclockwise with constant angular velocity ω and traces out the orbit

$$x(\phi) = a(1-\cos 2\phi) \quad (83a)$$

$$y(\phi) = a(2\phi - \sin 2\phi) \quad (83b)$$

$$(4a - s)^2 = 8a(2a - x), \quad (83c)$$

which are the parametric equations for the cycloid shown in Fig. 2(a). An equivalent expression for the cycloid is

$$(x-a)^2 + (y - 2a\phi)^2 = a^2, \quad (84)$$

which is the path traced by a point on the rim of a circle of fixed radius a which rolls on the straight line $x = 0$ and whose center moves along $x = a$ at constant speed $\dot{y} = 2a\omega$. In the moving frame, Eq. (84) is a fixed circle of radius a. At $\phi = \pi/2$, the penny reaches the cycloid minimum at $x(\pi/2) = 2a$ with maximum speed $v_{\max} = 4a\omega$. It then rolls uphill with a constant turning (spinning) rate ω, until at $\phi = \pi$, it comes to rest at its initial level $x_0 = 0$, but it is displaced sideways by $y(\pi) = 2\pi a$ at the cusp. Although instantaneously at rest at $\phi = \pi$, it has an acceleration downhill so that it rolls backward while turning along the second segment $\pi \leq \phi \leq 2\pi$ of the cycloid, until its motion is again reversed at $\phi = 2\pi$. The pattern is repeated continually, with reversals in rolling occurring between each successive segments, $n\pi \leq \phi \leq (n+1)\pi$. The segments $n = 1, 3, 5\ldots$, are "reversal" lanes, where $\dot{\psi} < 0$, in contrast to the forward lanes, $n = 0, 2, 4\ldots$ where $\dot{\psi} > 0$. The forward and backward

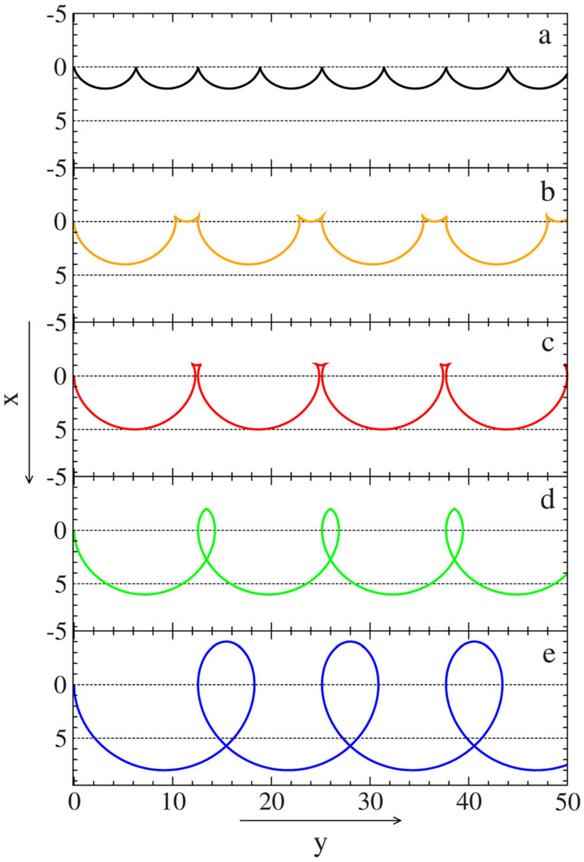

Fig. 2. Nonholonomic penny rolling and turning upright along orbits Eq. (79) on an inclined plane. Coordinates are $x/a, y/a$. Orbits (a)–(e) represent increasing values of $v_0/\omega a = R'/a =$ (a) 0, (b) 2, (c) 3, (d) 4, and (e) 6. They are mainly circularly-expanded cycloids (enhanced online). [URL: http://dx.doi.org/10.1119/1.3563538.1]; [URL:http://dx.doi.org/10.1119/1.3563538.2]; [URL:http://dx.doi.org/10.1119/1.3563538.3]; [URL:http://dx.doi.org/10.1119/1.3563538.4]; [URL:http://dx.doi.org/10.1119/1.3563538.5]

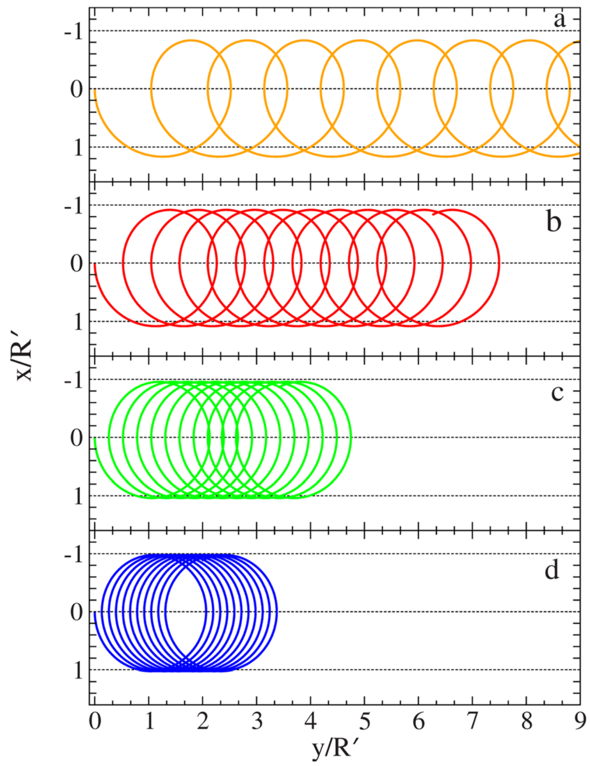

Fig. 3. Continuation of Fig. 2 for the nonholonomic penny rolling and turning along orbits Eq. (79) on an inclined plane. Coordinates are now $x/R', y/R'$. Orbits (a)–(d) represent increasing values of $v_0/\omega a = R'/a =$ (a) 12, (b) 24, (c) 48, (d) 96, and $\phi \leq 21\pi$. They are mainly cycloidal-perturbed circles with centers moving adiabatically with respect to more-rapid circular motion. The minima and maxima are at $(1 + 2a/R')$ and $-(1 - 2a/R')$, respectively. They are mainly circles with moving centers (enhanced online). [URL:http://dx.doi.org/10.1119/1.3563538.6]; [URL:http://dx.doi.org/10.1119/1.3563538.7]; [URL:http://dx.doi.org/10.1119/1.3563538.8] [URL: http://dx.doi.org/10.1119/1.3563538.9]

lanes are identical in size and the length of each lane (segment of the cycloid) is $s = 8a$ with enclosed area $3\pi a^2$. The orbit in Fig. 2(a) always oscillates with x between 0 and $2a$ and the horizontal distance covered by each oscillation is $\Delta Y = 2\pi a$. For large initial rates ω of spinning, the range decreases as $2a = g_d/2\omega^2$ and the oscillations in time with period π/ω become increasingly rapid and less perceptible to the eye so that the averaged orbital velocity is zero and the penny appears to move horizontally along the line $\langle x\rangle = a = g_d/4\omega^2$ across the plane at constant drift speed $\langle \dot{y}\rangle = 2a\omega = g_d/2\omega$.

Case 2. $0 < R' < 4a$, that is, the averaged orbital velocity $\langle v\rangle = v_0 < 4a\omega = 2\langle \dot{y}\rangle$. As the initial speed v_0 increases from zero, the R'-circular terms in Eq. (79) perturb the cycloidal orbit. The penny starts with velocity $v_0\hat{I}$, and rolls along the circularly-expanded cycloid, and reaches the primary minimum at $x_+ = x(\pi/2) = 2a + R'$, at maximum speed $v_+ = v_0 + 4a\omega$. On its uphill journey, it passes its initial horizontal level $x_0 = 0$ where $\phi = \pi$ at speed v_0 and penetrates into the uphill region $x < 0$, shown in Fig. 2(b). The penny then stops instantaneously at $x_{\text{rest}}(\phi_1) = -R'^2/8a$, where $\phi_1 = \pi + \gamma$ with $\gamma = \sin^{-1}(R'/4a) < \pi/2$. The penny then proceeds to roll backward along a much smaller secondary segment to reach a secondary minimum at $x_- = x(3\pi/2) = 2a - R'$, at speed $v_- = |v_0 - 4a\omega|$. The rolling backward ceases at $\phi_2 = 2\pi - \gamma$ where the penny stops instantaneously and proceeds to roll forward down to its initial level x_0 at $\phi = 2\pi$. The downhill range $\Delta X = x_+(\pi/2) - x_{\text{rest}}(\phi_1) = 2a + R' + R'^2/8a$ increases with R'.

Reversals in rolling always occur between the $\{0, \phi_1\}$ forward and $\{\phi_1, \phi_2\}$ reverse lanes which, in contrast to Case 1, now differ in size. The distances covered in the various segments are

$$s(0, \pi) = \pi R' + 8a \tag{85a}$$

$$s(\pi, \phi_1) = s(\phi_2, 2\pi) = \gamma R' - 4a(1 - \cos\gamma) \tag{85b}$$

$$s(\phi_1, \phi_2) = 8a\cos\gamma - (\pi - 2\gamma)R' \tag{85c}$$

$$s(\pi, 2\pi) = (4\gamma - \pi)R' + 8a(2\cos\gamma - 1). \tag{85d}$$

The reverse lane $\{\phi_1, \phi_2\}$ is traveled at reduced speeds and is therefore much shorter than that for the pure cycloid, as in Figs. 2(b) and 2(c). The ratio of the line element ds to that for the pure cycloid is $1 + R'/(4a\sin\phi)$. Expansion of ds therefore occurs in the $\{0, \pi\}$ segment while contraction occurs in the $\{\pi, 2\pi\}$ segment. For higher v_0, the expansion and contraction each become more pronounced, as shown by comparing Figs. 2(b) and 2(c). The initial level x_0 is crossed at $\phi = n\pi$ for all R'. When $R' < 2a$, there are additional crossings at $\phi_3 = [\pi + \sin^{-1}(R'/2a)]$ and $\phi_4 = [2\pi - \sin^{-1}(R'/2a)]$. When $R' = 2a$, these additional crossings converge to $3\pi/2, 7\pi/2, ..$ and produce the minima at $x_0 = x_-$, as shown in Figs. 2(b). For $2a < R' < 4a$, these minima rise to

$x_- = -(R' - 2a) < 0$ and the additional crossings disappear. The reversal $\{\phi_1, \phi_2\}$ lane is maintained for $R' < 4a$. The above patterns and angles $\phi = (0, \pi, \phi_1, \phi_2, \phi_3, \phi_4, 2\pi)$ have period 2π.

Case 3. $R' = 4a$, that is, $\langle v \rangle = v_0 = 2\langle \dot{y} \rangle$. As R' increases to $4a$, $\phi_{1,2} \to 3\pi/2$, $s(\phi_1, \phi_2) \to 0$, $s(0, \pi) = \pi R' + 8a = 4a(\pi + 2)$, $s(\pi, 2\pi) = \pi R' - 8a = 4a(\pi - 2)$ and $s(0, 2\pi) = 2\pi R'$. The maxima at $\{\phi_1, \phi_2\}$ in Figs. 2(b) and 2(c) have now combined into one maxima at $x_-(3\pi/2) = -(R' - 2a) = -2a$ and $y_-(3\pi/2) = R' + 3\pi a$ where both $\mathbf{v}_{\text{rest}}$ and $\dot{\mathbf{v}}_{\text{rest}}$ are zero, as shown in Fig. 2(d). The reversal lanes have disappeared, and the motion is continuous. The penny stops momentarily at the maximum, but it keeps turning counterclockwise at angular speed ω, thereby picking up acceleration, which enables it to roll and turn down the hill until it reaches the minimum at $x_+ = R' + 2a = 6a$, as displayed in Fig. 2(d). This case marks the onset of "looping-the-loop" where, in contrast to Figs. 2(b) and 2(c) for $R' < 4a$, $x(2\pi) < x(\pi)$ for $R' \geq 4a$. Also the mean orbiting velocity $\langle v \rangle = v_0$ has increased to twice the drift speed $\langle \dot{y} \rangle = 2a\omega$.

Case 4. $R' > 4a$, that is, $\langle v \rangle = v_0 > 2\langle \dot{y} \rangle$. The velocity is now always positive. The penny has initial rolling speed sufficiently high to keep rolling at the highest level $x_- = x[(3/2 + 2n)\pi)] = -(R' - 2a)$ without stopping down to the lowest level $x_+ = x[(1/2 + 2n)\pi)] = R' + 2a$, as displayed in Figs. 2(e) and 3(a)–3(d). The range covered downhill is, $\Delta X = 2R' = 2v_0/\omega$, independent of gravity and depends only on the initial conditions. Succeeding maxima and minima are separated by $\Delta Y = 4\pi a$, which is independent of v_0. The lower segment $\{0, \pi\}$ has path length $s(0, \pi) = (\pi R' + 8a)$, which is greater than $s(\pi, 2\pi) = (\pi R' - 8a)$ for the upper segment $\{\pi, 2\pi\}$. Also, $s(0, 2\pi) = 2\pi R'$.

Case 5. $R' \gg 4a$, that is, $\langle v \rangle = v_0 \gg 2\langle \dot{y} \rangle$. With further increase in R', the circular terms now increasingly dominate the a-cycloid gravitational terms in Eq. (79) as illustrated in Figs. 3(a)–3(d), where the orbits become more circular and "slinky" in character. For $R' \gg a$, the orbit Eq. (79) tends to

$$x(\phi) = a + R' \sin\phi, \tag{86a}$$

$$y(\phi) = 2a\phi + R'(1 - \cos\phi), \tag{86b}$$

which are the parametric equations for a prolate (extended) cycloid (with $R' \geq 2a$), which is the path of a point at distance $R > a$ from, and rigidly connected to, the center of a circle of radius a which is rolling on the straight line $x = 0$. The orbit Eq. (80) also tends to

$$[x - a]^2 + [y - (R' + 2a\phi)]^2 = R'^2, \tag{87}$$

which is a circle of fixed radius $R' = v_0/\omega$, whose center $(a, R' + 2a\phi)$ moves adiabatically with respect to the circular speed v_0 along the y-axis at constant speed $\dot{y} = 2a\omega \ll v_0$, the root cause of the "slinky" behavior. The path lengths $(\pi R' \pm 8a)$ of each successive segments $\{0, \pi\}$ and $\{\pi, 2\pi\}$ approach $\pi R'$. It is only when $a = g_d/4\omega^2 \to 0$ that the center's speed $2a\omega$ reduces to zero and the circles of Fig. 3(d) eventually coalesce to one fixed circle of constant radius $R' = (v_0/\omega)$. In this limit Eq. (87) reduces to the appropriate result Eq. (70) for upright spinning motion on a horizontal plane.

A remarkable property of all the orbits displayed in Figs. 2 and 3 is that each trajectory, when averaged over a full period $2\pi/\omega$ in t, or 2π in φ is along the same horizontal line

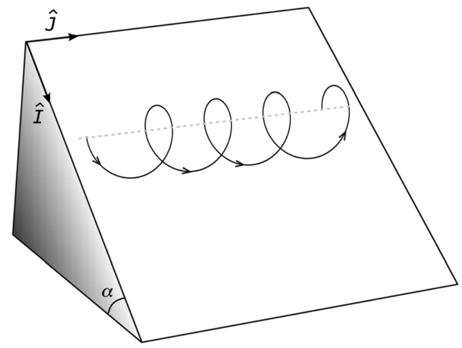

Fig. 4. Schematic of skater sliding and turning along a prolate cycloid on an inclined plane with speed $v(t) = v_0 + 4a_0\omega \sin\omega t$, where $v_0 > 4a_0\omega$ is the initial speed, ω is the constant frequency for angular turning, and $a_0 = g \sin\alpha/4\omega^2$. On average, the skater follows the horizontal line $\langle x \rangle = a$ across the plane at constant speed $\langle \dot{y} \rangle = 2a_0\omega$, irrespective of v_0.

$\langle x \rangle = a = g_d/4\omega^2$ across the plane at constant mean speed $\langle \dot{y} \rangle = 2a\omega = g_d/2\omega$, irrespective of v_0. On average, the penny does not roll further down the plane past a. Also, as ω increases, the oscillations in x become so rapid that the penny is perceived to move along $x = a = g_d/4\omega^2$ at constant speed $2a\omega$. The distance between the minima of Figs. 2(b) and 2(c) and the minima and maxima of the remaining orbits is the range of $\Delta X = 2R' = 2v_0/\omega$, which is unaffected by gravity, depending only on the initial conditions. The separation $\Delta Y = 4\pi a$ between the succeeding maxima (and minima) depends on gravity and is independent of v_0.

Ice skater/snowboarder on inclined plane. If there is no rolling but only sliding, only three generalized coordinates (x, y, ϕ), constrained only by the "knife-edge" condition, (56b), are needed. Examples are an ice skater or a plate with center of mass located at the knife-edge. Neimark has provided the solution for the $v_0 = 0$ case.[19] The solution for general v_0 can be determined *ab-initio* from Eq. (29) or deduced simply by setting the inertia coefficient $\beta = 0$ in the general solution of Sec. VI A for the nonholonomic rolling/turning penny. The orbit is given by Eq. (68), but with $a(\beta = 0) = a_0 = g \sin\alpha/4\omega^2$ while $R' = v_0/\omega$. The skater begins with velocity $v_0\hat{I}$, keeps turning at the initial turning rate ω, and then traces out the various cycloid/circle combinations, as displayed in Figs. 2–4, with speed $v(t) = v_0 + 4a_0\omega \sin\omega t$ along the path $s(t) = v_0 t + 4a_0(1 - \cos\omega t)$. As v_0 increases up to $4a_0\omega$, the skater traces out orbits with primary and secondary minima separated by $2R'$, as in Figs. 2(a)–2(c), and the downhill X-range is $\Delta X = 2a_0 + R' + R'^2/8a_0$. When $v_0 \geq 4a_0\omega$, "looping-the-loop" between minima and maxima separated by $\Delta X = 2R' = 2v_0/\omega$, the range downhill, are also displayed, as in Figs. 2(d) and 2(e) and Fig. 3. The downhill length of the inclined plane must be greater than ΔX for the full orbits to be traversed. On average, the skater follows the horizontal line $\langle x \rangle = a_0$ across the plane at constant speed $\langle \dot{y} \rangle = 2a_0\omega$, irrespective of v_0.

The force actuating the knife-edge constraint, (56b), is the sideways friction (72c) acting at P along $\hat{j}$, transversely to the skating direction $\hat{i}$. This sideways friction force, fully offsets the transverse component $-(Mg \sin\alpha \sin\phi)\hat{j}$ of gravity at the center of mass and also supplies the centripetal force mv^2/ρ, where the radius of curvature is v/ω. When starting from rest, $v_0 = 0$, the overall distance for one cycloidal segment is $8a_0$ traveled in the time $T = \pi/\omega = 2\pi(a_0/g \sin\alpha)^{1/2}$, and each segment encloses an area $3\pi a^2$ with the line $x = x_0$. Note that the skater may start from rest at any point along the pure cycloid and the travel time from initial rest to the final rest positions remains fixed at T, an interesting illustration of the *tautochrone* problem of finding the path, the cycloid $(4a - s)^2 = 8a(2a - x)$, down which a bead placed at rest anywhere will fall to the bottom and up again in the same amount of time.

Cart Wheels. The solution[39] for an assembly of two identical thin wheels with centers joined by a uniform axle is identical with that for the nonholonomic penny, but with $g_d = g \sin\alpha/(1 + 2\beta)$. The present general solution shows that the cart's center of mass at the center of the axle follows the orbits displayed in Figs. 2 and 3. This assembly may therefore be used to demonstrate the motion of the penny kept upright by the applied torque, Eq. (71).

The solution, Eq. (68), for the nonholonomic penny on an inclined plane is quite general with various applications. The five cases studied above provide an instructive and interesting case study, which has not been previously discussed in the literature.

VII. SUMMARY

We have shown how the elusive problem of utilizing the d'Alembert-Lagrange principle for nonholonomic constraints Eqs. (12) and (34) with general dependence on velocity and acceleration can be solved. The property of possible displaced states compatible with general velocity and acceleration constraints allows us to provide a set of linear conditions on the virtual displacements required for adjoining to the d'Alembert-Lagrange equation. We then derived equations of state, Eqs. (29) and (49), for dynamical systems under general velocity and acceleration constraints, Eqs. (12) and (34), respectively. These equations of state agree with those obtained[30] from Gauss' principle. The nonholonomic displacement conditions imply new transpositional relations that differ from the commutation rule traditionally accepted in Lagrangian dynamics.

The theory was tested by considering the non-integrable quadratic constraint Eq. (8). Solutions for the nonholonomic penny on an inclined plane were also obtained by embedding the linear-velocity constraints in the d'Alembert-Lagrange principle, as in Sec. VI A, to be tested with those obtained from quadratic velocity and acceleration forms of the original linear-velocity constraints in Eqs. (29) and (49) appropriate to general nonholonomic adjoined constraints. The geometric orbits of the nonholonomic penny for various initial velocities were found to exhibit interesting and instructive features. It is hoped that the present paper will serve as a welcome addition to the literature of nonholonomic systems.

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Sommerfelds Spring. Arnold Summerfeld published Coupled Oscillations of a Helical Spring in 1923 (J.O.S.A. & R.S.I., 7, 531-535). He had proposed the device in 1905, and the apparatus was made by the Central Scientific Company. In the paper he extends the discussion of the Wilberforce pendulum (1894), in which the up and down oscillation of the spring alternates with a torsional oscillation. If the two oscillations have frequencies close to each other (the tuning is done with the small masses), the two modes of oscillation exchange energy for a long time. The tag on the apparatus shows that it was sold by Cenco in 1930; by 1936 it was out of the catalogue, so it does not appear to have been commercially successful. The instrument is in the Greenslade Collection. (Notes and photograph by Thomas B. Greenslade, Jr., Kenyon College)